Weak projectives of finite semigroups

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Abstract

It is described weak projectives in the category of finite semigroups. These are precisely finite weak projectives in the category of compact right topological semigroups.

Keywords: Projective; Absolute coretract; Finite semigroup; Compact right topological semigroup

In [4], P.G. Trotter characterized projectives in the category $\mathcal{FR}$ of finite regular semigroups. In particular, he proved that projectives in $\mathcal{FR}$ are bands. In the category $\mathcal{F}$ of finite semigroups, there are no projectives [3]. The aim of this paper is to describe weak projectives in $\mathcal{F}$. An object $S$ in some category is a projective (a weak projective) if for every morphism $f : S \rightarrow Q$ and every epimorphism (surjective epimorphism) $g : T \rightarrow Q$ there exists a morphism $h : S \rightarrow T$ such that $g \circ h = f$.

The author came to the weak projectives in $\mathcal{F}$ from the finite weak absolute coretracts in the category $C$ of compact right topological semigroups (morphisms in $C$ are continuous homomorphisms). A semigroup endowed with a topology is right topological if all its right shifts are continuous. A significant example of compact right topological semigroup is the Stone–Čech compactification $\beta S$ of a discrete semigroup $S$ (see [1]). An object $S$ in some category is an absolute coretract (a weak absolute coretract) if for every epimorphism (surjective epimorphism) $f : T \rightarrow S$ there exists a morphism $g : S \rightarrow T$ such that $f \circ g = \text{id}_S$. The weak absolute coretracts in $C$ arose in solving the following two questions.

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The first of them is concerned with topological groups. For every topological group $(G, \tau)$, the subset $\tau^*$ of all nonprincipal ultrafilters on $G$ converging to the identity in the topology $\tau$ is a closed subsemigroup in $\beta G$ which is called a semigroup of ultrafilters of $(G, \tau)$. Which finite semigroups can be semigroups of ultrafilters of topological groups? It turns out that, under Continuum Hypothesis, for each finite weak absolute coretract $C$ in $\mathcal{C}$, there is a group topology $\tau$ on countable Boolean group with $\tau^*$ isomorphic to $C$, and that each finite semigroup of ultrafilters of a countable topological group is an idempotent weak absolute coretract in $\mathcal{F}$ [8]. (Observe that each countable topological group with finite semigroups of ultrafilters contains an open Boolean subgroup and cannot be constructed without additional set-theoretic assumptions [5,6].)

The second question is concerned with the semigroup $\beta \mathbb{N}$. Until now it is unknown whether there are elements of finite order in $\beta \mathbb{N}$ other than idempotents. In [7] it was proved that there are no non-trivial finite groups in $\beta \mathbb{N}$ (see also [1, Section 7.1]). Which finite bands exist in $\beta \mathbb{N}$? It is well known that $\beta \mathbb{N}$ contains closed subsemigroups admitting a continuous homomorphism onto any finite semigroup. Hence, $\beta \mathbb{N}$ contains isomorphic copies of any finite weak absolute coretract in $\mathcal{C}$.

In [9] author described finite idempotent weak absolute coretracts in $\mathcal{C}$ and proved that these are precisely idempotent weak absolute coretracts in $\mathcal{F}$. In this paper we prove the following Main Theorem on the collection $\mathcal{P}$ of finite bands from [9].

**Main Theorem.** Let $S$ be a finite semigroup. The following statements are equivalent:

1. $S$ is isomorphic to some semigroup of $\mathcal{P}$;
2. $S$ is a projective in $\mathcal{FR}$;
3. $S$ is a weak projective in $\mathcal{F}$;
4. $S$ is a weak absolute coretract in $\mathcal{F}$;
5. $S$ is an absolute coretract in $\mathcal{F}$;
6. $S$ is a weak projective in $\mathcal{C}$;
7. $S$ is a weak absolute coretract in $\mathcal{C}$;
8. $S$ is an absolute coretract in $\mathcal{C}$.

We prove the Main Theorem by circuits $(1) \Rightarrow (6) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ and $(1) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1)$. The implications $(8) \Rightarrow (5) \Rightarrow (4)$, $(6) \Rightarrow (7)$, and $(6) \Rightarrow (3)$ are trivial. Since, by Hall’s theorem, epimorphisms in $\mathcal{FR}$ are surjective (see [4, Theorem 2.1]), $(3) \Rightarrow (2)$ is also clear. So we need to prove $(1) \Rightarrow (6)$, $(2) \Rightarrow (1)$, $(4) \Rightarrow (1)$, and $(7) \Rightarrow (8)$.

We begin with the construction of the collection $\mathcal{P}$.

Denote by $U$ the semigroup of words of the form $i_1 i_2 \cdots i_p \lambda_p \lambda_{p-1} \cdots \lambda_1$, where $i_q, \lambda_q \in \omega$, $1 \leq q \leq p < \omega$, with the operation

$$i_1 \cdots i_p \lambda_p \cdots \lambda_1 \cdot j_1 \cdots j_q \rho_q \cdots \rho_1 = \begin{cases} i_1 \cdots i_p \rho_p \cdots \rho_1 & \text{if } p = q, \\ i_1 \cdots i_p \lambda_p \cdots \lambda_{q+1} \rho_q \cdots \rho_1 & \text{if } p > q, \\ i_1 \cdots i_p \lambda_p+1 \cdots j_q \rho_q \cdots \rho_1 & \text{if } p < q. \end{cases}$$
For every $p \in \mathbb{N}$, denote by $U_p$ the subsemigroup of $U$ of words of length $2p$. The semigroup $U$ is a band decomposing into the decreasing chain of its rectangular components $U_p$. For every subsemigroup $S$ of $U$, put $S_p = S \cap U_p$.

For every $p \in \mathbb{N}$, $q \in \{1, p\}$ and $u = i_1 \cdots i_p \lambda_p \cdots \lambda_1 \in U_p$, put $u' = i_1 \cdots i_p$, $u'' = \lambda_p \cdots \lambda_1$, $u'_q = i_q$, and $u''_q = \lambda_q$.

Denote by $\mathcal{P}$ the collection of finite subsemigroups $S$ of $U$ satisfying the following conditions for every $p \in \mathbb{N}$:

(i) if $u \in S_p$, both $u'_q \neq 0$ and $u''_q \neq 0$;
(ii) if $u \in S_p$ and $u'_q \neq 0$ for some $q \in \{1, p - 1\}$, there exists $v \in S_q$ such that $v'$ is the initial segment of $u'$; and dually, if $u \in S_p$ and $u''_q \neq 0$ for some $q \in \{1, p - 1\}$, there exists $v \in S_q$ such that $v''$ is the final segment of $u''$;
(iii) either $u'_q = 1$ for all $u \in S_q$ with $q \geq p$ or $u''_q = 1$ for all $u \in S_q$ with $q \geq p$.

We indicate also a complete system of non-isomorphic representatives of $\mathcal{P}$.

Denote by $\mathcal{M}$ the set of all matrices $M = (m_{p,q})_{p,q} \in \mathbb{N}$ without the main diagonal $(m_{p,p})$, where $l \in \mathbb{N}$ and $m_{p,q} \in \omega$, satisfying the following conditions for every $p \in \{1, l\}$:

(a) $m_{0,p} \leq m_{1,p} \leq \cdots \leq m_{p-1,p} \in \mathbb{N}$ and $m_{p,0} \leq m_{p,1} \leq \cdots \leq m_{p,p-1} \in \mathbb{N}$;
(b) either $m_{p-1,p} = 1$ and $m_{p-1,p+1} = \cdots = m_{p,1} = 0$ or $m_{p,p-1} = 1$ and $m_{p+1,p-1} = \cdots = m_{l-1,p-1} = 0$.

For each $M = (m_{p,q})_{p,q} \in \mathcal{M}$, denote by $S(M)$ the subsemigroup of $\bigcup_{p=1}^l U_p$ consisting of all $u \in U_p$, $p \in \{1, l\}$, which are satisfying the following conditions:

(1) both $u'_q \neq 0$ and $u''_q \neq 0$;
(2) for every $q < r \leq p$, if $u'_t = 0$ for all $t \in [q + 1, r - 1]$, then $u'_r \leq m_{q,r}$, and dually, if $u''_t = 0$ for all $t \in [q + 1, r - 1]$, then $u''_r \leq m_{r,q}$.

It is obvious that for every $M \in \mathcal{M}$, $S(M) \in \mathcal{P}$. We claim that every $S \in \mathcal{P}$ is isomorphic to $S(M)$ for some $M \in \mathcal{M}$.

Indeed, let $l = \max\{p: S_p \neq \emptyset\}$ and for every $p \in \{1, l\}$, let $I_p = \{u'_q: u \in S_p\}$ and $A_p = \{u''_q: u \in S_p\}$. For every $q < p \leq l$, let $I_{q,p}$ be the set of all $i \in I_p$ such that there exists $u \in S_p$ with $u'_q = i$ and $u''_q = 0$ for all $r \in [q + 1, p - 1]$ and let $m_{q,p} = |I_{q,p}|$. Analogously, for every $q < p \leq l$, let $A_{q,p}$ be the set of all $\lambda \in A_p$ such that there exists $u \in S_p$ with $u''_q = \lambda$ and $u'_q = 0$ for all $r \in [q + 1, p - 1]$ and let $m_{p,q} = |A_{p,q}|$. For every $p \in \{1, l\}$, choose bijections $f_p : I_p \rightarrow \{1, \ldots, m_{p-1,p}\}$ and $g_p : A_p \rightarrow \{1, \ldots, m_{p,p-1}\}$ such that whenever $q < r$, $i \in I_{q,p}$, and $j \in I_{r,p} \setminus I_{q,p}$, one has $f(i) < f(j)$, and dually, whenever $q < r$, $\lambda \in A_{q,p}$, and $\mu \in A_{p,r} \setminus A_{q,p}$, one has $g(\lambda) < g(\mu)$. An easy check shows that $M = (m_{p,q})_{p,q} \in \mathcal{M}$ and that

$$S \ni i_1 \cdots i_p \lambda_p \cdots \lambda_1 \mapsto f_1(i_1) \cdots f_p(i_p)g_p(\lambda_p) \cdots g_1(\lambda_1) \in S(M)$$

is the required isomorphism.
Let now $M = (m_{p,q})_{l \times l} \in M$ and $S = S(M)$. For every $q < p \leq l$, define families $\mathcal{F}_{q,p}$ and $\mathcal{F}_{p,q}$ of subsets of $S_p$ downstairs induction by $q$ putting

$$\mathcal{F}_{q,p} = \left\{ S_q u S_p : u \in S_p \setminus \bigcup_{r=q+1}^{p-1} S_r \right\}$$

and

$$\mathcal{F}_{p,q} = \left\{ S_p u S_q : u \in S_p \setminus \bigcup_{r=q+1}^{p-1} S_p \right\},$$

where

$$S_{r,p} = \bigcup \mathcal{F}_{r,p} \quad \text{and} \quad S_{p,r} = \bigcup \mathcal{F}_{p,r}.$$

It is easy to see that

$$m_{q,p} = |\mathcal{F}_{q,p}| \quad \text{and} \quad m_{p,q} = |\mathcal{F}_{p,q}|.$$

Consequently, $M$ uniquely determined by $S$.

The following theorem is the implication $(1) \Rightarrow (6)$. Its coretract version was proved in [9].

**Theorem 1.** Every semigroup of $P$ is a weak projective in $C$.

**Proof.** Let $S \in P$, let $f : S \rightarrow Q$ be a homomorphism, and let $g : T \rightarrow Q$ be a surjective continuous homomorphism. We adjoin the identities $\emptyset, 1_Q, 1_T$ to $S, Q, T$, respectively, and extend $f, g$ in the obvious way. We shall inductively construct the homomorphism $h : S \rightarrow T$ such that $g \circ h = f$.

Put $S_0 = \{ \emptyset \}$, $S_0^p = \bigcup_{q=0}^{p} S_q$,

$$e_p = \left\{ \begin{array}{ll} \emptyset & \text{if } p = 0, \\ 1 \cdots 1 & \text{if } p > 0 \end{array} \right.$$  

For every $u \in S_p$, put

$$\hat{u} = \left\{ \begin{array}{ll} \emptyset & \text{if } u \in S_0 \text{ or } u'_p = \cdots = u'_q = 0, \\ u'_1 \cdots u'_q 1 \cdots 1 & \text{if } q = \max \{ r < p : u'_r \neq 0 \}, \end{array} \right.$$  

where $q = \max \{ r < p : u'_r \neq 0 \}$, otherwise,

$$\tilde{u} = \left\{ \begin{array}{ll} \emptyset & \text{if } u \in S_0 \text{ or } u''_p = \cdots = u''_q = 0, \\ 1 \cdots 1 u''_q \cdots u''_1 & \text{if } q = \max \{ r < p : u''_r \neq 0 \}, \end{array} \right.$$  

where $q = \max \{ r < p : u''_r \neq 0 \}$, otherwise.

**Lemma 1.**

(a) For every $u \in S_p$, $\hat{u}u = uu = u$.
(b) If $q < p$, $u \in S_p$, and $v \in S_q$, then $(\tilde{u}v) = \tilde{u}v$ and $(\hat{u}v) = v\hat{u}$.
(c) For every $u, v \in S_p$, $\hat{u}v = e_{p-1}$. 
The proof of Lemma 1 is an easy check [1].

For every \( u \in S_p \), we also put \( R(u') = \{ v \in S_p : v' = u' \} \) and \( L(u'') = \{ v \in S_p : v'' = u'' \} \). Observe that these are respectively minimal right and minimal left ideals in \( S_p \) containing \( u \).

We shall use the fact that every compact right topological semigroup has the smallest ideal which is a completely simple semigroup (see [1, Section 2.2]).

Define \( h \) on \( S_0 \) by \( h(\emptyset) = 1_r \). Suppose that \( h \) has been defined on \( S_0^{p-1} \). We shall show that \( h \) can be extended to \( S_p \).

Let \( I_p = \{ u'_p : u \in S_p \} \) and for every \( u \in S_p \), let \( \mu(u) = \min \{ q < p : u'_{q+1} = \cdots = u'_{p-1} = 0 \} \). For each \( i \in I_p \), we choose \( w_i \in S_p \) such that \( (w_i)'_p = i \) and \( \mu(w_i) = \min \{ \mu(u) : u \in S_p \text{ with } u'_p = i \} \), and then choose a minimal right ideal \( R_p(i) \) in \( g^{-1}(f(S_p)) \) with \( g(R_p(i)) \subseteq f(R((w_i)')) \). We observe that, for any \( u \in S_p \) with \( u'_p = i \), we have \( \hat{u} R((w_i)') \subseteq R(u') \), and so \( g(h(\hat{u})R_p(i)) \subseteq f((\hat{u})f((w_i)')) \subseteq f(R(u')) \). Hence, for any \( u \in S_p \), we have \( g(h(\hat{u})R_p(u'')) \subseteq f(R(u')) \). We define minimal left ideals \( L_p(\lambda) \) in \( g^{-1}(f(S_p)) \) in the dual way. For every \( u \in S_p \), we define \( h(u) \) to be the idempotent of the group \( h(\hat{u})R_p(u'')L_p(u'')h(\hat{u}) \). Then \( gh(u) = f(u) \), because

\[
gh(u) \in g(h(\hat{u})R_p(u''))g(L_p(u'')h(\hat{u})) \subseteq f(R(u'))f(L(u'')) = f(\{ u \}).
\]

Let \( v \in S_0^{p-1} \). We shall show that \( h(u)h(v) = h(uv) \).

We have \( h(u)h(v) = h(\hat{u})R_p(u'')L_p(u'')h(\hat{v})h(v) \). We have also \( \hat{u} = \hat{(uv)} \), \( u' = (uv)' \), \( u'' = (uv)'_p \), and \( h(\hat{u})h(\hat{v}) = h(\hat{uv}) = h(\hat{uv}) \). So \( h(u)h(v) \) and \( h(uv) \) belong to the same group in \( g^{-1}(f(S_p)) \). It will therefore be sufficient to show that \( h(u)h(v) \) is idempotent. To establish this, we shall show that \( h(u)h(v)h(u) = h(u) \).

We write \( h(u) = h(\hat{u})wh(\hat{u}) \) for some \( w \in R_p(u'')L_p(u'') \). Then

\[
h(u)v(h(u) = h(\hat{u})wh(\hat{u})h(\hat{v})wh(\hat{u}) = h(\hat{u})wh(\hat{uv})wh(\hat{u}).
\]

Since \( \hat{uv} = (\hat{uv})\hat{u} = e_{p-1} = \hat{u} \), then

\[
h(u)h(v)h(u) = h(\hat{u})wh(\hat{u})wh(\hat{u})h(\hat{u})wh(\hat{u}) = h(\hat{u})wh(\hat{uv})wh(\hat{u}).
\]

This establishes that \( h(uv) = h(u)h(v) \). Similarly, \( h(uu) = h(u)h(u) \).

Let now \( v \in S_p \). Again, we have \( h(u)h(v) = h(\hat{u})R_p(u'')L_p(v'')h(\hat{v})h(v) \) and we have also \( \hat{u} = \hat{uv} \), \( u'_p = (uv)'_p \), \( v'' = (uv)'_p \), and \( \hat{v} = \hat{uv} \). So \( h(u)h(v) \) and \( h(uv) \) belong to the same group. We shall again show that \( h(u)h(v) \) is idempotent by proving that \( h(u)h(v)h(u) = h(u) \).

We know that either \( w'' = 1 \) for all \( w \in S_p \) or \( w'' = 1 \) for all \( w \in S_p \). In the first case, \( h(v) \in L_p(1)h(\hat{v}) \) and \( h(u) \in L_p(1)h(\hat{u}) \). So \( h(v)h(\hat{u}) \) and \( h(u)h(\hat{v}) \) belong to the same minimal left ideal \( L_p(1)h(e_{p-1}) \) in \( g^{-1}(f(S_p)) \). We have seen that these elements are idempotent, and so \( h(u)h(v)h(u) = h(u)h(v) \). Thus \( h(u)h(v)h(u) = h(u)h(v)h(\hat{u})h(\hat{u})h(\hat{u}) = h(u)h(\hat{v})h(\hat{u})h(\hat{u})h(\hat{u}) = h(u)h(e_{p-1})h(u) \). This statement holds with \( v \) replaced by \( u \), and so \( h(u) = h(u)h(e_{p-1})h(u) = h(u)h(v)h(u) \).
Similarly, we can prove that $h(u)h(v)h(u) = h(u)$ if we assume that $w'_p = 1$ for all $w \in S_p$. ✷

Recall that a Bernside semigroup $B(k, 1, 3)$ is the free semigroup on $k$ generators in the variety of semigroups defined by the identity $x = x^3$. It is finite for every $k \in \mathbb{N}$ (see [2, Chapter 10, Theorem 3]).

The following theorem was proved in [9], but now we give more direct and short proof.

**Theorem 2.** Let $S$ be a finite band. If $S$ is a coretract of a Bernside semigroup $B(k, 1, 3)$, then $S$ is isomorphic to some semigroup of $P$.

**Proof.** Let $B = B(k, 1, 3)$ and let $f : B \to S$ be a coretraction. We may suppose that $S$ is a subsemigroup of $B$ and that $f|_S = \text{id}_S$. Let $F$ be the free semigroup on a $k$-element alphabet $A$ and let $h : F \to B$ be the canonical homomorphism. Observe that $h(u) = h(v)$ if and only if $v$ can be obtained from $u$ by a succession of operations in each of which a word $w_1ww_2$ is replaced by $w_1w_3w_2$, or vice versa. (Here words $w_1, w, w_2$ allowed to be empty.)

Let $w \in F$, $C \subseteq A$, and $\rho \subseteq C^2$. We shall use the following notation.

- $ct(w)$ is the set of letters in $w$.

- $w|_C$ is the word obtained from $w$ by removing all letters in $A \setminus C$.

- $\alpha(w, C)$ is the first letter in $w|_C$.

- $\beta(w, C)$ is the last letter of $w|_C$.

Observe that if $h(u) = h(v)$, then $\alpha(u, C) = \alpha(v, C)$ and $\beta(u, C) = \beta(v, C)$.

- $\sigma(w, C, \rho)$ is the quantity of pairs of neighboring letters in $w|_C$ which belong to $\rho$.

Observe that if $h(u) = h(v)$, then $\sigma(u, C, \rho) \equiv \sigma(v, C, \rho) \mod(2)$. To prove this, it suffices to consider the case $u = w_1w_2w_3$, $v = w_4w_5w_6$. Put $\sigma(t) = \sigma(t, C, \rho)$. Then

$$
\sigma(v) = \begin{cases} 
\sigma(u) + 2\sigma(w) + 2 & \text{if } w|_C \neq \emptyset \text{ and } (\beta(w, C), \alpha(w, C)) \in \rho, \\
\sigma(u) + 2\sigma(w) & \text{otherwise}.
\end{cases}
$$

**Lemma 2.** $S$ is a chain of its rectangular components.

**Proof.** Suppose the contrary. Then there exist $u, v \in h^{-1}(S)$ with $a \in ct(u) \setminus ct(v)$ and $b \in ct(v) \setminus ct(u)$. Put $\sigma(w) = \sigma(w, \{a, b\}, \{(a, b)\})$. Then $\sigma(uv) = 1$ and $\sigma(uuv) = 2$, although $h(uuv) = h(uv)$, a contradiction. ✷
Let \( S_1 > S_2 > \cdots > S_l \) be the rectangular components of \( S \). Put
\[
A_p = \{ a \in A : fh(a) \in S_p \}.
\]
Observe that for every \( u \in h^{-1}(S) \), we have that \( h(u) \in S_p \) if and only if \( p = \max\{ q \leq l : ct(u) \cap A_p \neq \emptyset \} \). Indeed, if \( u = a_1 \cdots a_n \), then \( h(u) = fh(a_1) \cdots fh(a_n) \).

Next, put
\[
A^q_p = \bigcup_{r=p}^q A_r, \quad S^q_p = \bigcup_{r=p}^q S_r \quad (p < q),
\]
\[
M_p = \{ \alpha(u, A^1_p) : u \in h^{-1}(S^1_p) \}, \quad N_p = \{ \beta(u, A^1_p) : u \in h^{-1}(S^1_p) \}.
\]
Observe that \( M_p \cap A_p \neq \emptyset \) and \( N_p \cap A_p \neq \emptyset \).

**Lemma 3.** For every \( p \in \{1, l\} \), one of the sets \( M_p, N_p \) is a singleton.

**Proof.** Choose \( u \in h^{-1}(S_l) \). Let \( a = \alpha(u, A^1_p) \) and \( b = \beta(u, A^1_p) \). Put \( \sigma(u) = \sigma(u, A^1_p, ((a, b))) \). Since \( \sigma(uu) = 2\sigma(u) + 1 \equiv \sigma(u) \mod 2 \), \( \sigma(u) \) is odd. Suppose that there exist \( v_1, v_2 \in h^{-1}(S^1_p) \) with \( \alpha(v_1, A^1_p) \neq a \) and \( \beta(v_2, A^1_p) \neq b \). Put \( v = v_1v_2 \). Since \( \sigma(vv) = 2\sigma(v) \equiv \sigma(v) \mod 2 \), \( \sigma(v) \) is even. Then \( \sigma(uvv) = 2\sigma(u) + \sigma(v) \) is also even. On the other hand, in \( S \), as in every chain of rectangular bands, the following statement holds true:
if \( x, z \in S_p, y \in S_r, \) and \( r \leq q \), then \( xyz = xz \). Therefore \( h(uvv) = h(uuu) = h(u) \), and so \( \sigma(uvv) \equiv \sigma(u) \mod 2 \), a contradiction. \( \square \)

**Lemma 4.** If \( x \in S_p, y \in S_q, z \in S_r, \) and \( q \leq p, r \), then \( xyz = xz \).

**Proof.** It is convenient for us to adjoin identities \( \emptyset, 1_F, 1_B, 1_S \) to \( F, B, S \) and to extend \( h, f \) in the obvious way. Put also \( S_0 = \{1_S\} \). Then the lemma is obviously true if \( q = 0 \).
Fix \( q > 0 \) and assume that the lemma holds for all smaller values of \( q \). Take \( u \in h^{-1}(x) \), \( v \in h^{-1}(y) \), and \( w \in h^{-1}(z) \). By Lemma 3, one of the sets \( M_q, N_q \) is a singleton. Suppose that \( N_q = \{a\} \). Then we can write \( u = u_1a \) and \( v = v_1a \), where \( ct(v_1), ct(v_2) \subseteq A_1^{-1} \).
Since \( x = fh(u) \) and \( y = fh(v) \), it follows from this that \( x = x_1sxz \) and \( y = y_1sz \), where
\[
s = fh(a) \in S_q, \quad x_2 = fh(u_2), \quad y_2 = fh(v_2) \in S_0^{-1}\]
and \( x_1 = fh(v_1) \in S_0^q \). So \( xyz = x_1zszz \) and \( zx = x_1szz \). It is clear that \( xyz = xz \). By our inductive assumption, \( syz = sz \) and \( sxz = za \). Hence \( xyz = x_1szz \) and \( xz = x_1sz \). The case \( |M_q| = 1 \) is similar. \( \square \)

We enumerate sets \( M_p \cap A_p \) and \( N_p \cap A_p \) as \( \{a_{pi} : 1 \leq i \leq p \} \) and \( \{b_{pi} : 1 \leq \lambda \leq n_p \} \), respectively. Define functions \( \phi_p \) and \( \theta_p \) on \( S^q_p \) as follows. Let \( x \in S^q_p \). Pick \( u \in h^{-1}(x) \) and put
\[
\phi_p(x) = \begin{cases} 0 & \text{if } \alpha(u, A^1_p) \notin A_p, \\ i & \text{if } \alpha(u, A^1_p) = a_{pi}, \\ \end{cases}
\]
\[
\theta_p(x) = \begin{cases} 0 & \text{if } \beta(u, A^1_p) \notin A_p, \\ \lambda & \text{if } \beta(u, A^1_p) = b_{p\lambda}. \\ \end{cases}
\]
We now define the map $\psi : S \to U$ putting for every $x \in S_p$,

$$
\psi(x) = \phi_1(x)\phi_2(x) \cdots \phi_p(x)\theta_p(x)\theta_{p-1}(x) \cdots \theta_1(x).
$$

It is clear that both $\phi_p(x) \neq 0$ and $\theta_p(x) \neq 0$. By Lemma 3, either $\phi_p(y) = 1$ for all $y \in S_p'$ or $\theta_p(y) = 1$ for all $y \in S_p'$.

To check that $\psi$ is injective, let $x \in S_p$. Let $p_1 < p_2 < \cdots < p_k = p$ are all $r \in [1, p]$ with $\phi_r(x) \neq 0$, let $q_1 < q_2 < \cdots < q_l = p$ are all $r \in [1, p]$ with $\theta_r(x) \neq 0$, let $\phi_{p_j}(x) = i_j$, and let $\theta_{q_k}(x) = \lambda_k$. Pick $u \in h^{-1}(x)$. Then

$$
u = a_{p_1i_1}u_1a_{p_2i_2}u_2 \cdots u_{k-1}a_{p_{k-1}i_{k-1}}w_{q_{k-1}i_{k-1}}v_1 \cdots v_2b_{q_2i_2}v_3b_{q_1i_1},
$$

where $\text{ct}(u) \subseteq A_i^{p_j}$ and $\text{ct}(v) \subseteq A_i^{q_k}$. But then, by Lemma 4,

$$
x = fh(a_{p_1i_1}a_{p_2i_2} \cdots a_{p_{k-1}i_{k-1}}b_{q_{k-1}i_{k-1}} \cdots b_{q_2i_2}b_{q_1i_1})
$$

and, consequently, $x$ is uniquely determined by $\psi(x)$.

To see that $\psi$ is homomorphism, let $x \in S_p$ and $y \in S_q$. It suffices to check that

(a) $\phi_r(xy) = \phi_r(x)$ if $r \leq p$;
(b) $\phi_r(xy) = \phi_r(y)$ if $p < r \leq q$;
(c) $\theta_r(xy) = \theta_r(y)$ if $r \leq q$;
(d) $\theta_r(xy) = \theta_r(x)$ if $q < r \leq p$.

Let $u \in h^{-1}(x)$, $v \in h^{-1}(y)$, and $w = uv$. If $r \leq p$, then $\alpha(w, A_i^r)$ occurs in $u$, because $\text{ct}(u) \cap A_i^r \neq \emptyset$, so $\phi_r(xy) = \phi_r(x)$. If $p < r \leq q$, then $\alpha(w, A_i^r)$ occurs in $v$, because $\text{ct}(v) \cap A_i^r = \emptyset$, so $\phi_r(xy) = \phi_r(y)$. The check of (c) and (d) is similar.

It remains to verify that the semigroup $\psi(S)$ satisfies condition (ii) in the definition of the class $\mathcal{P}$. Let $x \in S_p$ and let $\phi_p(x) = a \neq 0$ for some $q \in [1, p]$. Pick $u \in h^{-1}(x)$ and write it in the form $u = vaw$, where $ct(v) \subseteq A_i^{q-1}$. Define $y \in S_q$ by $y = fh(va)$. Since $x = fh(vaw)$, $yx = x$. By statement (a), $\phi_r(xy) = \phi_r(y)$ for all $r \leq q$. Hence $(\psi(y)')'$ is the initial segment of $(\psi(x))'$.

From Theorem 2 and Trotter’s theorem [4, Theorem 2.6] it follows the implication $(2) \Rightarrow (1)$. From Trotter’s theorem we also deduce the next result.

**Theorem 3.** Each weak projective in $\mathcal{F}$ is a band.

**Proof.** Let $S$ be a weak projective in $\mathcal{F}$ and let $S = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n$ be its principal series. By Trotter’s theorem, it suffices to prove that for every $i < n$, the quotient $S_i/S_{i+1}$ is not a semigroup with zero multiplication. Assume the contrary and let $m$ be the smallest such $i$. Take any integer $k > |\bigcup_{i=m+1}^n S_i|$. Let $Q$ and $T$ be cyclic monoids defined by relations $a^3 = a^2$ and $b^{k+2} = b^2$, respectively. Define homomorphisms $f : S \to Q$ and...
Then there is no a homomorphism $h : S \to T$ such that $g \circ h = f$, a contradiction. \qed

The implication (4) ⇒ (1) follows from Theorems 2 and 3. It remains to prove (7) ⇒ (8). For this, by Theorem 3, it suffices to prove the following proposition.

**Proposition 1.** Let $S$ be a category of semigroups containing all finite bands. Then in $S$, every epimorphism into a finite band is surjective.

**Proof.** Assume, by the contrary, there is a finite band $S$ and a non-surjective epimorphism $f : T \to S$ in $S$. Let $J$ be a maximal component among all rectangular components $I$ of $S$ with $I \setminus f(T) \neq \emptyset$. Then either there exists $L$-class $L$ in $J$ with $L \cap f(T) = \emptyset$ or there exists $R$-class $R$ in $J$ with $R \cap f(T) = \emptyset$. Obviously, it suffices to consider the first case.

Define the equivalence $\theta_J$ on $J$ by

$$
\theta_J = \{(x, y) \in J^2 : x Ly \text{ or both } L_x \cap f(T) \neq \emptyset \text{ and } L_y \cap f(T) \neq \emptyset\}.
$$

It is partitioned $J$ into the subset $A$ being the union of $C$-classes meeting $f(T)$, and the $C$-classes disjoint $f(T)$. If $(x, y) \in \theta_J$ and $z \in J$, then $(xz, yz) \in \theta_J$ because $xzLyz$, and $zLy \in \theta_J$ because both $xzLx$ and $yzLy$. Hence $\theta_J$ is a congruence on $J$. For any another rectangular component $I$ of $S$, put

$$
\theta_I = \begin{cases} 
\forall I = I^2 & \text{if } I < J, \\
\Delta_I = \{(x, x) : x \in I\} & \text{otherwise.}
\end{cases}
$$

Define the equivalence $\theta$ on $S$ by $\theta = \bigcup_J \theta_I$. Make sure that $\theta$ is congruence, that is for any $(x, y) \in \theta$ and $z \in S$, both $(xz, yz) \in \theta$ and $(x, yz) \in \theta$. Obviously, only the case $(x, y) \in \theta_J$ and $z \in I > J$ needs the verification.

If $xLyz$ because $C$ is a right congruence, and $xzLyz$ because both $xzLx$ and $yzLy$. Let now $L_x \cap f(T) \neq \emptyset$ and $L_y \cap f(T) \neq \emptyset$. Then there are $a, b \in T$ such that $f(a) \in L_x$ and $f(b) \in L_y$. Since $I \subseteq f(T)$, there is $c \in T$ such that $f(c) = z$. But then $f(ac) = f(a)f(c) \in Lxz$ and $f(bc) = f(b)f(c) \in L_yz$. Hence, $Lxz \cap f(T) \neq \emptyset$ and $Lyz \cap f(T) \neq \emptyset$, and so $(xz, yz) \in \theta$. The fact that $(xz, yz) \in \theta$ is obvious because $Lxz = Lx$ and $Lyz = Ly$.

Let $\pi : S \to S/\theta$ be the canonical homomorphism. Define the homomorphism $\pi' : S \to S/\theta$ by

$$
\pi'(x) = \begin{cases} 
\pi(x) & \text{if } x \in I \neq J, \\
A & \text{if } x \in J.
\end{cases}
$$

Then $\pi \circ f = \pi' \circ f$ and $\pi \neq \pi'$, a contradiction. \qed
Observe that as distinguished from the equivalence (6) ⇔ (7), (3) ⇔ (4) is a simple fact. It is a partial case of the following proposition.

**Proposition 2.** Let \( S \) be a category of finite semigroups closed under finite direct product and subsemigroups. Then in \( S \), every weak absolute coretract is a weak projective.

**Proof.** Let \( S \) be an absolute coretract in \( S \), let \( f : S \to Q \) be a homomorphism, and let \( g : T \to Q \) be a surjective homomorphism. We need to construct a homomorphism \( h : S \to T \) with \( g \circ h = f \).

Let \( I \) be the set of all maps \( i : S \to R_i \), where \( R_i = S \) or \( R_i = T \). Define the injection \( e : S \to \prod_I R_i \) by \( e(x) = (i(x))_{i \in I} \). Let \( X = e(S) \) and let \( F \) be the subsemigroup of \( \prod_I R_i \) generated by \( X \). Then each map \( X \to R_i \), where \( R_i = S \) or \( R_i = T \) can be extended to a homomorphism \( F \to R_i \).

Take any bijection \( X \to S \) and extend it to a homomorphism \( \alpha : F \to S \). Then, for every \( x \in X \) choose \( \gamma(x) \in g^{-1}(f \circ \alpha(x)) \), and extend the map \( X \ni x \mapsto \gamma(x) \in T \) to a homomorphism \( \gamma : F \to T \). We have \( g \circ \gamma = f \circ \alpha \). Since \( S \) is an absolute coretract, there exists a homomorphism \( \beta : S \to F \) with \( \alpha \circ \beta = \text{id}_S \). Define \( h : S \to T \) by \( h = \gamma \circ \beta \). \( \square \)

Let \( \mathcal{FG} \) be a category of finite Clifford semigroups. From Hall’s theorem it follows that both in \( \mathcal{FR} \) and in \( \mathcal{FG} \), weak projectives are projectives and weak absolute coretracts are absolute coretracts. By Theorems 1 and 2, in \( \mathcal{FR} \), idempotent absolute coretracts are projectives. By Proposition 2, in \( \mathcal{FG} \), all absolute coretracts are projectives.

**Question 1.** Is there an absolute coretract in \( \mathcal{FR} \) other than bands?

**Question 2.** Is there a projective in \( \mathcal{FG} \) other than bands?

**References**


