Bielecki's Method, Existence and Uniqueness Results for Volterra Integral Equations in $L^p$ Space

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Submitted by V. Lakshmikantham

Received August 8, 1989

1. Introduction

Bielecki’s method [3, 4] of weighted norm has been used very frequently to obtain global existence and uniqueness results for wide classes of differential, differential-delay, integral, integro-differential, integro-functional, and many other functional equations. For a review of the results obtained by the mentioned method, see C. Corduneanu’s paper [6]. For some extension of Bielecki’s method, see also [7]. At the present time, there is a huge number of papers which make use of Bielecki’s method frequently not quoting its author. In fact the method became a standard technique in dealing with the mentioned problems.

It is important to observe that up to now Bielecki’s method was used, as a rule, for fixed point equations considered in spaces of continuous or bounded and measurable functions (see [2, 8]). P. R. Beesack [2] tried to obtain by Bielecki’s method an existence and uniqueness result for multidimensional Volterra integral equations in $L^2$ space taking the weighted norm

$$\|u\|^2 = \int_a^b \omega(t) |u(t)|^2 \, dt$$

and he found that the method does not work because the equation which he obtained for the weight function $\omega$ sometimes has no solution. Finally, he proved the result by adopting the classical successive approximation method.

The aim of the present paper is to show that Bielecki’s method works fairly well for such equations considered even in $L^p$ spaces. One thing we need to change is the definition of the norm by taking $\omega$ outside of the integration symbol, replacing $b$ by $x$ and employing supremum operation (see our definition (3) of the norm in $L^p$ space). We will show also that an
extension of Bielecki's method similar to that formulated in [7] for equations considered in continuous function space works fairly well in the case discussed in the present paper. Finally, a comparison result will be formulated which applies in the cases when Bielecki's method fails. It is worth noting that these approaches are applications of the general results formulated in [9].

2. INTEGRAL EQUATION WITH ONE DIMENSIONAL INDEPENDENT VARIABLE

Consider the Volterra integral equation of the form

\[ u(x) = g(x) + \int_0^x f(x, s, u(s)) \, ds, \quad x \in [0, a] = I_a, \quad a > 0. \tag{1} \]

Let \( L^p(I_a, \mathbb{R}^N) \), \( p > 1 \), denote the space of all measurable functions \( u \) defined on \( I_a \) with values in \( \mathbb{R}^N \) such that

\[ \int_0^a |u(s)|^p \, ds < +\infty, \]

here \( |\cdot| \) stands for any fixed norm in \( \mathbb{R}^N \).

We assume that \( g \in L^p(I_a, \mathbb{R}^N) \), \( f: I_a \times I_x \times \mathbb{R}^N \to \mathbb{R}^N \) is a measurable function and

\[ \int_0^a \left( \int_0^x |f(x, s, 0)|^p \, ds \right) \, dx < +\infty. \]

It is assumed also that there is a nonnegative and measurable function \( L \) defined on \( I_a \times I_x \), \( x \in I_a \), such that

\[ \int_0^a \left[ \int_0^x (L(x, t))^{q} \, dt \right]^{p/q} \, dx < +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \]

and

\[ |f(x, s, u) - f(x, s, v)| \leq L(x, s) |u - v| \tag{2} \]

for all \( u, v \in \mathbb{R}^N \) and a.e. in \( I_a \times I_x \).

Let \( \omega: I_a \to R_+, \quad R_+ = [0, +\infty) \), be a continuous function, assume that \( \omega(x) > 0, \quad x \in I_a \). Put

\[ \|u\|_{\mu, \omega} = \left( \sup \left\{ \omega^{-1}(x) \int_0^x |u(s)|^p \, ds; \quad x \in I_a \right\} \right)^{1/p}. \tag{3} \]
Note that for \( \omega(x) \equiv 1 \) we obtain the classical norm \( \| u \|_p \) in \( L^p(I_a, \mathbb{R}^N) \) space which is a Banach space. However, it is easy to see that Eq. (3) defines a norm for any \( \omega \). Indeed, multiplying the Minkowski inequality by \( (\omega(x))^{-1/p} \) we obtain

\[
\left( \omega^{-1}(x) \int_0^x |u(s) + v(s)|^p \, ds \right)^{1/p} \leq \left( \omega^{-1}(x) \int_0^x |u(s)|^p \, ds \right)^{1/p} + \left( \omega^{-1}(x) \int_0^x |v(s)|^p \, ds \right)^{1/p}.
\]

Now taking the supremum with respect to \( x \in I_a \), we obtain the triangle inequality

\[
\| u + v \|_{p, \omega} \leq \| u \|_{p, \omega} + \| v \|_{p, \omega}.
\]

It is clear that \( \| \cdot \|_{p, \omega} \) has the other norm properties. Moreover, the inequality

\[
c_1 \| u \|_p \leq \| u \|_{p, \omega} \leq c_2 \| u \|_p
\]

is true for

\[
c_1 = (\sup\{\omega(x) : x \in I_a\})^{-1/p}, \quad c_2 = (\inf\{\omega(x) : x \in I_a\})^{-1/p}.
\]

This means that the norm \( \| \cdot \|_{p, \omega} \) is equivalent to \( \| \cdot \|_p \). Now we can formulate

**Theorem 1.** Under the conditions assumed the operator \( F \) defined by the right hand side of equation (1) is a contradiction in \( L^p(I_a, \mathbb{R}^N) \) with respect to the norm \( \| \cdot \|_{p, \omega} \) and \( \omega \) defined by

\[
\omega(x) = \exp \left( \lambda \int_0^x M(s) \, ds \right), \quad M(s) = \left( \int_s^x L^q(s, t) \, dt \right)^{p/q}, \quad \lambda > 1. \tag{4}
\]

Equation (1) has in \( L^p(I_a, \mathbb{R}^N) \) a unique solution which can be obtained as the limit of successive approximations.

**Proof.** Observe first that \( F(L^p(I_a, \mathbb{R}^N)) \subset L^p(I_a, \mathbb{R}^N) \). Now using the Hölder inequality we obtain
\[(Fu)(t) - (Fv)(t)\|^p \leq \left( \int_0^t |f(t, s, u(s)) - f(t, s, v(s))| \, ds \right)^p
\leq \left( \int_0^t L(t, s) |u(s) - v(s)| \, ds \right)^p
\leq \left( \int_0^t (L(t, s))^\alpha \, ds \right)^{p/\alpha} \cdot \int_0^t |u(s) - v(s)|^p \, ds
\leq M(t) \int_0^t |u(s) - v(s)|^p \, ds.
\]

Integrating this inequality with respect to \( t \) we find
\[
\int_0^\infty |(Fu)(t) - (Fv)(t)|^p \, dt
\leq \int_0^\infty \left( M(t) \int_0^t |u(s) - v(s)|^p \, ds \right) \, dt
= \int_0^\infty \left[ M(t) \exp \left( \lambda \int_0^t M(s) \, ds \right) \exp \left( -\lambda \int_0^t M(s) \, ds \right) \int_0^t |u(s) - v(s)|^\alpha \, ds \right] \, dt
\leq \| u - v \|_{p,\omega}^p \cdot \int_0^\infty M(t) \exp \left( \lambda \int_0^t M(s) \, ds \right) \, dt
\leq \frac{1}{\lambda} \| u - v \|_{p,\omega}^p \exp \left( \lambda \int_0^t M(s) \, ds \right).
\]

This inequality implies the following one
\[
\exp \left( -\lambda \int_0^\infty M(s) \, ds \right) \int_0^\infty |(Fu)(t) - (Fv)(t)|^p \, dt \leq \frac{1}{\lambda} \| u - v \|_{p,\omega}^p
\]
which means that
\[
\| Fu - Fv \|_{p,\omega}^p \leq \frac{1}{\lambda} \| u - v \|_{p,\omega}^p
\]
and
\[
\| Fu - Fv \|_{p,\omega} \leq \lambda \cdot \lambda \cdot \| u - v \|_{p,\omega},
\]
with \( \lambda = \lambda^{-1/p} < 1 \). The proof is complete.
Remark 1. Note that in our considerations the interval $I_a$ can be replaced by $I_\infty = [0, +\infty) = \mathbb{R}_+$. It is also easy to see that only small adjustments are necessary in order to obtain a similar result for the case when the space $L^p(I_a, \mathbb{R}^N)$ is replaced by $L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^N)$.

3. INTEGRAL EQUATION WITH MULTIDIMENSIONAL INDEPENDENT VARIABLE

Let us now assume that in Eq. (1) $x = (x_1, \ldots, x_n)$, $s = (s_1, \ldots, s_n)$, $a = (a_1, \ldots, a_n)$, $I_a = I_{a_1} \times \cdots \times I_{a_n}$, $ds = ds_n \cdots ds_1$, and

$$\int_0^x = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n}.$$

We keep formally the same assumptions about $g$, $f$, and $L$ as before. We also formally use the same definition of the norm in $L^p(I_a, \mathbb{R}^N)$ given by Eq. (3). Similar to before we obtain the evaluations

$$| (Fu)(t) - (Fv)(t) |^p \leq M(t) \int_0^t |u(s) - v(s)|^p ds,$$

$$\int_0^x | (Fu)(t) - (Fv)(t) |^p dt \leq \left( M(t) \int_0^t |u(s) - v(s)|^p ds \right) dt.$$

Hence for any positive and continuous function $\omega$ we obtain

$$\omega^{-1}(x) \int_0^x | (Fu)(t) - (Fv)(t) |^p dt$$

$$\leq \omega^{-1}(x) \int_0^x M(t) \omega(t) \sup_{t \in I_a} \left( \omega^{-1}(t) \int_0^t |u(s) - v(s)|^p ds \right) dt$$

$$= \| u - v \|^p_{p, \omega} \cdot \omega^{-1}(x) \int_0^x M(t) \omega(t) dt.$$

It is clear from this that $F$ will be a contraction in $L^p(I_a, \mathbb{R}^N)$ with respect the norm $\| \cdot \|_{p, \omega}$ if the function $\omega$ is a solution of the inequality

$$\omega^{-1}(x) \int_0^x M(t) \omega(t) dt \leq \frac{1}{\lambda}$$

for some $\lambda > 1$.

It is not difficult to find positive and continuous functions $\omega$ for which (5) holds. One may use the following.
Lemma [1]. If a function $D: \mathbb{R}_+ \rightarrow \mathbb{R}$ has a nondecreasing derivative $D'$ and the function $M: I_\alpha \rightarrow \mathbb{R}$ is $L$-integrable then

$$\int_0^x M(s) D' \left( \int_0^s M(t) \, dt \right) \, ds \leq D \left( \int_0^x M(t) \, dt \right) - D(0).$$

for any $x \in I_\alpha$.

Observe that here $x$, $t$, and $s$ are multidimensional variables. One can see easily that for the one dimensional case the symbol $\leq$ can be replaced by $=$.

Let us take $D(z) = \exp z$ then by the lemma we obtain

$$\lambda \int_0^x M(s) \exp \left( \lambda \int_0^s M(t) \, dt \right) \, ds \leq \left( \lambda \int_0^x M(t) \, dt \right) - 1.$$

This means that we can take

$$\omega(x) = \exp \left( \lambda \int_0^x M(t) \, dt \right)$$

and (5) will be satisfied.

Let us note that there are many functions $\omega$ for which (5) holds. For instance if $M$ is bounded, say by the number $\bar{M}$ then one can take

$$\omega(x) = \exp \left( \lambda \sqrt{\bar{M}} x^1 (x_1 + x_2 + \cdots + x_n) \right)$$

or

$$\omega(x) = \exp (\lambda \bar{M} x_1 x_2 \cdots x_n).$$

In the case when $M$ is not bounded in order to find a function $\omega$ for which (5) holds one can consider also the integral equation

$$\omega(x) = \lambda \int_0^x M(t) \omega(t) \, dt + 1.$$  

(6)

In the one dimensional case the unique solution of this equation is given by the formula (4). In the multidimensional case it is not so; however, a unique solution of (6) exists and it is given by the corresponding Neumann series [1]. The same is true for $1$ in Eq. (6) replaced by any $c > 0$.

Taking this discussion into account we can formulate

**Theorem 2.** In the multidimensional case under the conditions assumed the operator $F$ is a contraction in $L^p(I_\alpha, \mathbb{R}^N)$ with to the norm $\| \cdot \|_{p,c_0}$ with $\omega$ defined by (5') or being a solution of Eq. (6). Equation (1) has in
\(L^p(I, \mathbb{R}^N)\) a unique solution which can be found as the limit of successive approximations.

Remark 2. Observe that in multidimensional case the same comment as in Remark 1 holds for \(L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)\).

Remark 3. Note that in the case \(p = 1\) the function \(L\) in condition (2) has to be bounded with respect to the variable \(s\) say \(L(x, s) \leq M(x)\). In this case we obtain the evaluation

\[
|(Fu)(t) - (Fv)(t)| \leq M(t) \int_0^t |u(s) - v(s)| \, ds
\]

which after integration with respect to \(t\) over \(I\), and introducing a weight function \(\omega\) leads us to the inequality (5). This means that the case \(p = 1\) can be treated as that for \(p > 1\).

4. An Extension of Bielecki's Method for Abstract Functional Equations in \(L^p_{\text{loc}}\) Spaces

Let \(L^p_{\text{loc}}(G, \mathbb{R}^N)\) be the space of all locally \(L^p\) integrable \(\mathbb{R}^N\) vector valued functions defined on a measurable subset \(G\) of \(\mathbb{R}^N\). Let an operator \(F: L^p_{\text{loc}}(G, \mathbb{R}^N) \to L^p_{\text{loc}}(G, \mathbb{R}^N)\) be given.

Consider the equation

\[
u(x) = (Fu)(x), \quad \text{a.e. in } G.
\]  

We are interested in establishing the existence and uniqueness of the solution of Eq. (7). We will show that the approach is quite similar to that one presented in [7] for operators \(F\) defined on the space of continuous functions and works fairly well for our case.

Consider some \(u_0 \in L^p_{\text{loc}}(G, \mathbb{R}^N), \omega_0 \in L^p_{\text{loc}}(G, \mathbb{R}_+)\) and define

\[
V(\omega_0) = \{\omega : \omega \in L^p_{\text{loc}}(G, \mathbb{R}_+), 0 \leq \omega(x) \leq c\omega_0(x), \text{a.e. in } G, c > 0\}
\]

\[
D(u_0, \omega_0) = \{u : u \in L^p_{\text{loc}}(G, \mathbb{R}^N), |u(x) - u_0(x)| \leq c\omega_0(x), \text{a.e. in } G, c > 0\}.
\]

We adopt the following

Assumption A. Assume that

(i) There is a nondecreasing operator \(\Omega: V(\omega_0) \to V(\omega_0)\) such that

\[
|(Fu)(x) - (Fv)(x)| \leq \Omega(|u - v|(x), \text{ a.e. in } G,
\]

for every \(u, v \in D(u_0, \omega_0)\). Here \(|u - v|\) denotes the function \(G \ni x \to |u(x) - v(x)|\).
(ii) There is a function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \), upper semicontinuous from the right, such that \( \phi(s) < \phi(0) = 0 \), and
\[
\Omega(s \omega_0)(x) \leq \phi(s) \omega(x), \quad s \geq 0, \quad \text{a.e. in } G, \tag{9}
\]

(iii) There is a \( q_0 > 0 \) such that
\[
|u_0(x) - (Fu_0)(x)| := q(x) \leq q_0 \omega_0(x) \quad \text{a.e. in } G. \tag{10}
\]

Now we can formulate

**THEOREM 3.** If Assumption A is fulfilled then there exists in \( D(u_0, \omega_0) \) a unique solution of Eq. (7), say \( u^* \). The solution \( u^* \) is the limit of the sequence of iterations of \( u_0 \) by \( F \), i.e., \( F^k u_0 \to u^* \) in \( L^p_{\text{loc}}(G, \mathbb{R}^N) \).

**Proof.** Define in \( D(u_0, \omega_0) \) the metric
\[
d(u, v) = \inf \{ c : |u(x) - v(x)| \leq c \omega(x), \text{a.e. in } G, c \geq 0 \}. \tag{11}
\]
It is easy to check that \( D(u_0, \omega_0) \) is a complete metric space. Observe that for \( u \in D(u_0, \omega_0) \) we have
\[
|(Fu)(x) - u_0(x)| \leq |(Fu)(x) - (Fu_0)(x)| + |(Fu_0)(x) - u_0(x)|
\leq \Omega(|u - u_0|)(x) + q(x) \leq \Omega(c \omega_0)(x) + q_0 \omega_0(x)
\leq (\phi(c) + q_0) \omega_0(x),
\]
which means that \( F(D(u_0, \omega_0)) \subset D(u_0, \omega_0) \).

Now take any \( u, v \in D(u_0, \omega_0) \) then for every \( \varepsilon > 0 \) we have
\[
|(Fu)(x) - (Fv)(x)| \leq \Omega(|u - v|)(x) \leq \Omega((d(u, v) + \varepsilon) \omega_0)(x)
\leq \phi(d(u, v) + \varepsilon) \omega_0(x)
\]
which means that
\[
d(Fu, Fv) \leq \phi(d(u, v) + \varepsilon).
\]
Because \( \varepsilon \) is arbitrary and \( \phi \) is upper semicontinuous from the right we conclude that \( F \) is a nonlinear contraction in \( D(u_0, \omega_0) \), i.e.,
\[
d(Fu, Fv) \leq \phi(d(u, v)). \tag{12}
\]

The assertion of our theorem is a consequence of the Boyd–Wong result of [5].

**Remark 4.** Note that for \( \phi(s) = \alpha s, 0 \leq \alpha < 1 \) the operator \( F \) is a classi-
cal contraction in $D(u_0, \omega_0)$. This takes place if the operator $\Omega$ has the properties

$$\Omega(s\omega_0)(x) \leq s\Omega(\omega_0)(x), \quad s \geq 0, \quad \Omega(\omega_0)(x) \leq \omega_0(x) \text{ a.e. in } G,$$

for some $x \in [0, 1)$.

5. How Does One Find $\omega_0$?

It may be difficult to find a function $\omega_0$ for which Assumption A holds. To avoid this difficulty we will use the following stronger assumption

Assumption B. Assume that:

(i) there is a nondecreasing operator $\Omega: L^p_{\text{loc}}(G, \mathbb{R}^+) \rightarrow L^p_{\text{loc}}(G, \mathbb{R}^+)$ such that for every $u, v \in L^p_{\text{loc}}(G, \mathbb{R}^+),$

$$|(Fu)(x) - (Fv)(x)| \leq \Omega(|u - v|(x)) \text{ a.e. in } G,$$

(ii) $\Omega(s\omega) \leq s\Omega(\omega), \quad s \geq 0, \omega \in L^p_{\text{loc}}(G, \mathbb{R}^+),$

(iii) there exist $\lambda > 1$ and $\omega_0 \in L^p_{\text{loc}}(G, \mathbb{R}^+)$ such that

$$\omega_0(x) \geq \lambda \Omega(\omega_0)(x) + q(x), \quad \text{a.e. in } G.$$

One can see easily that Assumption A is an immediate consequence of Assumption B, it is enough to take $\phi(s) = s/\lambda$, so the assertion of Theorem 3 holds if Assumption A is replaced by Assumption B.

It is also clear that (see [7]) a solution of the inequality (15) exists:

(a) if $\Omega(q)(x) \leq \gamma q(x)$, a.e. in $G$, for some $\gamma \in (0, 1)$; in this case

$$\omega_0(x) = (1 - \lambda \gamma)^{-1} q(x) \text{ with } \lambda \gamma < 1,$$

or

(b) if the operator $\Omega$ is linear and the Neumann series of $\lambda \Omega,$

$$\sum_{i=0}^{\infty} \lambda^i (\Omega^i q)(x), \quad x \in G,$$

converges to some $\bar{\omega} \in L^p_{\text{loc}}(G, \mathbb{R}^+)$, now $\omega_0(x) = \bar{\omega}(x)$.

6. The Case $\lambda = 1$

Now the question is what can be done when Assumption B holds but only with $\lambda = 1$. This is the case for which Bielecki's method does not work because we are able to show only that $F$ is nonexpanding in the metric
space $D(u_0, \omega_0)$. To assure the assertion of Theorem 3 we have to make use of the comparison method (see [9]). Take

**Assumption C.** Assume that the nondecreasing operator $\Omega: L^p_{\text{loc}}(G, \mathbb{R}_+^+) \rightarrow L^p_{\text{loc}}(G, \mathbb{R}_+^+)$ has the properties:

(i) $\omega_k \in L^p_{\text{loc}}(G, \mathbb{R}_+^+)$, $\omega_{k+1}(x) \leq \omega_k(x)$, $k = 0, 1, \ldots, \omega_k(x) \rightarrow \omega(x)$, a.e. in $G$, implies

$$\Omega(\omega_k)(x) \rightarrow \Omega(\omega)(x) \quad \text{a.e. in } G \quad \text{as } k \to \infty.$$ 

(ii) $\Omega(s \omega) \leq s \Omega(\omega)$, $s \geq 0$, $\omega \in L^p_{\text{loc}}(G, \mathbb{R}_+^+)$,

(iii) $\omega_0 \in L^p_{\text{loc}}(G, \mathbb{R}_+^+)$ is such that

$$\omega_0(x) \geq \Omega(\omega_0)(x) + q(x), \quad \text{a.e. in } G, \quad (16)$$

with $q$ defined by Eq. (10),

(iv) $\omega(x) \equiv 0$, a.e. in $G$ is in $V(\omega_0)$ the only solution of the equation

$$\omega(x) = \Omega(\omega)(x), \quad \text{a.e. in } G, \quad (17)$$

(v) for all $u, v \in D(u_0, \omega_0)$,

$$|(F^k u)(x) - (F^k v)(x)| \leq \Omega(|u - v|)(x), \quad \text{a.e. in } G. \quad (18)$$

Now we have

**Theorem 4.** If Assumption C is fulfilled then the assertion of Theorem 3 holds.

**Proof.** By Assumption C it is clear that $\Omega^k(\omega_0) \rightarrow \bar{\omega}$ a.e. in $G$, $\bar{\omega} \leq \omega_0$, and $\bar{\omega} = \Omega(\bar{\omega})$, so $\bar{\omega}(x) \equiv 0$, a.e. in $G$.

By the induction rule we easily obtain

$$|F^{k+1}(u_0) - F^k(u_0)| (x) \leq \Omega^k(\omega_0)(x), \quad \text{a.e. in } G, \quad (19)$$

for $k, l = 0, 1, \ldots$.

Indeed, this true for all $l = 0, 1, \ldots$ and $k = 0$ because

$$|F^l(u_0) - u_0| (x) \leq \omega_0(x)$$

is a consequence of (16), (18), the definition of $q$, and the induction rule. This rule applied again with respect to $k$ results in (19). The convergence of the sequence of iterations $\{F^k u_0\}$ follows immediately from (19).

To prove the uniqueness of the solution of Eq. (7) we observe that if
there are in $D(u_0, \omega_0)$ two solutions $u^*$ and $u^{**}$ of this equation, then for some $c > 0$,

$$|u^*(x) - u^{**}(x)| \leq c\omega_0(x)$$

and

$$|u^*(x) - u^{**}(x)| = (Fu^*)(x) - (Fu^{**})(x)|$$

$$\leq \Omega(|u^* - u^{**}|)(x) \leq \Omega(c\omega_0)(x) \leq c\Omega(\omega_0(x),$$

and by induction we obtain

$$|u^*(x) - u^{**}| \leq \Omega^k(\omega_0)(x), \quad \text{a.e. in } G, k = 0, 1, ...$$

which implies that $u^* = u^{**}$. Thus, the theorem is proved.

Remark 4. If we drop the condition (ii) in Assumption C then $V(\omega_0)$ and $D(u_0, \omega_0)$ in this assumption and in Theorem 4 should be replaced by $S_1(0, \omega_0), S_2(u_0, \omega_0)$, respectively, where

$$S_1(0, \omega_0) = \{ \omega : \omega \in L^p_{\text{loc}}(G, \mathbb{R}^N), 0 \leq \omega(x) \leq \omega_0(x), \text{a.e. in } G \},$$

$$S_2(u_0, \omega_0) = \{ u : u \in L^p_{\text{loc}}(G, \mathbb{R}^N), |u(x) - u_0(x)| \leq \omega_0(x), \text{a.e. in } G \}.$$

7. Uniqueness of Solution in the Whole Space $L^p_{\text{loc}}(G, \mathbb{R}^N)$

In Sections 4, 5, and 6 of this paper we obtained the uniqueness of solution of Eq. (7) only in the subset $D(u_0, \omega_0)$ of the space $L^p_{\text{loc}}(G, \mathbb{R}^N)$. To obtain the uniqueness in whole space $L^p_{\text{loc}}(G, \mathbb{R}^N)$ we need to modify Assumption C as follows:

Assumption D. Assume that

(i) the condition (i) of Assumption C holds,

(ii) the condition (18) holds for all $u, v \in L^p_{\text{loc}}(G, \mathbb{R}^N)$

(iii) $\omega(x) \equiv 0$, a.e. in $G$ is in $L^p_{\text{loc}}(G, \mathbb{R}^+)$ the only solution of Eq. (17).

(iv) for every $r \in L^p_{\text{loc}}(G, \mathbb{R}^+)$ there is a function $\omega_r \in L^p_{\text{loc}}(G, \mathbb{R}^+)$ such that

$$\omega_r(x) \geq \Omega(\omega_r)(x) + r(x) \quad \text{a.e. in } G.$$

Now we can claim

Theorem 5. If Assumption D holds, then there is in $L^p_{\text{loc}}(G, \mathbb{R}^N)$ a unique
solution of Eq. (7), say $u^*$. The solution $u^*$ is the limit of the sequence
$\{ F^* u_0 \}$ which converges to $u^*$ for arbitrary $u_0 \in L^p_{\text{loc}}(G, \mathbb{R}^N)$.

Proof. The existence of solution is guaranteed by Theorem 4. What is
left to be proved is the uniqueness. Assume that there are two solutions of
Eq. (7), say $u^*$, $u^{**}$. We have then

$$ |u^*(x) - u^{**}(x)| = |(Fu^*)(x) - Fu^{**}(x)| \leq \Omega(|u^* - u^{**}|)(x). \quad (20) $$

Let $\omega^*$ be such that

$$ \omega^*(x) \geq \Omega(\omega^*)(x) + |u^*(x) - u^{**}(x)|, \quad \text{a.e. in } G. $$

$\omega^*$ exists according to condition (iv) of Assumption $D$.

We have then

$$ |u^*(x) - u^{**}(x)| \leq \omega^*(x), \quad \text{a.e. in } G, $$

and as a consequence of this and (20) we obtain

$$ |u^*(x) - u^{**}(x)| \leq \Omega(\omega^*)(x) \leq \omega^*(x), \quad \text{a.e. in } G. $$

Using (20) again by induction we obtain

$$ |u^*(x) - u^{**}(x)| \leq \Omega^k(\omega^*)(x), \quad \text{a.e. in } G. $$

However, the sequence $\{ \Omega^k(\omega^*) \}$ is nonincreasing so it converges to $\omega$
which by the property of $\Omega$ (see condition (i) of the Assumption $D$) is the
fixed point of $\Omega$. But by condition (iii) of the Assumption $D$ $\tilde{\omega} = 0$, this
means that $u^* = u^{**}$ and the uniqueness is proved.

8. Concluding Remarks

Equation (7) includes a great variety of special cases, among them there
are integral or integro-functional equations of both Volterra and Fredholm
type. Both the extended Bielecki's method and comparison method reduce
the existence and uniqueness problems for Eq. (7) to the discussion of the
properties of the related comparison operator $\Omega$. For instance in the case
considered in Section 2 of this paper the comparison operator $\Omega$ has the form

$$ \Omega(\omega)(x) = \int_0^x L(x, s) \omega(s) \, ds. $$
Under the assumptions taken there for every $\lambda > 1$ and every $r \in L^p(I_a, \mathbb{R}_+)$ there is a solution $\omega_0 \in L^p(I_a, \mathbb{R}_+)$ of equation

$$\omega(x) = \lambda \int_0^x L(x, s) \omega(s) \, ds + r(x)$$

which can be represented in the form of the Neumann series of the operator $\lambda \Omega$, so Theorem 5 applies. It is clear that more restrictive conditions appear when the integral equation is of the Fredholm type or the integro-functional type.

What is interesting is that the extended Bielecki's method as well as the comparison method gives us the possibility of investigating the behavior of solutions when $x \to \infty$ (clearly in the case when $I_a$ is replaced by $I_{\infty}$).

Observe also that if one would like to be more detailed with the investigation of solutions of the equations of type (7), then one can replace the comparison space $L^p_{\text{loc}}(G, \mathbb{R}_+)$ by $L^p_{\text{loc}}(G, \mathbb{R}_+^\omega)$ and use the vector valued norm instead of the scalar one. However, it is not the aim of this paper to go into details of this type. For an abstract approach consult Ref. [9].

ACKNOWLEDGMENT

The paper was written during the Spring Semester 1989 while the author was visiting the Department of Mathematics, University of Texas at Arlington. His stay was essentially supported by Professor C. Corduneanu. The author is sincerely thankful to Professor C. Corduneanu for the support and for many interesting discussions related to the subject of the paper.

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