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# State-dependent Foster–Lyapunov criteria for subgeometric convergence of Markov chains

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**Abstract**

We consider a form of state-dependent drift condition for a general Markov chain, whereby the chain *subsampled* at some deterministic time satisfies a geometric Foster–Lyapunov condition. We present sufficient criteria for such a drift condition to exist, and use these to partially answer a question posed in Connor and Kendall (2007) [2] concerning the existence of so-called 'tame' Markov chains. Furthermore, we show that our 'subsampled drift condition' implies the existence of finite moments for the return time to a small set.


*MSC:* 60J10; 37A25

*Keywords:* Markov chains; Foster–Lyapunov functions; State-dependent drift conditions; Regularity; Tame chains; Networks of queues

## 1. Introduction and notation

Let $\{\Phi_n, n \geq 0\}$ be a time-homogeneous Markov chain on a state space $\mathsf{X}$, with transition kernel $P$. Our goal in this paper is to develop a new criterion for determining the ergodic properties of $\Phi$. Specifically, we consider a form of *state-dependent* drift condition for $\Phi$, whereby the chain *subsampled* at some deterministic time satisfies a geometric Foster–Lyapunov condition. Drift conditions are classical tools to prove the stability of Markov chains. Most of the

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literature addresses the case when the drift inequality is satisfied by the kernel $P$ [14,10,8,7]. Nevertheless, depending upon the application, it may be far easier to prove a drift condition for a state-dependent iterated kernel than for $P$ itself (see e.g. [15] and Section 4.2). State-dependent drift conditions for phi-irreducible chains were originally studied by Meyn and Tweedie [15], who gave criteria for $\Phi$ to be Harris-recurrent, positive Harris-recurrent and geometrically ergodic. The drift condition that we develop in this paper is different from those in [15], as will be highlighted in Section 3.2, and can be used to infer a greater range of convergence rates for $\Phi$ (including both geometric and subgeometric rates). When applied to subgeometric rates, the present work extends the theory about subgeometric chains: in [21] (resp. [7]), it is discussed how a nested family of drift conditions (resp. a single drift) on the kernel $P$ is related to the control of modulated moments of the return time to some set. In this paper, we provide similar drift criteria in terms of state-dependent iterates of the transition kernel. Control of such return times is a key step to establish ergodicity and, more generally, limit theorems for the chains. In this paper, we also address a converse result and show how a state-dependent drift condition can be deduced from the convergence of the iterates of $P$. Such a converse result exists for geometric chains (see [14]) but, to the best of our knowledge, this is pioneering work for subgeometric chains.

We begin with a little notation; the unfamiliar reader can refer to [14]. For any non-negative function $f$ and $n \in \mathbb{N}$ we write $P^n f(x)$ for $\int P^n(x, \mathrm{d}y) f(y)$ where $P^n(x, \mathrm{d}y)$ denotes the $n$-step transition probability kernel; and for a signed measure $\mu$ we write $\mu(f) = \int \mu(\mathrm{d}y) f(y)$. The norm $\|\mu\|_f$ is defined as $\sup_{\{g:|g| \leq f\}} |\mu(g)|$. This generalises the total variation norm, $\|\cdot\|_{\mathrm{TV}} \equiv \|\cdot\|_1$. The first return time to a set $\mathcal{A}$ is denoted by $\tau_{\mathcal{A}} := \inf\{n \geq 1, \Phi_n \in \mathcal{A}\}$ and the hitting time on $\mathcal{A}$ is denoted by $\sigma_{\mathcal{A}} := \inf\{n \geq 0, \Phi_n \in \mathcal{A}\}$. A set $\mathcal{C}$ is called *small* (or $\nu$-small) if there exist some non-trivial measure $\nu$ and constant $\varepsilon > 0$ such that $P(x, \cdot) \geq \varepsilon \nu(\cdot)$ for all $x \in \mathcal{C}$. Any $\sigma$-finite measure $\pi$ satisfying $\pi = \pi P$ is called *invariant*. An aperiodic chain $\Phi$ that possesses a finite invariant measure $\pi$ is *ergodic* and for any $x$, $\|P^n(x, \cdot) - \pi\|_{\mathrm{TV}} \to 0$ as $n \to \infty$ [14, Theorem 13.0.1]. This condition is also equivalent to the existence of a moment for the return time to some accessible small set $\mathcal{C}$: $\sup_{x \in \mathcal{C}} \mathbb{E}_x[\tau_{\mathcal{C}}] < \infty$. $\Phi$ is said to be *geometrically ergodic* if there exist a function $V : \mathsf{X} \to [0, \infty)$ and constants $\gamma \in (0, 1)$, $R < \infty$, such that for any $x \in \mathsf{X}$, $\|P^n(x, \cdot) - \pi(\cdot)\|_V \leq R\, V(x) \gamma^n$. This is equivalent to the existence of a scale function $V : \mathsf{X} \to [1, \infty)$, a small set $\mathcal{C}$, and constants $\beta \in (0, 1)$, $b < \infty$, such that

$$PV(x) \leq \beta V(x) + b \mathbb{1}_{\mathcal{C}}(x), \tag{1}$$

where $\mathbb{1}_{\mathcal{A}}$ is the indicator function of the set $\mathcal{A}$. This geometric drift condition is also equivalent to the existence of an exponential moment for the return time to the set $\mathcal{C}$: $\sup_{x \in \mathcal{C}} \mathbb{E}_x[\beta^{-\tau_{\mathcal{C}}}] < \infty$ [14, Chapter 15]. More generally, $\Phi$ is said to be *$(f, r)$-ergodic* if there exist functions $r : \mathbb{N} \to [1, \infty)$ and $f : \mathsf{X} \to [1, \infty)$ such that, for all $x$ in a full and absorbing set, $r(n) \|P^n(x, \cdot) - \pi(\cdot)\|_f \to 0$ as $n \to \infty$.

In this paper we will be studying *subgeometrically ergodic* chains; the class $\Lambda$ of subgeometric rates $r = \{r(n), n \geq 0\}$ is defined in [19] as follows: call $\Lambda_0$ the set of rate functions $r_0 = \{r_0(n), n \geq 0\}$ such that $r_0(0) \geq 1$, $n \mapsto r_0(n)$ is non-decreasing and $\lim_{n \to \infty} \ln r_0(n)/n \downarrow 0$. Then $r \in \Lambda$ iff $r$ is non-negative, non-decreasing and there exists $r_0 \in \Lambda_0$ such that $\lim_{n \to \infty} r(n)/r_0(n) = 1$. Sufficient drift conditions for $(f, r)$-ergodicity, relative to the one-step transition kernel $P$ exist in the literature [21,7]. The converse result is, to the best of our knowledge, an open question (except when $f = 1$; see [21]).

The remainder of this paper is laid out as follows. In Section 2 we consider when it is possible to find functions $n : \mathsf{X} \to \mathbb{N}_\star$ and $V : \mathsf{X} \to [1, \infty)$, and $\beta \in (0, 1)$ such that

$$P^{n(x)}V(x) \leq \beta V(x) + b\mathbb{1}_{\mathcal{C}}(x). \tag{2}$$

That is, such that the chain $\Phi$ subsampled at time $n(x)$ exhibits a geometric drift condition. The sufficient conditions presented are ultimately based on the existence of moments for the return time to a small set. In Section 3 we consider the inverse problem: starting from a drift condition of the form (2), what can be said about the existence of moments of $\tau_{\mathcal{C}}$? In Section 4 these results are applied to the classification of tame chains, and to a discrete-time process forming part of a perfect simulation algorithm for such chains. In Section 5, it is shown how these results can be used to prove the subgeometric ergodicity of strong Markov processes. Proofs of the main results are provided in the Appendix.

## 2. Foster–Lyapunov drift inequalities under subsampling

We first consider when it is possible to deterministically subsample a chain $\Phi$ at rate $n$, in order to produce a Foster–Lyapunov drift condition, i.e.an inequality of the form $P^{n(x)}V(x) \leq \beta V(x) + b\mathbb{1}_{\mathcal{C}}(x)$ for some function $V : \mathsf{X} \to [1, \infty)$, constants $\beta \in (0, 1)$, $b < \infty$ and a measurable set $\mathcal{C}$. The main result of this section is the following generalisation of [1, Theorem 5.26]. It shows how a Foster–Lyapunov drift condition may be established for a subsampled chain from knowledge of the rate of convergence of the signed measures $\{P^n(x, \cdot) - P^n(x', \cdot), n \geq 0\}$.

**Theorem 2.1.** *Assume that there exists a non-decreasing function* $r : \mathbb{N} \to (0, \infty)$ *with* $\lim_{k\to\infty} r(k) = \infty$*, some measurable functions* $W, V : \mathsf{X} \to [1, \infty)$ *and a constant* $C < \infty$ *such that*

$$\forall (x, x') \in \mathsf{X} \times \mathsf{X}, \qquad r(k)\ \|P^k(x, \cdot) - P^k(x', \cdot)\|_W \leq C\left(V(x) + V(x')\right), \tag{3}$$

$$\exists x_0 \in \mathsf{X}, \qquad \sup_{k\geq 0} P^k W(x_0) < \infty. \tag{4}$$

*Let* $\beta \in (0, 1)$ *and* $n : \mathsf{X} \to \mathbb{N}$ *satisfy* $n(x) \geq r^{-1}\left(\frac{C}{\beta}\ \frac{V(x)}{W(x)}\right)$*, where* $r^{-1}(t) := \inf\{x \in \mathbb{N}, r(x) \geq t\}$ *denotes the generalised inverse of* $r$*. Then there exists* $b < \infty$ *such that* $P^{n(x)}W(x) \leq \beta W(x) + b$*. In addition, for any* $\beta < \beta' < 1$ *with* $\mathcal{C} := \{x \in \mathsf{X}, W(x) \leq b(\beta' - \beta)^{-1}\}$*,*

$$P^{n(x)}W \leq \beta' W + b\mathbb{1}_{\mathcal{C}}. \tag{5}$$

**Proof.** From (3) and (4), we have for any $x \in \mathsf{X}$, $k \in \mathbb{N}$,

$$P^k W(x) \leq \frac{C}{r(k)}V(x) + P^k W(x_0) + \frac{C}{r(k)}V(x_0) \leq \frac{C}{r(k)}V(x) + b,$$

where $b := \sup_{k\geq 0} P^k W(x_0) + C\frac{V(x_0)}{r(0)}$. By definition of $x \mapsto n(x)$, $CV(x)/r(n(x)) \leq \beta W(x)$. This yields $P^{n(x)}W \leq \beta W + b = \beta' W + \left(\beta - \beta'\right) W + b$, and $\left(\beta - \beta'\right) W + b \leq 0$ on $\mathcal{C}^c$. Since $\lim_{k\to\infty} r(k) = \infty$, the set $\{x \in \mathbb{N},\ r(x) \geq t\}$ is non-empty whatever $t \geq 0$. □

### 2.1. Uniformly ergodic chains

When assumption (3) holds for some *bounded* function $V$, we have $\lim_n \sup_{(x,x')\in\mathsf{X}\times\mathsf{X}} \|P^n(x, \cdot) - P^n(x', \cdot)\|_{\mathrm{TV}} = 0$. Then classical results on the Dobrushin coefficient imply that $P$ admits a unique invariant distribution $\pi$ and $\sup_{x\in\mathsf{X}} \|P^n(x, \cdot) - \pi\|_{\mathrm{TV}} \leq \rho^n$ for some $\rho \in (0, 1)$. Hence the drift condition (1) holds for some bounded function $\tilde{V}$ [14, Theorem 16.0.1], and this implies (5) with $n(x) = c$ for any $c \in \mathbb{N}$. We are able to retrieve this result too from our result. Applying Theorem 2.1 with $W = \omega$ and $n(x) = n_\star$ such that $n_\star \geq r^{-1}\left(C\,\beta^{-1}\,\sup_{\mathsf{X}} V\right)$ we have $P^{n_\star} W \leq \beta W + b$. By classical computations (see e.g. the proof of [14, Theorem 16.1.4]), this yields $P\tilde{W} \leq \beta^{1/n_\star}\,\tilde{W} + b\beta^{1/n_\star - 1} n_\star^{-1}$ with $\omega \leq \tilde{W} \leq \omega\beta^{-1}$; hence (5) holds with $n(x) = c$ for some (and thus any) constant $c$.

In what follows, we do not impose boundedness on $V$, thus allowing chains which are not necessarily uniformly ergodic.

### 2.2. *Sufficient conditions for assumptions (3)–(4)*

When $P$ is phi-irreducible and aperiodic, assumption (4) is implied by any one of the following equivalent conditions (see [14, Theorem 14.0.1]):

(i) $P$ possesses a unique invariant probability $\pi$ and $\pi(W) < \infty$;
(ii) there exists a small set $\mathcal{C}$ such that $\sup_{x\in\mathcal{C}} \mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{C}}-1} W(\Phi_k)\right] < \infty$;
(iii) there exist a function $U : \mathsf{X} \to (0, \infty]$ finite at some $x_\star \in \mathsf{X}$, a constant $b < \infty$ and a small set $\mathcal{C}$ such that $PU \leq U - W + b\mathbb{1}_{\mathcal{C}}$.

The main difficulty is to prove (3); Proposition 2.2 provides sufficient conditions.

**Proposition 2.2.** *Let $P$ be a phi-irreducible and aperiodic transition kernel. Assume that there exist a small set $\mathcal{C}$, measurable functions $W, V : \mathsf{X} \to [1, \infty)$ and a constant $b < \infty$ such that $\sup_{\mathcal{C}} V < \infty$, and*

$$\begin{cases} PV(x) \leq V(x) - W(x) + b\mathbb{1}_{\mathcal{C}}(x), \\ PW(x) \leq W(x) + b\mathbb{1}_{\mathcal{C}}(x). \end{cases} \tag{6}$$

*Then* (i) *there exists $x_0 \in \mathsf{X}$ such that $\sup_{n\geq 0} P^n W(x_0) < \infty$; and* (ii) *there exists a constant $C < \infty$ such that for any $(x, x') \in \mathsf{X}^2$,*

$$n\,\|P^n(x,\cdot) - P^n(x',\cdot)\|_W \leq C\{V(x) + V(x')\}.$$

When the first inequality in (6) holds, then it also holds by replacing $W$ with $\tilde{W} := 1$, which trivially satisfies the second inequality in (6). In this setting Proposition 2.2 is [14, Theorem 13.4.4]. Nevertheless, the interest of satisfying the second inequality with an unbounded function $W$ is that this allows for control of $\|P^n(x,\cdot) - P^n(x',\cdot)\|_W$ with a stronger norm than total variation. Proposition 2.2 also provides a rate of convergence for $\|P^n(x,\cdot) - P^n(x',\cdot)\|_W$ to zero; this rate is stronger than that which could be deduced from the control of $\|P^n(x,\cdot) - \pi\|_W$ under similar assumptions [14, Chapter 14].

The following two results show that (6) holds when we have a subgeometric-type drift inequality, or when we are able to control modulated moments of the return time to a small set $\mathcal{C}$. A proof of Proposition 2.3 can be found in [1, Lemma 5.9].

**Proposition 2.3.** *Assume that there exist a set $\mathcal{C}$, a constant $b < \infty$, a measurable function $V : \mathsf{X} \to [1, \infty)$ and a continuously differentiable increasing concave function $\phi : [1, \infty) \to (0, \infty)$, such that*

$$PV \leq V - \phi \circ V + b\mathbb{1}_{\mathcal{C}}, \qquad \sup_{\mathcal{C}} V < \infty, \qquad \inf_{[1,\infty)} \phi > 0.$$

*Then (6) holds with $W \propto \phi \circ V$.*

**Proposition 2.4.** *Assume that there exist a set $\mathcal{C}$, a non-decreasing rate function $r : \mathbb{N} \to (0, \infty)$ such that $r(0) \geq 1$ and $\sup_{x \in \mathcal{C}} \mathbb{E}_x \left[\sum_{k=1}^{\tau_{\mathcal{C}}} r(k)\right] < \infty$. Then (6) holds with*

$$V(x) = \mathbb{E}_x \left[ \sum_{k=0}^{\sigma_{\mathcal{C}}} r(k) \right], \qquad W(x) = \mathbb{E}_x \left[ r(\sigma_{\mathcal{C}}) \right].$$

### 2.3. Examples

A phi-irreducible aperiodic chain satisfying the conditions of Proposition 2.3 for a small set $\mathcal{C}$, is ergodic at a subgeometric rate [7]. In that case conditions (3) and (4) hold with $r(k) = k$ and $W \sim \phi \circ V$, for which $r^{-1}\left(C\beta^{-1} V/W\right) \sim C\beta^{-1} V/\phi \circ V$. This yields the following examples of subsampling rate $n$ and the scale function $W$; hereafter, $c' > 0$.

*Logarithmically ergodic chains.* Assume that $\phi(t) \sim c\,[1 + \ln t]^{\alpha}$ for some $\alpha > 0$ and $c > 0$. Then (5) holds with $n(x) \geq c' \frac{V(x)}{[1+\ln V(x)]^{\alpha}}$ and $W := [1 + \ln V]^{\alpha}$.

*Polynomially ergodic chains.* Assume that $\phi(t) \sim ct^{1-\alpha}$ for some $\alpha \in (0, 1)$ and $c > 0$. Then (5) holds with $n(x) \geq c' V^{\alpha}(x)$ and $W := V^{1-\alpha}$.

*Subgeometrically ergodic chains.* Assume that $\phi(t) \sim ct[\ln t]^{-\alpha}$ for some $\alpha > 0$ and $c > 0$. Then (5) holds with $n(x) \geq c'[\ln V(x)]^{\alpha}$ and $W := V[\ln V]^{-\alpha}$.

The results above are coherent with the geometric case: on the one hand, when a transition kernel satisfies the drift inequality $PV \leq \beta V + b\mathbb{1}_{\mathcal{C}}$, then it also satisfies (1) with the same drift function $V$, and $n(x) = c$ for any constant $c \in \mathbb{N}_\star$; on the other hand, when $\alpha \to 0$, the polynomial and subgeometric drift conditions 'tend' to the geometric drift condition (in the sense that $\phi(t) \to t$). From the above discussion, when $\alpha \to 0$, $[c'V^{\alpha}, V^{1-\alpha}] \to [c', V]$ and $[c'\,(\ln V)^{\alpha}, V(\ln V)^{-\alpha}] \to [c', V]$, thus showing coherence in the results.

## 3. State-dependent drift criteria for regularity

We now discuss how a state-dependent Foster–Lyapunov drift condition is related to the existence of a moment of the return time to measurable sets. Such controls are related to the *regularity* of the chain [21,14, chapter 14] which, under general conditions, is known to imply limit theorems such as strong laws of large numbers, mean ergodic theorems, functional central limit theorems and laws of iterated logarithm (see [14, chapters 14 to 17]). We provide conditions for the control of (subgeometric) moments of the return time to a small set, expressed in terms of a family of nested drift conditions (Proposition 3.1) or in terms of a single drift condition (Theorem 3.2).

### 3.1. Family of nested drift conditions

Proposition 3.1 extends the conditions provided by Tuominen and Tweedie [21], expressed in terms of the one-step transition kernel, to the case of the state-dependent transition kernels.

**Proposition 3.1.** *Let $f : \mathsf{X} \to [1, \infty)$ and $n : \mathsf{X} \to \mathbb{N}$ be measurable functions and $\{r(k), k \geq 0\}$ be a non-negative sequence. Assume that there exist measurable functions $\{V_k, k \geq 0\}$ and $\{S_k, k \geq 0\}$, $V_k, S_k : \mathsf{X} \to [1, \infty)$, and a measurable set $\mathcal{C}$ such that for any $k \geq 0$, $x \in \mathsf{X}$,*

$$\mathbb{E}_x \left[ V_{k+n(\Phi_0)}(\Phi_{n(\Phi_0)}) \right] \leq V_k(x) - \mathbb{E}_x \left[ \sum_{j=0}^{n(\Phi_0)-1} r(k+j) f(\Phi_j) \right] + S_k(x) \mathbb{1}_{\mathcal{C}}(x).$$

*Then for any $x \in \mathsf{X}$, $\mathbb{E}_x \left[\sum_{k=0}^{\tau_{\mathcal{C}}-1} r(k)\; f(\Phi_k)\right] \leq V_0(x) + S_0(x)\; \mathbb{1}_{\mathcal{C}}(x)$.*

However, this criterion is more of theoretical than practical interest since it is quite difficult to check. We now propose a criterion based on a single drift condition.

### 3.2. Single drift condition

We consider the case when

$$\mathbb{E}_x\left[W(\Phi_{n(\Phi_0)})\right] \leq \beta W(x) + b\mathbb{1}_{\mathcal{C}}(x), \tag{7}$$

for some $\beta \in (0,1)$ and measurable positive functions $n, W \geq 1$. The case $n(x) = c$ on $\mathsf{X}$ corresponds to the usual Foster–Lyapunov drift condition (see e.g. [14, Chapter 15] and the references therein). The drift condition extends earlier work by Meyn and Tweedie [15, Theorem 2.1] who address the cases when the drift condition is of the form

$$\begin{aligned}
&\mathbb{E}_x\left[W(\Phi_{n(\Phi_0)})\right] \leq W(x) + b\mathbb{1}_{\mathcal{C}}(x),\\
&\mathbb{E}_x\left[W(\Phi_{n(\Phi_0)})\right] \leq W(x) - n(x) + b\mathbb{1}_{\mathcal{C}}(x),\\
&\mathbb{E}_x\left[W(\Phi_{n(\Phi_0)})\right] \leq \beta^{n(x)} W(x) + b\mathbb{1}_{\mathcal{C}}(x), \quad \beta \in (0,1),
\end{aligned} \tag{8}$$

without assuming any relations between $n$ and $W$. In [15], it is established that for a phi-irreducible and aperiodic kernel $P$, these drift inequalities imply respectively Harris-recurrence, positive Harris-recurrence and geometric ergodicity provided the set $\mathcal{C}$ is small and $W$ is bounded on $\mathcal{C}$. Our drift inequality (7) differs from (8) in the rate of contraction $\beta$ which does not depend on the subsampling rate $n(x)$.

**Theorem 3.2.** *Assume that there exist measurable functions $W : \mathsf{X} \to [1,\infty)$ and $n : \mathsf{X} \to \mathbb{N}_\star$, constants $\beta \in (0,1)$ and $b < \infty$, and a measurable set $\mathcal{C}$ such that (7) holds. If there exists a strictly increasing function $R : (0,\infty) \to (0,\infty)$ satisfying one of the following conditions*

(i) *$t \mapsto R(t)/t$ is non-increasing and $R \circ n \leq W$,*
(ii) *$R$ is a convex continuously differentiable function such that $R'$ is log-concave and $R^{-1}(W) - R^{-1}(\beta W) \geq n$,*

*then there exists a constant $C$ such that $\mathbb{E}_x[R(\tau_{\mathcal{C}})] \leq C\{W(x) + b\mathbb{1}_{\mathcal{C}}(x)\}$.*

The conclusion of Theorem 3.2 is unchanged if $R$ is modified on some bounded interval $[0,t]$: hence it is sufficient to define $R$ such that the above conditions on $(R, n, W)$ hold for any $x$ such that $n(x)$ – or equivalently $W(x)$ – is large enough. We provide at the end of this section examples of pairs $(n, W)$ and the associated rate $R$.

A sufficient condition for the existence of $R$ satisfying (i) is that, outside some level set of $n$, there exists a strictly increasing concave function $\xi$ such that $\xi \circ n = W$. Then we can set $R = \xi$. Since concave functions are sub-linear, the case (i) addresses the case when $n \gg W$ (outside some bounded set). A sufficient condition for the existence of a function $R$ satisfying $R^{-1}(W) - R^{-1}(\beta W) \geq n$ is that $(1-\beta)\ W(x)\ [R^{-1}]'(W(x)) \geq n(x)$; when there exists $\xi$ such that $n(x) = \xi \circ W(x)$ and $t \mapsto \xi(t)/t$ is non-increasing, we can choose $R^{-1}(t) \sim \int_1^t u^{-1}\xi(u)\mathrm{d}u$. Hence case (ii) addresses the case when $n/W$ is decreasing (outside some bounded set).

Existence of an invariant probability distribution is related to $\mathbb{E}_x[\tau_{\mathcal{C}}]$, the first moment of the return time to a small set $\mathcal{C}$ [14, Theorem 10.0.1]. Theorem 3.2(i) shows that the control of this moment can be deduced from a condition of the form (7) provided $n(x) \leq W(x)$ (choose $R(t) = t$).

Given $(n, W)$ and a drift inequality of the form (7), Theorem 3.2 provides a moment of the return time to $\mathcal{C}$ which depends upon the initial value $x$ at most as $W(x)$ (outside $\mathcal{C}$). From the drift inequality (7), we are able to deduce a family of similar drift conditions with $n$ unchanged: for example, Jensen's inequality implies that for any $\alpha \in (0, 1)$,

$$\mathbb{E}_x \left[ W^\alpha(\Phi_{n(\Phi_0)}) \right] \le \beta^\alpha W^\alpha(x) + b^\alpha \mathbb{1}_{\mathcal{C}}(x).$$

Application of Theorem 3.2 with this new pair $(n, W^\alpha)$, will allow the control of a moment of $\tau_{\mathcal{C}}$ which depends on $x$ at most as $W^\alpha(x)$.

**Corollary 3.3** (*Of* Theorem 3.2). *Assume in addition that $P$ is phi-irreducible and aperiodic, $\mathcal{C}$ is small with $\sup_{\mathcal{C}} W < \infty$, and $R$ is a subgeometric rate.*

(i) *For any accessible set $\mathcal{D}$, there exists $C < \infty$ such that $\mathbb{E}_x [R(\tau_{\mathcal{D}})] \le C\, W(x)$;*
(ii) *If $P$ admits a unique invariant probability measure $\pi$ such that $\pi(W) < \infty$, there exists an accessible small set $\mathcal{D}$ such that $\sup_{x\in\mathcal{D}} \mathbb{E}_x \left[\sum_{k=0}^{\tau_{\mathcal{D}}-1} R(k)\right] < \infty$.*

Examples of moments that can be obtained from Theorem 3.2 (possibly combined with Corollary 3.3) are the geometric, subgeometric, polynomial and logarithmic rates.

*Geometric rates.* If $n(x) = 1$: setting $R^{-1}(t) = \ln(t)/\ln(\kappa)$ with $1 \le \kappa \le \beta^{-1}$ it is easily verified that condition (ii) of the theorem is satisfied. We therefore deduce that $\mathbb{E}_x[R(\tau_{\mathcal{C}})] = \mathbb{E}_x[\kappa^{\tau_{\mathcal{C}}}] \le C\{W(x) + b\mathbb{1}_{\mathcal{C}}\}$. In particular, $\mathbb{E}_x[\beta^{-\tau_{\mathcal{C}}}] < \infty$, in agreement with the well-known equivalence between the geometric drift condition and the exponential moment of $\tau_{\mathcal{C}}$ mentioned in Section 1.

*Subgeometric rates.* If $n(x) \propto [\ln V(x)]^\alpha$ for some $\alpha > 0$ and $W \propto V[\ln V]^{-\alpha}$: then $n(x) \sim \xi \circ W(x)$ with $\xi(t) \sim [\ln t]^\alpha$. For some convenient $c$, the function $R(t) \sim \exp(ct^{1/(1+\alpha)})$ satisfies the condition $[R^{-1}]'(t) \sim \xi(t)/[(1-\beta)t]$ and also condition (ii) of Theorem 3.2.

*Polynomial rates.* If $n(x) \propto V^\alpha(x)$, for some $\alpha \in (0, 1]$ and $W \propto V^{1-\alpha}$: when $\alpha \le 1/2$ (respectively $\alpha \ge 1/2$) condition (ii) (resp. condition (i)) of Theorem 3.2 is satisfied with $R(t) \sim t^{(1-\alpha)/\alpha}$. We thus have $\mathbb{E}_x \left[\tau_{\mathcal{C}}^{1/\alpha-1}\right] \le C\, V^{1-\alpha}(x)$.

*Logarithmic rates.* If $n(x) \propto V[\ln V(x)]^{-\alpha}$ for some $\alpha > 0$ and $W \propto [\ln V]^\alpha$. Then $n \gg W$ and condition (i) is verified with $R(t) \sim [\ln t]^\alpha$. Hence $\mathbb{E}_x\, [(\ln \tau_{\mathcal{C}})^\alpha] \le C\, [1 + \ln V]^\alpha(x)$.

As an application of Corollary 3.3 and of the discussion in Section 2.3, we can deduce moments of the return time to $\mathcal{C}$ from a single drift condition of the form $PV \le V - \phi\circ V + b\mathbb{1}_{\mathcal{C}}$, $\phi$ concave. For example, in the case $\phi(t) \sim t^{1-\alpha}$ for some $\alpha \in (0, 1)$, we have $\mathbb{E}_x[\tau_{\mathcal{C}}^{1/\alpha}] \le CV(x)$. This is in total agreement with that which has been established in the literature using other approaches [10,8,7]. This agreement illustrates the fact that the sufficient conditions provided in Sections 2 and 3 are quite minimal.

## 4. Application to tame chains

### 4.1. Tame chains

The class of *tame* Markov chains was introduced by Connor and Kendall [2], who showed that a perfect simulation algorithm exists for such chains.

**Definition 4.1.** The chain $\Phi$ is *tame* if there exists a scale function $W : \mathsf{X} \to [1, \infty)$, a small set $\mathcal{C}$, and constants $\beta \in (0, 1)$, $b < \infty$ such that the following two conditions hold:

(i) there exist $\delta \in (0, 1)$ and a deterministic function $n : \mathsf{X} \to \mathbb{N}$ satisfying

$$n(x) \le W^{\delta}(x) \tag{9}$$

$$\text{such that} \quad \mathbb{E}_x \left[ W(\Phi_{n(x)}) \right] \le \beta W(x) + b \mathbb{1}_C(x); \tag{10}$$

(ii) the constant $\delta$ in (9) satisfies $\ln \beta < \delta^{-1} \ln(1 - \delta)$.

In other words, $\Phi$ is tame if for all $x \in \mathsf{X}$ there exists a deterministic time $n(x)$ such that the chain subsampled at this time exhibits a geometric drift condition (10). Furthermore, $n(x)$ should be sufficiently small compared to the scale function $W$ (9). Part (ii) of Definition 4.1 is a technical condition required for construction of the simulation algorithm.

Clearly the class of tame chains includes all geometrically ergodic chains. In addition, it is shown in [2] that $\Phi$ is tame if it satisfies a polynomial drift condition of the form

$$PV \le V - cV^{(1-\alpha)} + b \mathbb{1}_C, \tag{11}$$

where $0 < \alpha < 1/4$. However, this condition is not necessary: in [2] there is an example of a random walk satisfying (11) with $\alpha = 1/2$, which is explicitly shown to be tame. The results of Section 2.2 now enable us to generalise this sufficient condition.

**Proposition 4.2.** *Suppose that $\Phi$ satisfies the assumptions of Proposition 2.3 with $\phi(t) \sim ct^{1-\alpha}$ where $\alpha \in (0, 1/2)$. Then $\Phi$ is tame.*

**Proof.** Choose $\delta \in (0, 1)$ such that $\delta > \alpha/(1 - \alpha)$, and then $\beta \in (0, 1)$ such that $\ln \beta < \delta^{-1} \ln(1 - \delta)$. As noted in Section 2.3, the results of Proposition 2.3 and Theorem 2.1 show that if $\phi(t) \sim ct^{1-\alpha}$ for $\alpha < 1/2$, then $\mathbb{E}_x \left[ W(\Phi_{n(x)}) \right] \le \beta W(x) + b \mathbb{1}_C(x)$ with $n(x) = c_\beta V^{\alpha}(x)$, $W = V^{1-\alpha}$, and where $c_\beta \propto \beta^{-1}$. The choice of $\delta$ and $\beta$ ensures that $\Phi$ satisfies all parts of Definition 4.1, as required. □

Note that any chain with subgeometric drift $\phi(t) \sim ct[\ln t]^{-\alpha}$ is tame. However, the logarithmically ergodic chains identified in Section 2.3 do not satisfy $\phi(t) > t^{\alpha}$ for any value of $\alpha \in (0, 1)$, and so are not covered by Proposition 4.2.

### *4.2. Dominating process for tame chains*

In this section we describe a non-trivial example of a Markov chain $D$ for which there is no obvious one-step drift, but for which it is simple to establish a subsampled drift condition. The chain $D$ finds application in the perfect simulation algorithm of [2], as will be explained below.

Let $\beta \in (0, e^{-1})$, $\kappa > 0$ and $n^\star$ be a function from $[1, \infty) \to \mathbb{N}$. To begin our construction of $D$, let $U$ be the system workload of a $D/M/1$ queue, sampled just before arrivals, with arrivals every $\ln \beta^{-1}$ units of time, and service times being independent and of unit rate Exponential distribution. This satisfies

$$U_{n+1} = \max \left\{ U_n + E_{n+1} - \ln \beta^{-1},\ 0 \right\},$$

where $\{E_n\}_{n \ge 1} \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$.

Define $Y = \kappa \exp(U)$, for some $\kappa > 0$. The set $[\kappa, \kappa/\beta]$ is a small set for $Y$, and

$$\mathbb{P}[Y_1 > v | Y_0 = u] = \frac{\beta u}{v}, \quad \text{for } v \ge \max\{\beta u, \kappa\}. \tag{12}$$

Finally, let $D$ be the two-dimensional process $D = (Z, M)$ on $\mathsf{X} := \{(z, m) : z \in [1, \infty), m \in \{1, \ldots, n^\star(z)\}\}$ with transitions controlled by:

$$\mathbb{P}[Z_{k+1} = Z_k,\ M_{k+1} = M_k - 1 | Z_k,\ M_k] = 1\ , \quad \text{if } M_k \geq 2;$$
$$\mathbb{P}[Z_{k+1} \in E | Z_k = z,\ M_k = 1] = \mathbb{P}[Y_1 \in E | Y_0 = z], \quad \text{for all measurable } E \subseteq [1, \infty);$$
$$\mathbb{P}[M_{k+1} = n^*(Z_{k+1}) | Z_k,\ Z_{k+1}, M_k = 1] = 1.$$

Thus the first component of $D$ is simply a slowed down version of $Y$, and the second component is a forward recurrence time chain, counting down the time until the first component jumps (determined by the function $n^*$).

**Proposition 4.3.** *Let $\beta \in (0, e^{-1})$, $\kappa > 0$ and $n^*$ be a measurable function $n^\star : [1, \infty) \to \mathbb{N}$. Set $\mathcal{C} := \{(z, m) : z \in [\kappa, \kappa/\beta], m \in \{1, \ldots, n^\star(z)\}\}$ and denote by $\tau_{\mathcal{C}}$ the return time to $\mathcal{C}$ for the Markov chain $D$. Let $\alpha_\beta$ be the unique solution in $(0, 1)$ of the equation $\ln \beta = \ln(1 - \alpha)/\alpha$.*

(i) *When $n^\star(z) \sim z^\gamma$ for some $\gamma \geq 0$: for any $\alpha \in (0, \alpha_\beta)$ and*
- *any $\eta \in (\gamma/\alpha, 1]$ when $\gamma \in [0, \alpha_\beta)$*
- *any $\eta > \gamma/\alpha$ when $\gamma \geq \alpha_\beta$,*

*there exists a constant $C$ such that $\mathbb{E}_{(z,m)}[\tau_{\mathcal{C}}^{1/\eta}] \leq C\, z^\alpha$ for any $(z, m) \in \mathsf{X}$.*

(ii) *When $n^\star(z) \sim [\ln z]^\gamma$ for some $\gamma > 0$: for any $\alpha \in (0, \alpha_\beta)$ and $\eta > 0$ satisfying $\eta < \{(1 + \gamma)\alpha^{-1} \ln((1 - \alpha)/\beta^\alpha)\}^{1/(1+\gamma)}$, there exists a constant $C$ such that $\mathbb{E}_{(z,m)}[\exp(\eta\alpha\tau_{\mathcal{C}}^{1/(1+\gamma)})] \leq C\, z^\alpha$ for any $(z, m) \in \mathsf{X}$.*

(iii) *When $n^\star(z) \sim 1$, for any $\alpha \in (0, \alpha_\beta)$ there exists a constant $C$ such that for any $(z, m) \in \mathsf{X}$, $\mathbb{E}_{(z,m)}[\{(1 - \alpha)\beta^{-\alpha}\}^{\tau_{\mathcal{C}}}] \leq C\, z^\alpha$.*

**Proof.** We first of all establish a drift condition of the form (7) for the chain $D$. Let $V(z, m) = z^\alpha$, with $\alpha \in (0, \alpha_\beta)$. Then $\mathbb{E}[V(Z_m, M_m)|Z_0 = z, M_0 = m] = \mathbb{E}[Y_1^\alpha | Y_0 = z]$. When $z \notin [\kappa, \kappa/\beta]$,

$$\mathbb{E}[V(Z_m, M_m)|Z_0 = z, M_0 = m] = \int_{\beta z}^{\infty} y^\alpha \frac{\beta z}{y^2}\, \mathrm{d}y = \frac{\beta^\alpha}{1 - \alpha} z^\alpha.$$

Since $\alpha < \alpha_\beta$, $\alpha \ln \beta < \ln(1 - \alpha)$ and the chain $D$ satisfies the drift condition

$$P^m V(z, m) \leq \beta' V(z, m), \quad \text{with } \beta' = \frac{\beta^\alpha}{1 - \alpha} < 1, \tag{13}$$

whenever $z \notin [\kappa, \kappa/\beta]$. If $z \in [\kappa, \kappa/\beta]$ however, then

$$\begin{aligned}\mathbb{E}[V(Z_m, M_m)|Z_0 = z, M_0 = m] &= \int_{\kappa}^{\infty} y^\alpha \frac{\beta z}{y^2}\, \mathrm{d}y + \left(1 - \frac{\beta z}{\kappa}\right)\kappa^\alpha \\ &\leq \beta' z^\alpha + \left(\frac{1 - \beta^\alpha}{1 - \alpha}\right)\kappa^\alpha.\end{aligned}$$

It follows that $D$ satisfies $P^m V(z, m) \leq \beta' V(z, m) + b' \mathbb{1}_{\mathcal{C}}$.

We may now apply Theorem 3.2 to establish moments of the return time of $D$ to the set $\mathcal{C}$. For example, suppose that $n^*(z) \sim z^\gamma$, for some $\gamma \in [0, \alpha_\beta)$ and set $R(z) := z^{1/\eta}$ for some $\eta \in (\gamma/\alpha, 1]$. It follows that $R$ satisfies the conditions of Theorem 3.2(ii), with

$$\begin{aligned}R^{-1}(V(z, m)) - R^{-1}(\beta' V(z, m)) - m &\geq R^{-1}(z^\alpha) - R^{-1}(\beta' z^\alpha) - n^*(z) \\ &= \left(1 - \beta'^\eta\right) z^{\alpha\eta} - z^\gamma \geq 0.\end{aligned}$$

(Here we have used the fact that $m \leq n^*(z)$, by definition of $\mathsf{X}$.) Thus,

$$\mathbb{E}_{(z,m)}[\,\tau_{\mathcal{C}}^{1/\eta}\,] \leq C\{z^{\alpha} + b'\mathbb{1}_{\mathcal{C}}(z,m)\}.$$

If $n^*(z) \sim z^{\gamma}$ with $\gamma \geq \alpha_{\beta}$ then the same argument shows that the function $R(z) = z^{1/\eta}$, with $\eta > \gamma/\alpha$, satisfies Theorem 3.2(i). Parts (ii) and (iii) follow similarly by taking $R(z) \sim \exp(\eta\alpha z^{1/(1+\gamma)})$ and $R(z) \sim \exp(\eta\alpha z)$ (for some $\eta > 0$) respectively. □

In Proposition 4.3(i), $1/\eta \geq 1$ iff $\gamma \in [0, \alpha_{\beta})$. When $1/\eta \geq 1$ and $D$ is phi-irreducible, aperiodic and $\mathcal{C}$ is small, this shows that $D$ possesses an invariant probability distribution and is ergodic. The convergence to $\pi$ (in total variation norm) occurs at the polynomial rate $n^{1/\eta-1}$ (see [21]). When $1/\eta < 1$, we cannot deduce from the control of this moment the existence of $\pi$.

The chain $D$ is of interest for the following reason. Suppose that $\Phi$ is a tame chain satisfying $P^{n(x)}W(x) \leq \beta W(x) + b\mathbb{1}_{\mathcal{C}'}$ where $n(x) = n^{\star} \circ W(x) \leq W^{\delta}(x)$ for some $\delta \in (0, 1)$. Connor and Kendall [2] show that the chains $Z$ and $W(\Phi)$ can be coupled so that $Z$ dominates $W(\Phi)$ at the times when $Z$ jumps. Thus the chain $D$ 'pseudo-dominates' $W(\Phi)$, and this coupling can be exploited to produce a perfect simulation algorithm for $\Phi$. Proposition 4.3 allows us to calculate ergodic properties of $D$, and hence bound the expected run time of the algorithm: this issue is not addressed in [2].

## 5. Subgeometric ergodicity of strong Markov processes

In this section we provide sufficient conditions for ergodicity of a strong Markov process. In [16,9,6], the conditions are (mainly) expressed in terms of a drift inequality on the generator of the process. Our key Assumption A1 is in terms of the time the process rescaled in time and space enters a ball of radius $\rho$, $\rho \in (0, 1)$. Proposition 5.1 finds application in, for example, queueing theory as discussed below.

Let $\{\Phi_t, t \in \mathbb{R}_+\}$ be a strong Markov process taking values in $\mathsf{X} \subseteq \mathbb{R}^d$. It is assumed that $(\Omega, \mathcal{A}, \mathcal{F}_t, \Phi_t, \mathbb{P}_x)$ is a Borel-right process on the space $\mathsf{X}$ endowed with its Borel $\sigma$-field $\mathcal{B}(\mathsf{X})$. We assume that the sub-level sets $\{x \in \mathsf{X},\ |x| \leq \ell\}$ are compact subsets of $\mathsf{X}$ ($|\cdot|$ is a norm on $\mathbb{R}^d$).

A1. $\lim_{|x|\to\infty} |x|^{-(p+1)}\mathbb{E}_x\left[|\Phi_{\lfloor t_0|x|^{1+\tau}\rfloor}|^{p+1}\right] = 0$ for some $t_0 > 0$, $p \geq 0$ and $0 \leq \tau \leq p$.

A2. For any $t_{\star} > 0$, there exists $C$ such that for any $x \in \mathsf{X}$, $\sup_{t \leq t_{\star}} \int P^t(x, \mathrm{d}y)|y|^{p+1} \leq C\,|x|^{p+1}$.

A3. Every compact subset of $\mathsf{X}$ is small for the process and the skeleton $P$ is phi-irreducible.

A4. There exist $q \geq 0$ and $C$ such that for any $x$, $\mathbb{E}_x\left[\sum_{k=0}^{\lfloor t_0|x|^{1+\tau}\rfloor-1} |\Phi_k|^q\right] \leq C|x|^{p+1}$.

Recall that a set $\mathcal{C}$ is said to be small (for the process) if there exist $t > 0$ and a measure $\nu$ on $\mathcal{B}(\mathsf{X})$ such that $P^t(x,\cdot) \geq \mathbb{1}_{\mathcal{C}}(x)\nu(\cdot)$.

A1 is a condition on the process $\{\Phi_t, t \geq 0\}$ rescaled in time and space. Such a transformation is largely used in the queueing literature for the study of the stability of networks. This approach is referred to as the *fluid model* (see e.g. [20,13] for a rigorous definition; see also [4,5,13] and the references therein for applications to queueing). In these applications, A1 is proved by showing that the fluid model is stable (see e.g. [5, Proposition 5.1]). Condition A2 is a control of the $L^p$-moment of the system. A3 is related to the phi-irreducibility of the Markov process, a property which is necessary when ergodicity holds. A4 is required to prove the existence of a steady-state value for the moments $\mathbb{E}_x[|\Phi_t|^s]$ for $s > 0$, when $t \to \infty$. Examples of Markov processes satisfying A1–A4 are given in [4,5].

**Proposition 5.1.** *Assume that* A1–A3 *hold. Then the Markov process possesses a unique invariant probability* $\pi$ *and for any* $x \in \mathsf{X}$, $\lim_{t\to\infty}(t+1)^{(p-\tau)/(1+\tau)} \, \|P^t(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} = 0$. *If in addition* A4 *holds, then* $\int |y|^q \pi(\mathrm{d}y) < \infty$ *and for any* $x \in \mathsf{X}$ *and any* $0 \le \kappa \le 1$,

$$\lim_{t\to+\infty} (t+1)^{\kappa(p-\tau)/(1+\tau)} \sup_{\{g:|g(x)|\le 1+|x|^{(1-\kappa)q}\}} |\mathbb{E}_x[g(X_t)] - \pi(g)| = 0.$$

Proposition 5.1 provides a polynomial rate of convergence and convergence of power moments. More general rates of convergence can be obtained by replacing in A1–A2 the power functions $|x|^{1+\tau}$, $|x|^{p+1}$ by more general functions $W$; more general moments can be obtained by replacing the power function $|x|^q$ in A4 by a general function $f$. These extensions are easily obtained from the proof of Proposition 5.1; details are omitted.

Proposition 5.1 extends [3, Theorem 3.1] which addresses the positive Harris-recurrence of the process. It also extends [5, Theorem 6.3] by providing (i) a continuum range of rates of convergence (and thus a continuum range of rate functions) and (ii) an explicit norm of convergence.

**Proof of Proposition 5.1.** The reader unfamiliar with basic results on Markov processes may refer to [17]. For a measurable set $\mathcal{C}$ and a delay $\delta > 0$, let $\tau_{\mathcal{C}}(\delta) := \inf\{t \ge \delta, \Phi_t \in \mathcal{C}\}$ denote the $\delta$-delayed hitting time on $\mathcal{C}$; by convention, we write $\tau_{\mathcal{C}}$ for $\tau_{\mathcal{C}}(0)$. Let $\beta \in (0,1)$ and set $W(x) := 1 + |x|^{p+1}$. By A1, there exists $\ell > 0$ such that $t_0\ell \in \mathbb{N}$ and for any $x \notin \mathcal{C} := \{x, |x|^{1+\tau} \le \ell\}$, $\mathbb{E}_x\left[\left|\Phi_{\lfloor t_0|x|^{1+\tau}\rfloor}\right|^{p+1}\right] \le 0.5\,\beta\,|x|^{p+1}$. We can assume without loss of generality that $\ell$ is large enough so that $\mathbb{E}_x\left[W(\Phi_{\lfloor t_0|x|^{1+\tau}\rfloor})\right] \le \beta\, W(x)$ for $x \notin \mathcal{C}$. Set $n(x) := \max(\ell t_0, \lfloor t_0|x|^{1+\tau}\rfloor)$. By A2, there exists $b < \infty$ s.t.

$$\mathbb{E}_x\left[W(\Phi_{n(x)})\right] \le \beta\, W(x) + b\mathbb{1}_{\mathcal{C}}(x). \tag{14}$$

By Theorem 3.2, there exists $C < \infty$ such that $\mathbb{E}_x\left[\{\tau_{\star,\mathcal{C}}\}^{(p+1)/(\tau+1)}\right] \le C W(x)$ where $\tau_{\star,\mathcal{C}} := \inf\{n \ge 1, \Phi_n \in \mathcal{C}\}$ is the return time to $\mathcal{C}$ of the skeleton $P$. Hence, there exists a delay $0 < \delta \le 1$ such that $\mathbb{E}_x[\tau_{\mathcal{C}}(\delta)] \le C\, W(x)$; $\sup_{\mathcal{C}} W < \infty$ and $\mathcal{C}$ is small for the process (by A3), so $\{\Phi_t, t \ge 0\}$ is positive Harris-recurrent and possesses a unique invariant probability measure $\pi$. By [18, Proposition 6.1] and [14, Section 5.4.3], the skeleton $P$ is aperiodic and any compact set is small for the skeleton $P$. A3 and the above properties on the skeleton $P$ imply $\lim_{n\in\mathbb{N}}(n+1)^{(p-\tau)/(1+\tau)} \, \|P^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} = 0$ for any $x \in \mathsf{X}$ [21, Theorem 2.1], which in turn implies that $\lim_{t\in\mathbb{R}^+}(t+1)^{(p-\tau)/(1+\tau)} \, \|P^t(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} = 0$. Set $f(x) := 1 + |x|^q$. A4 and (14) imply that

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_{\star,\mathcal{C}}-1} f(\Phi_k)\right] \le \mathbb{E}_x\left[\sum_{k=0}^{\bar\tau_{\mathcal{C}}-1} \mathbb{E}_{\bar\Phi_k}\left[\sum_{k=0}^{n(\bar\Phi_k)-1} f(\Phi_k)\right]\right]$$
$$\le C\,\mathbb{E}_x\left[\sum_{k=0}^{\bar\tau_{\mathcal{C}}-1} W\left(\bar\Phi_k\right)\right] \le C'\, W(x)$$

where $\bar\tau_{\mathcal{C}}$ is the return time to $\mathcal{C}$ of the discrete-time chain $\{\bar\Phi_n, n \ge 0\}$ with transition kernel $P^{n(x)}(x,\cdot)$. Hence $\pi(f) < \infty$ by [14, Theorem 14.3.7]. As in [7], we obtain $(f,r)$-modulated moments of $\tau_{\star,\mathcal{C}}$ by using Young's inequality (see e.g. [11]). This yields ergodic properties for the skeleton $P$ and the desired limits for $\{P^t, t \ge 0\}$. □

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## Appendix. Proofs

### *A.1. Proof of Proposition 2.2*

In the proof, $C$ is constant and its value may change upon each appearance. We use the following properties

R1. If $\mathcal{D}$ is petite for a phi-irreducible and aperiodic transition kernel, then it is also small [14, Theorem 5.5.7].

R2. If a transition kernel is phi-irreducible and aperiodic, then any skeleton is phi-irreducible and aperiodic [14, Proposition 5.4.5].

R3. If $\mathcal{D}$ is $\nu$-small for a phi-irreducible and aperiodic transition kernel, then we can assume without loss of generality that $\nu$ is a maximal irreducibility measure [14, Proposition 5.5.5].

R4. If there exist measurable functions $f, V : \mathsf{X} \to [1, \infty)$, a measurable set $\mathcal{C}$ and a constant $b$ such that $PV \leq V - f + b\mathbb{1}_{\mathcal{C}}$, then for any stopping time $\tau$ [14, Proposition 11.3.2]

$$\mathbb{E}_x\left[\tau\right] \leq \mathbb{E}_x\left[\sum_{k=0}^{\tau-1} f(\Phi_k)\right] \leq V(x) + b\mathbb{E}_x\left[\sum_{k=0}^{\tau-1} \mathbb{1}_{\mathcal{C}}(\Phi_k)\right].$$

If in addition, there exist $m \geq 1$ and $c > 0$ such that $c\mathbb{1}_{\mathcal{C}}(x) \leq P^m(x, \mathcal{D})$, then

$$\begin{aligned}\mathbb{E}_x\left[\tau_{\mathcal{D}}\right] &\leq \mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{D}}-1} f(\Phi_k)\right] \leq V(x) + \frac{b}{c}\,\mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{D}}-1} \mathbb{1}_{\mathcal{D}}(\Phi_{k+m})\right] \\ &\leq V(x) + \frac{b(m+1)}{c}.\end{aligned}$$

The conditions (6), $\sup_{\mathcal{C}} V < \infty$ and R4 imply that $\sup_{x\in\mathcal{C}} \mathbb{E}_x[\tau_{\mathcal{C}}] < \infty$. Hence, $\mathcal{C}$ is an accessible set. By R1 and R3, there exist $m \geq 1$ and a maximal irreducibility measure $\nu_m$ such that $P^m(x, \cdot) \geq \mathbb{1}_{\mathcal{C}}(x)\nu_m(\cdot)$.

*Step* 1: *From $P$ to the strongly aperiodic transition kernel $P^m$*. By (6), there exists a constant $C$ such that $W^{(m)} := \sum_{k=0}^{m-1} P^k W \leq C\ W$. Write $n = km + l$ for $k \in \mathbb{N}$ and $l \in \{0, \dots, m-1\}$ so that

$$\begin{aligned}n\,\|P^n(x,\cdot) - P^n(x',\cdot)\|_W &\leq n\,\|P^{km}(x,\cdot) - P^{km}(x',\cdot)\|_{W^{(m)}} \\ &\leq C\,(k+1)\,\|P^{km}(x,\cdot) - P^{km}(x',\cdot)\|_W.\end{aligned}$$

By R2, $P^m$ is phi-irreducible and strongly aperiodic, and satisfies drift inequalities of the form (6). Indeed,

$$P^m V(x) \leq V(x) - \sum_{k=0}^{m-1} P^k W(x) + b\sum_{k=0}^{m-1} P^k(x,\mathcal{C}) \leq V(x) - W(x) + b\sum_{k=0}^{m-1} P^k(x,\mathcal{C});$$

this implies that there exist a small set $\mathcal{D}$ (for $P^m$) and a constant $\bar{b} < \infty$ such that $P^m V(x) \leq V(x) - 0.5\ W(x) + \bar{b}\mathbb{1}_{\mathcal{D}}(x)$ (see e.g.[14, proof of Lemma 14.2.8]). By R4, for any accessible set $\mathcal{A}$ (for $P^m$), there exists $C$ such that

$$\mathbb{E}_x[\tau_{\mathcal{A}}^{(m)}] \leq CV(x), \tag{15}$$

where $\tau_{\mathcal{A}}^{(m)} := \inf\{n \geq 1, \Phi_{nm} \in \mathcal{A}\}$. Furthermore, we also have $P^m W(x) \leq W(x) + b\sum_{k=0}^{m-1} P^k(x, \mathcal{C})$. This implies for any accessible (for $P^m$) measurable set $\mathcal{A}$

$$(k+1)\ \mathbb{E}_x\left[W(\Phi_{km})\mathbb{1}_{k\leq\tau_{\mathcal{A}}^{(m)}}\right] \leq CV(x); \tag{16}$$

(the proof is postponed below). Following the same approach as in [21] or [14, Chapter 14], we use the *splitting technique* and associate to the chain with transition kernel $P^m$ a split chain that possesses an atom.

*Step* 2: *From* $P^m$ *to an atomic transition kernel* $\check{P}$. A detailed construction of the split chain and the connection between the original chain and the split chain can be found in [14, Chapter 5]. We use the same notation as in [14]. The split chain $\{(\Phi_n, d_n), n \geq 0\}$ is a chain taking values in $\mathsf{X} \times \{0, 1\}$: its transition kernel is denoted by $\check{P}$. Based on the connection between $P^m$ and $\check{P}$, Proposition 2.2 holds provided there exists $C < \infty$ with

$$m\ (n+1) \sup_{\{f, \sup_{\mathsf{X}} |f|[W]^{-1} \leq 1\}} |\check{P}^n f(x) - \check{P}^n f(y)| \leq C\ \{V(x) + V(y)\}. \tag{17}$$

We prove (17). $\check{P}$ is phi-irreducible and aperiodic and possesses an accessible atom $\alpha := \mathcal{C} \times \{1\}$. Let $\tau_\alpha$ be the return time to the atom $\alpha$. From (15), we have $\check{\mathbb{E}}_{x^\star}[\tau_\alpha] \leq CV(x)$ (see e.g. [21, Proposition 3.7] or [19, Lemma 2.9]); and by (16)

$$(k+1)\ \check{\mathbb{E}}_{x^\star}\left[W(\Phi_k)\mathbb{1}_{k\leq\tau_\alpha}\right] \leq CV(x). \tag{18}$$

(The proof of (18) is postponed below.)

Set $a_x(n) := \check{\mathbb{P}}_{x^\star}(\tau_\alpha = n)$, $u(n) := \check{\mathbb{P}}_\alpha((\Phi_n, d_n) \in \alpha)$ and $t_f(n) := \check{\mathbb{E}}_\alpha\left[f(\Phi_n)\mathbb{1}_{n\leq\tau_\alpha}\right]$, where $a * b(n) := \sum_{k=0}^n a(k)b(n-k)$. Then, by the first-entrance last-exit decomposition [14, Chapter 14], for any function $f$ such that $|f| \leq W$,

$$\begin{aligned}(n+1)\ |\check{P}^n f(x) - \check{P}^n f(y)| \leq{} & (n+1)\ \check{\mathbb{E}}_{x^\star}\left[|f|(\Phi_n)\mathbb{1}_{n\leq\tau_\alpha}\right] \\ & + (n+1)\ \check{\mathbb{E}}_{y^\star}\left[|f|(\Phi_n)\mathbb{1}_{n\leq\tau_\alpha}\right] \\ & + (n+1)\ \left|a_x * u - a_y * u\right| * t_{|f|}(n).\end{aligned}$$

By (18), the first two terms on the right-hand side are upper bounded by $C\{V(x) + V(y)\}$. Applying again (18), $\sup_{k\geq1} k\ t_{|f|}(k) \leq \sup_{k\geq1} k\ \check{\mathbb{E}}_\alpha\left[W(\Phi_k)\mathbb{1}_{k\leq\tau_\alpha}\right] \leq C \sup_{\mathcal{C}} V < \infty$, so

$$\begin{aligned}(n+1)\ \left|a_x * u - a_y * u\right| * t_f(n) &\leq \left(\sup_{k\geq1} k\ t_W(k)\right) \quad \sup_{n\geq1}(n+1)\ \left|a_x * u - a_y * u\right|(n) \\ &\leq C\ \sup_{n\geq1}(n+1)\ \left|a_x * u - a_y * u\right|(n).\end{aligned}$$

Since $\sup_{\mathcal{C}} V < \infty$, $\check{\mathbb{E}}_\alpha[\tau_\alpha] < \infty$; standard results from renewal theory imply (see e.g. [12]) $\sup_{n\geq1}(n+1)\ \left|a_x * u - a_y * u\right|(n) \leq C\{\check{\mathbb{E}}_{x^\star}[\tau_\alpha] + \check{\mathbb{E}}_{y^\star}[\tau_\alpha]\}$. The right-hand side is upper bounded by $C\{V(x) + V(y)\}$. This concludes the proof of (17) and thus the overall proof. □

**Proof of inequality (16).** Using $P^m W(x) - W(x) \le b \sum_{j=0}^{m-1} P^j(x, \mathcal{C})$,

$$\begin{aligned}
&(n+1)\, \mathbb{E}_x \left[ W(\Phi_{nm}) \mathbb{1}_{n \le \tau_{\mathcal{A}}^{(m)}} \right] - W(x) \\
&= \sum_{k=1}^{n} \mathbb{E}_x \left[ (k+1) W(\Phi_{km}) \mathbb{1}_{k \le \tau_{\mathcal{A}}^{(m)}} - k W(\Phi_{(k-1)m}) \mathbb{1}_{k-1 \le \tau_{\mathcal{A}}^{(m)}} \right] \\
&\le \sum_{k=1}^{n} \mathbb{E}_x \left[ W(\Phi_{km}) \mathbb{1}_{k \le \tau_{\mathcal{A}}^{(m)}} \right] + \sum_{k=1}^{n} \mathbb{E}_x \left[ k\{W(\Phi_{km}) - W(\Phi_{(k-1)m})\} \mathbb{1}_{k-1 \le \tau_{\mathcal{A}}^{(m)}} \right] \\
&\le \mathbb{E}_x \left[ \sum_{k=1}^{\tau_{\mathcal{A}}^{(m)}} W(\Phi_{km}) \right] + b\, \mathbb{E}_x \left[ \sum_{k=0}^{\tau_{\mathcal{A}}^{(m)}} (k+1) \sum_{j=0}^{m-1} P^j(\Phi_{km}, \mathcal{C}) \right]. \qquad (19)
\end{aligned}$$

Since $\mathcal{A}$ is an accessible set for $P^m$ and $\mathcal{D}$ is small for $P^m$, there exist $c, r > 0$ such that $C \mathbb{1}_{\mathcal{D}}(x) \le P^{mr}(x, \mathcal{A})$. Hence, by R4, the first term in (19) is upper bounded by $CV(x)$. $P$ is aperiodic and $\mathcal{C}$ is small: there exist $l \ge 1$ and a non-trivial measure $\nu$ such that $\nu(\mathcal{A}) > 0$ and $P^{lm-j}(x, \mathcal{A}) \ge \mathbb{1}_{\mathcal{C}}(x) \nu(\mathcal{A})$ for any $j \in \{0, \ldots, m-1\}$ (see e.g. [14, proof of Lemma 14.2.8]). Hence, $P^{lm}(x, \mathcal{A}) \ge P^j(x, \mathcal{C}) \nu(\mathcal{A})$ for any $j \in \{0, \ldots, m-1\}$, and this yields $m P^{lm}(x, \mathcal{A}) \ge \sum_{j=0}^{m-1} P^j(x, \mathcal{C}) \nu(\mathcal{A})$. Therefore, there exists $C < \infty$ such that

$$\sum_{j=0}^{m-1} P^j(x, \mathcal{C}) \le C\ P^{lm}(x, \mathcal{A}) = C\, \mathbb{E}_x \left[ \mathbb{1}_{\mathcal{A}}(\Phi_{lm}) \right].$$

This yields

$$\begin{aligned}
&b\, \mathbb{E}_x \left[ \sum_{k=0}^{\tau_{\mathcal{A}}^{(m)}} (k+1) \sum_{j=0}^{m-1} P^j(\Phi_{km}, \mathcal{C}) \right] \le C\, \mathbb{E}_x \left[ \sum_{k=0}^{\tau_{\mathcal{A}}^{(m)}} (k+1)\, \mathbb{E}_{\Phi_{km}} \left[ \mathbb{1}_{\mathcal{A}}(\Phi_{lm}) \right] \right] \\
&\le C\, \mathbb{E}_x \left[ \sum_{k=l}^{\tau_{\mathcal{A}}^{(m)}+l} (k-l+1)\ \mathbb{1}_{\mathcal{A}}(\Phi_{km}) \right] \\
&\le C\, \mathbb{E}_x \left[ \mathbb{1}_{l \le \tau_{\mathcal{A}}^{(m)}} \sum_{k=\tau_{\mathcal{A}}^{(m)}}^{\tau_{\mathcal{A}}^{(m)}+l} (k-l+1)\ \mathbb{1}_{\mathcal{A}}(\Phi_{km}) \right] \\
&\quad + C\, \mathbb{E}_x \left[ \mathbb{1}_{l > \tau_{\mathcal{A}}^{(m)}} \sum_{k=l}^{2l} (k-l+1)\ \mathbb{1}_{\mathcal{A}}(\Phi_{km}) \right] \\
&\le C\ \left\{ \mathbb{E}_x \left[ \tau_{\mathcal{A}}^{(m)} \right] + 1 \right\} \le CV(x). \quad \square
\end{aligned}$$

**Proof of inequality (18).** In what follows, we write $\tau_{\mathcal{C}}$ as a shorthand notation for $\tau_{\mathcal{C} \times \{0,1\}}$, and $\Phi_{l:n} \notin \mathcal{C}$ for $\{\Phi_l \notin \mathcal{C}, \ldots, \Phi_n \notin \mathcal{C}\}$. Let $\{\tau^q, q \ge 1\}$ be the successive return times to $\mathcal{C} \times \{0, 1\}$. We write

$$\begin{aligned}
(k+1)\, \check{\mathbb{E}}_{x^\star} \left[ W(\Phi_k) \mathbb{1}_{k \le \tau_\alpha} \right] &= (k+1)\, \check{\mathbb{E}}_{x^\star} \left[ W(\Phi_k) \mathbb{1}_{k \le \tau_\alpha, \tau_\alpha = \tau_{\mathcal{C}}} \right] \\
&\quad + \sum_{q \ge 1} (k+1)\, \check{\mathbb{E}}_{x^\star} \left[ W(\Phi_k) \mathbb{1}_{\tau_{\mathcal{C}}^q < k \le \tau_\alpha} \mathbb{1}_{\Phi_{\tau_{\mathcal{C}}^q+1:k-1} \notin \mathcal{C}} \right]. \qquad (20)
\end{aligned}$$

The connection between $P^m$ and $\check{P}$ yields:

$$(k+1)\,\check{\mathbb{E}}_{x^\star}\left[W(\Phi_k)\mathbb{1}_{k\le\tau_\alpha,\tau_\alpha=\tau_{\mathcal{C}}}\right] \le (k+1)\,\mathbb{E}_x\left[W(\Phi_{km})\mathbb{1}_{k\le\tau_{\mathcal{C}}^{(m)}}\right] \le CV(x).$$

For the second term, let $q \ge 1$ and consider a general term of the series:

$$(k+1)\check{\mathbb{E}}_{x^\star}\left[W(\Phi_k)\mathbb{1}_{\tau_{\mathcal{C}}^q<k\le\tau_\alpha}\mathbb{1}_{\Phi_{\tau_{\mathcal{C}}^q+1:k-1}\notin\mathcal{C}}\right]$$

$$\le \sum_{l=0}^{k-1}(k+1-l)\,\check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\check{\mathbb{E}}_{\Phi_l,d_l}\left[W(\Phi_{k-l})\mathbb{1}_{\Phi_{1:k-l-1}\notin\mathcal{C}}\right]\right] \tag{21}$$

$$+\sum_{l=0}^{k-1} l\,\check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\check{\mathbb{E}}_{\Phi_l,d_l}\left[W(\Phi_{k-l})\mathbb{1}_{\Phi_{1:k-l-1}\notin\mathcal{C}}\right]\right]. \tag{22}$$

By definition of the split chain,

$$\begin{aligned}
&(k+1-l)\check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\,\check{\mathbb{E}}_{\Phi_l,d_l}\left[W(\Phi_{k-l})\mathbb{1}_{\Phi_{1:k-l-1}\notin\mathcal{C}}\right]\right]\\
&= \check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\check{\mathbb{E}}_{\Phi_l,d_l}\left[\check{\mathbb{E}}_{\Phi_1,d_1}\left[(k+1-l)\;W(\Phi_{k-l-1})\mathbb{1}_{\Phi_{0:k-l-2}\notin\mathcal{C}}\right]\right]\right]\\
&\le \check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\check{\mathbb{E}}_{\Phi_l,d_l}\left[\mathbb{E}_{\Phi_1}\left[(k+1-l)\;W(\Phi_{m(k-l-1)})\mathbb{1}_{\tau_{\mathcal{C}}^{(m)}\ge k-l-1}\right]\right]\right]\\
&\le C\,\check{\mathbb{E}}_{x^\star}\left[\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\check{\mathbb{E}}_{\Phi_l,d_l}\left[V(\Phi_1)\right]\right] \le \sup_{\mathcal{C}} RV\;\check{\mathbb{P}}_{x^\star}\left(\mathbb{1}_{\tau_{\mathcal{C}}^q=l}\mathbb{1}_{l<\tau_\alpha}\right).
\end{aligned}$$

Hence, (21) is upper bounded by $C\;\check{\mathbb{E}}_{x^\star}\left(\tau_{\mathcal{C}}^q<\tau_\alpha\right)$ and thus by $C\;(1-\epsilon)^{q-1}$. Furthermore, (22) is upper bounded by

$$C\;\sup_{\mathcal{C}} RW\;\sum_{l=0}^{k-1} l\,\check{\mathbb{P}}_{x^\star}\left(\tau_{\mathcal{C}}^q=l,\,l<\tau_\alpha\right) \le C\check{\mathbb{E}}_{x^\star}\left[\tau_{\mathcal{C}}^q\;\mathbb{1}_{\tau_{\mathcal{C}}^q<\tau_\alpha}\right].$$

The decomposition $\tau_{\mathcal{C}}^q = \sum_{r=1}^{q-1}\{\tau_{\mathcal{C}}^{r+1}-\tau_{\mathcal{C}}^r\}+\tau_{\mathcal{C}}$, and the inequalities $\check{\mathbb{P}}_{x^\star}(\tau_{\mathcal{C}}^r<\tau_\alpha)\le(1-\epsilon)^r$ and $\sup_{\mathcal{C}\times\{0,1\}}\check{\mathbb{E}}_{x,d}\left[\tau_{\mathcal{C}}\right]<\infty$, imply that $\check{\mathbb{E}}_{x^\star}\left[\tau_{\mathcal{C}}^q\;\mathbb{1}_{\tau_{\mathcal{C}}^q<\tau_\alpha}\right]\le C(1-\epsilon)^q\;\check{\mathbb{E}}_{x^\star}[\tau_{\mathcal{C}}]$.

The second term on the rhs of (20) is a convergent series. This concludes the proof. □

### A.2. *Proof of Proposition* 2.4

Under the stated assumptions on $r$, $\inf_{\mathsf{X}} V > 0$, and $\inf_{\mathsf{X}} W > 0$. For any measurable set $\mathcal{C}$ and any function $f:\mathsf{X}\to[1,\infty)$ the function $F(x) := \mathbb{E}_x\left[\sum_{k=0}^{\sigma_{\mathcal{C}}} f(\Phi_k)\right]$ satisfies

$$\begin{aligned}
PF(x) &= \mathbb{E}_x\left[\sum_{k=1}^{\tau_{\mathcal{C}}} f(\Phi_k)\right] = F(x)-f(x)+\mathbb{1}_{\mathcal{C}}(x)\,\mathbb{E}_x\left[\sum_{k=1}^{\tau_{\mathcal{C}}} f(\Phi_k)\right]\\
&\le\; F(x)-f(x)+b\mathbb{1}_{\mathcal{C}}(x)
\end{aligned}$$

where $b := \sup_{x\in\mathcal{C}}\mathbb{E}_x\left[\sum_{k=1}^{\tau_{\mathcal{C}}} f(\Phi_k)\right]$. □

### A.3. *Proof of Proposition* 3.1

Upon noting that $r$, $f$ are non-negative and that $\tau^{k+1} = \tau^k + \tau\circ\theta^{\tau^k}\mathbb{P}_x$-a.s.

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{C}}-1} r(k)\ f(\Phi_k)\right] \leq \mathbb{E}_x\left[\sum_{k=0}^{\tau^{\bar{\tau}_{\mathcal{C}}}-1} r(k)\ f(\Phi_k)\right]$$
$$\leq \mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1}\ \sum_{j=0}^{\tau\circ\theta^{\tau^k}-1} r(j+\tau^k)\ f(\Phi_{j+\tau^k})\right].$$

By definition of the random time $\tau$, $\tau \circ \theta^{\tau^k} = n(\Phi_{\tau^k})$ and this implies

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{C}}-1} r(k)\ f(\Phi_k)\right] \leq \mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1}\ \sum_{j=0}^{n(\Phi_{\tau^k})-1} r(j+\tau^k)\ f(\Phi_{j+\tau^k})\right]$$
$$= \mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1}\sum_{l\geq k}\mathbb{1}_{\tau^k=l}\mathbb{E}_{\Phi_l}\left[\sum_{j=0}^{n(\Phi_0)-1} r(j+l)\ f(\Phi_j)\right]\right].$$

By the drift assumption, $\mathbb{P}_x$-a.s.,

$$\mathbb{E}_{\Phi_l}\left[\sum_{j=0}^{n(\Phi_0)-1} r(j+l)\ f(\Phi_j)\right] \leq V_l(\Phi_l) - \mathbb{E}_{\Phi_l}\left[V_{l+n(\Phi_0)}(\Phi_{n(\Phi_0)})\right] + S_l(\Phi_l)\mathbb{1}_{\mathcal{C}}(\Phi_l),$$

so that

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{C}}-1} r(k)\ f(\Phi_k)\right] \leq \mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1}\left\{V_{\tau^k}(\Phi_{\tau^k}) - V_{\tau^k+n(\Phi_{\tau^k})}(\Phi_{\tau^k+n(\Phi_{\tau^k})})\right.\right.$$
$$\left.\left. + S_{\tau^k}(\Phi_{\tau^k})\mathbb{1}_{\mathcal{C}}(\Phi_{\tau^k})\right\}\right]$$
$$\leq \mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1}\left\{V_{\tau^k}(\Phi_{\tau^k}) - V_{\tau^{k+1}}(\Phi_{\tau^{k+1}}) + S_{\tau^k}(\Phi_{\tau^k})\mathbb{1}_{\mathcal{C}}(\Phi_{\tau^k})\right\}\right]$$
$$\leq V_0(x) + S_0(x)\mathbb{1}_{\mathcal{C}}(x). \quad \square$$

### A.4. *Proof of Theorem* 3.2

Define $\tau := n(\Phi_0)$ and the iterates $\tau^1 := \tau$, $\tau^{k+1} := \tau \circ \theta^{\tau^k} + \tau^k$ for $k \geq 1$, where $\theta$ denotes the shift operator. (By convention, $\tau^0 = 0$.) Finally, set $\bar{\tau}_{\mathcal{C}} := \inf\{k \geq 1,\ \Phi_{\tau^k} \in \mathcal{C}\}$.

**Proof of Theorem 3.2(i).** By definition of the random time $\tau$, $\mathbb{P}_x$-a.s., $\tau_{\mathcal{C}} \leq \tau^{\bar{\tau}_{\mathcal{C}}} = \sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1} n\left(\Phi_{\tau^k}\right)$. Since $t \mapsto R(t)/t$ is non-increasing, we have $R(a+b) \leq R(a)+R(b)$ for any $a, b \geq 0$. This property, combined with the fact that $R$ is increasing yields $\mathbb{P}_x$-a.s.

$$R\left(\tau_{\mathcal{C}}\right) \leq \sum_{l\geq 1}\mathbb{1}_{\bar{\tau}_{\mathcal{C}}=l}\ R\left(\sum_{k=0}^{l-1} n\left(\Phi_{\tau^k}\right)\right) \leq \sum_{l\geq 1}\mathbb{1}_{\bar{\tau}_{\mathcal{C}}=l}\ \sum_{k=0}^{l-1} R\circ n\left(\Phi_{\tau^k}\right) \leq \sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1} W\left(\Phi_{\tau^k}\right),$$

where we used $R(n(x)) \leq W(x)$ in the last inequality. The proof is concluded upon noting that from the drift assumption and the Comparison Theorem [14, Proposition 11.3.2],

$$(1-\beta)\,\mathbb{E}_x\left[\sum_{k=0}^{\bar{\tau}_{\mathcal{C}}-1} W(\Phi_{\tau^k})\right] \le W(x) + b\mathbb{1}_{\mathcal{C}}(x). \quad \square$$

**Proof of Theorem 3.2(ii).** Under the stated assumption, the inverse $H := R^{-1}$ exists. Set $r(t) := R'(t) = 1/(H' \circ R(t))$. Define a sequence of measurable functions $\{H_k, k \in \mathbb{N}\}$, $H_k : [1,\infty) \to (0,\infty)$ by $H_k(t) := \int_0^{H(t)} r(z+k)\,\mathrm{d}z = R(H(t)+k) - R(k)$. Then $H_k$ is increasing and concave. Indeed $H_k'(t) = \frac{r(H(t)+k)}{r(H(t))}$ and this is positive since $R$ is increasing. Since $R$ (and thus $H$) is increasing and $t$ $t \mapsto r(t+k)/r(t)$ is non-increasing (since $R'$ is log-concave), $H_k'$ is non-increasing. Jensen's inequality and the drift assumption imply for any $k \in \mathbb{N}$, $x \in \mathsf{X}$,

$$\begin{aligned} P(H_{k+n(x)} \circ W(\Phi_{n(x)})) &\le H_{k+n(x)}\left(PW(\Phi_{n(x)})\right) \le H_{k+n(x)}\left(\beta W(x) + b\mathbb{1}_{\mathcal{C}}(x)\right) \\ &\le H_{k+n(x)}(\beta W(x)) + H_{k+n(x)}(b)\mathbb{1}_{\mathcal{C}}(x) \\ &\le H_k(W(x)) - \sum_{j=0}^{n(x)-1} r(k+j) + H_{k+n(x)}(b)\mathbb{1}_{\mathcal{C}}(x) + \mathcal{R}_k(x), \end{aligned}$$

where we defined $\mathcal{R}_k(x) := H_{k+n(x)}(\beta W(x)) - H_k(W(x)) + \sum_{j=0}^{n(x)-1} r(k+j)$. We now prove that $\mathcal{R}_k(x) \le 0$ which will conclude the proof by applying Proposition 3.1 with $V_k = H_k \circ W$. We have

$$\begin{aligned} \mathcal{R}_k(x) &= \sum_{j=0}^{n(x)-1} r(j+k) + \int_{n(x)}^{H(\beta W(x))+n(x)} r(z+k)\mathrm{d}z - \int_0^{H(W(x))} r(z+k)\mathrm{d}z \\ &\le \sum_{j=0}^{n(x)-1} r(j+k) - \int_0^{n(x)} r(z+k)\mathrm{d}z, \end{aligned}$$

since by assumption, $H(W) \ge H(\beta W) + n$. Now, $R$ is convex and so $r$ is non-decreasing. Therefore, $\sum_{j=0}^{n(x)-1} r(j+k) \le \int_0^{n(x)} r(z+k)\mathrm{d}z$ and this concludes the proof. $\square$

### A.5. *Proof of Corollary* 3.3

(a) This follows from [19, Lemma 3.1] (see also [21, Proposition 3.1]). (b) By [14, Theorem 14.2.11], there exist measurable sets $\{A_n, n \ge 0\}$ whose union is full and such that $\sup_{x\in A_n} \mathbb{E}_x\left[\sum_{k=0}^{\tau_{\mathcal{D}}-1} W(\Phi_k)\right] < \infty$ for any accessible set $\mathcal{D}$. For an accessible small set $\mathcal{D}$, there exists $n$ such that $\tilde{\mathcal{D}} := \mathcal{D} \cap A_n$ is small and accessible. The proof follows by combining the results of (a) with $\sup_{x\in\tilde{\mathcal{D}}} \mathbb{E}_x\left[\sum_{k=0}^{\tau_{\tilde{\mathcal{D}}}-1} W(\Phi_k)\right] < \infty$. $\square$

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