Blow-up rates of large solutions for elliptic equations

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A R T I C L E   I N F O

Article history:
Received 13 May 2009
Revised 10 January 2010
Available online 17 March 2010

MSC:
35J25
35J65
35J67

Keywords:
Semilinear elliptic equations
Boundary blow-up
The first and second expansions of solutions near the boundary
The mean curvature of the boundary

A B S T R A C T

In this paper, we mainly study the boundary behavior of solutions to boundary blow-up elliptic problems for more general nonlinearities \( f \) (which may be rapidly varying at infinity) \( \Delta u = b(x)f(u), \) \( x \in \Omega, \) \( u|_{\partial\Omega} = +\infty, \) where \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N, \) and \( b \in C^\alpha(\bar{\Omega}) \) which is positive in \( \Omega \) and may be vanishing on the boundary and rapidly varying near the boundary. Further, when \( f(s) = s^p \pm f_1(s) \) for \( s \) sufficiently large, where \( p > 1 \) and \( f_1 \) is normalized regularly varying at infinity with index \( p_1 \in (0, p), \) we show the influence of the geometry of \( \Omega \) on the boundary behavior for solutions to the problem. We also give the existence and uniqueness of solutions.

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1. Introduction and the main results

In this paper, we mainly consider the first and second expansions of solutions near the boundary to the following boundary blow-up elliptic problems

\[ \Delta u = b(x)f(u), \quad x \in \Omega, \quad u|_{\partial\Omega} = +\infty, \]

where the last condition means that \( u(x) \to +\infty \) as \( d(x) = \text{dist}(x, \partial\Omega) \to 0, \) \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N \) \( (N \geq 2), \) \( b \) satisfies

\((b_1)\) \( b \in C^\alpha(\bar{\Omega}) \) for some \( \alpha \in (0, 1), \) is positive in \( \Omega \)

\* This work is supported in part by NNSF of PR China under Grant no. 10671169 and by Shandong Province Natural Science Foundation under Grant no. 2009ZRB01795.

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doi:10.1016/j.jde.2010.02.019
and $f$ satisfies

(f1) $f \in C^1(R)$, $f(s) > 0$, $\forall s \in R$, $f$ is increasing on $R$ (or (f01) $f \in C^1[0, \infty)$, $f(0) = 0$, $f$ is increasing on $[0, \infty)$);

(f2) $\int_1^\infty \frac{du}{f(u)} < \infty$;

(f3) there exists $C_f > 0$ such that $\lim_{s \to +\infty} f'(s) \int_s^\infty \frac{dv}{F(v)} = C_f$.

The problem (1.1) arises from many branches of mathematics and has been discussed by many authors, see, for instance, [1–17, 19–27, 29, 32–37] and the references therein.

For $b \equiv 1$ on $\Omega$ and $f$ satisfying (f1) (or (f01)), Keller [19] and Osserman [29] first supplied a necessary and sufficient condition

$$\int_0^\infty ds \sqrt{2F(s)} < \infty, \quad \forall a > 0, F(s) = \int_0^s f(\nu) d\nu,$$

(1.2)

for the existence of solutions to problem (1.1).

Loewner and Nirenberg [22] showed that if $f(u) = u^{p_0}$ with $p_0 = (N + 2)/(N - 2)$, $N > 2$, then problem (1.1) has a unique positive solution $u$ which satisfies

$$\lim_{d(x) \to 0} u(x)(d(x))^{(N-2)/2} = \left(N(N - 2)/4\right)^{(N-2)/4}.$$

Bandle and Marcus [2] established the following results: if $f$ satisfies (f01) and the condition that

(f4) there exist $\theta > 0$ and $S_0 \geq 1$ such that $f(\xi s) \leq \xi^{1+\theta} f(s)$ for all $\xi \in (0, 1)$ and $s \geq S_0/\xi$, then for any solution $u$ of problem (1.1)

$$\frac{u(x)}{\phi(d(x))} \to 1 \quad \text{as} \quad d(x) \to 0,$$

(1.3)

where $\phi$ satisfies

$$\int_0^\infty \frac{ds}{\sqrt{2F(s)}} = t, \quad \forall t > 0.$$

(1.4)

If $f$ further satisfies

(f5) $f(s)/s$ is increasing on $(0, \infty),$

then problem (1.1) has a unique solution.

Lazer and McKenna [21] showed that if $f$ satisfies (f1) (or (f01)) and

(f6) there exists $S_0 > 0$ such that $f'$ is non-decreasing on $[S_0, \infty)$, and

$$\lim_{s \to \infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty,$$

then for any solution $u$ of problem (1.1)

$$u(x) - \phi(d(x)) \to 0 \quad \text{as} \quad d(x) \to 0.$$

(1.5)

When $f$ satisfies (f01), (f3) and the condition that
there exist \( p > 1, S_0 > 0 \) such that \( f(s)/s^p \) is increasing on \([S_0, \infty)\) and \( b \in C^\alpha(\Omega) \) which is positive in \( \Omega \) and satisfies

\[
(b_0) \text{ there exist } b_0 > 0 \text{ and } \sigma \in (0, 2) \text{ such that }
\lim_{d(x) \to 0} b(x)(d(x))^{\sigma} = b_0.
\]

García Meliáán [16] showed (by using nonlinear transformations, a perturbation method and a comparison principle) that

(i) if \( C_f > 1 \), then for any solution \( u \) of problem (1.1)

\[
\lim_{d(x) \to 0} \frac{u(x)}{\psi(A(d(x))^{2-\sigma})} = 1,
\]

where

\[
A = \frac{b_0}{(2-\sigma)((2-\sigma)(C_f-1)+1)}
\]

and \( \psi \) satisfies

\[
\int_0^\infty \frac{ds}{f(s)} = t, \quad \forall t > 0; \tag{1.7}
\]

(ii) if \( C_f = 1 \) and \( h(t) := tf'(\psi(t)) \geq 1 \) for sufficiently small \( t > 0 \), then (i) still holds.

When \( f(u) = c_0 u^p \), Du [13] and López-Gómez [24] showed the first expansion and uniqueness of solutions to problem (1.1). Moreover, López-Gómez [25], and Cano-Casanova and López-Gómez [7] established the following optimal uniqueness result without the first expansion of solutions and the first expansion of solutions for more general weight \( b \).

(R1) Suppose \( \Omega \) is a ball or an annulus, \( b(x) = \beta(d(x)) \), \( \beta \in C[0, \infty) \) which is increasing in \((0, \infty)\) and \( f(u) := g(u)u \) satisfies the Keller-Osserman condition, where \( g \in C[0, \infty) \cap C^1(0, \infty) \) satisfies

\[
g(0) = 0, \quad g'(u) > 0, \quad \forall u > 0, \quad \lim_{u \to \infty} g(u) = \infty,
\]

and there exists \( \alpha_0 = \alpha_0(g) > 0 \) such that

\[
\xi^2 g(\xi^{-\alpha_0} u) \leq g(u), \quad \forall \xi > 1 \text{ and } u > 0.
\]

Then, problem (1.1) has a unique solution \( u \), which is radially symmetric;

(R2) if, in addition, \( f \) satisfies for some \( p > 1 \),

\[
c_0 := \lim_{t \to \infty} \frac{f(t)}{t^p} \in (0, \infty),
\]

and \( b \) satisfies

\[
b_0 := \lim_{t \to 0^+} \frac{H(t)H''(t)}{[H'(t)]^2} \in (0, \infty)
\]

In this case, problem (1.1) has a unique solution \( u \), which is radially symmetric;
is well defined for some \( R > 0 \), where \( H(t) \) stands for the function

\[
H(t) := \int_{t}^{R} \frac{ds}{A(s)}, \quad A(t) := \left( \int_{0}^{t} (b(\tau))^{1/(p+1)} \, d\tau \right)^{(p+1)/(p-1)}, \quad t \in (0, R].
\]

Then

\[
\lim_{d(x) \to 0} \frac{u(x)}{H(d(x))} = b_0^{-p/(p-1)} \left( \frac{p + 1}{p - 1} \right)^{(p+1)/(p-1)} c_0^{-1/(p-1)}.
\]

Bandle and Marcus [3] first studied the influence of the geometry of \( \Omega \) in the boundary behavior for the unique radially symmetric solution \( u \) of problem (1.1) in a ball and an annulus and showed that for \( f(u) = u^p, \ p > 1 \),

\[
u(x) = \left( \frac{2(p + 1)}{(p - 1)^2} \right)^{1/(p-1)} (d(x))^{-\frac{2}{p-1}} \left[ 1 + \frac{N - 1}{p + 3} H(\bar{x})d(x) + o(d(x)) \right], \quad (1.8)
\]

for all \( x \) in a neighborhood of \( \partial \Omega \), where \( \bar{x} \) is the nearest point to \( x \) on \( \partial \Omega \) and \( H(\bar{x}) \) denotes the mean curvature of \( \partial \Omega \) at \( \bar{x} \).

Their result was extended by García Meliáán, Letelier Albornoz and Sabina de Lis [15], del Pino and Letelier [10], Bandle [4], Bandle and Marcus [5], Anedda and Porru [1] for general boundary smooth domains, some more general nonlinearities and some weights.

Now we introduce a class of functions.

Let \( \Lambda \) denote the set of all positive non-decreasing functions in \( C^1(0, \delta_0) \ (\delta_0 > 0) \) which satisfy

\[
\lim_{t \to 0^+} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) := C_k \in [0, \infty), \quad K(t) = \int_{0}^{t} k(s) \, ds. \quad (1.9)
\]

Some basic examples of the functions in \( \Lambda \) are

(i) \( k(t) = t^{\sigma/2}, \ \sigma > 0, \ C_k = 2/(2 + \sigma); \)
(ii) \( k(t) = 1/(\ln t)^{\sigma}, \ \sigma > 0, \ C_k = 1; \)
(iii) \( k(t) = t^{\sigma/\ln(1 + t^{-1})}, \ \sigma > 0, \ C_k = 1/(1 + \sigma); \)
(iv) \( k(t) = e^{-t^{\sigma}}, \ \sigma > 0, \ C_k = 0; \)
(v) \( k(t) = e^{-\sigma \ln t}, \ \sigma > 0, \ C_k = 0; \)
(vi) \( k(t) = e^{-(\ln t)^\sigma}, \ \sigma > 1, \ C_k = 0. \)

The set \( \Lambda \) was first introduced by Cîrstea and Rădulescu [8] for studying the boundary behavior and uniqueness of solutions of problem (1.1) with the weight \( b \) satisfying \((b_1)\) and the assumption that

\((b_2)\) there exist some \( k \in \Lambda \) and \( b_0 > 0 \) such that

\[
\lim_{d(x) \to 0} \frac{b(x)}{k^2(d(x))} = b_0.
\]

In this paper, we relate the constants \( C_f \) and \( C_k \) in order to get the asymptotic expansion of solutions to problems (1.1). We also give the existence and uniqueness of solutions. Further, when \( b(x) = k^2(d(x)) \) near the boundary and \( f(s) = s^p \pm f_1(s) \) for sufficiently large \( s \), where \( p > 1 \) and \( f_1 \) satisfies
there exists $p_1 \in (0, p)$ such that

$$\lim_{s \to \infty} \frac{s f'_1(s)}{f_1(s)} = p_1.$$  

we show the influence of the geometry of $\Omega$ in the boundary behavior for solutions to problem (1.1).

Our main results are summarized in the following theorems.

**Theorem 1.1.** Let $f$ satisfy $(f_1)$ (or $(f_{01})$), $(f_2)$, $(f_3)$ and $b$ satisfy $(b_1)$ and $(b_2)$. If

$$C_k + 2C_f > 2,$$  

then for any solution $u$ of problem (1.1)

$$\lim_{d(x) \to 0} \frac{u(x)}{\psi(\tau_0 K^2(d(x)))} = 1,$$  

where $\psi$ is uniquely determined by (1.7) and

$$\tau_0 = \frac{b_0}{2(C_k + 2C_f - 2)}.$$  

**Remark 1.1.** By view of (1.10), one can see that if $C_f > 1$, then $C_k$ can be equal to zero and if $C_k > 0$, then $C_f$ can be equal to 1.

**Theorem 1.2.** Let $f$ satisfy $(f_{01})$, $f(s) = s^p \pm f_1(s)$ for $s$ sufficiently large, $p > 1$, $f_1$ satisfy $(f_8)$ and let $b(x) = k^2(d(x))$ near the boundary where $k$ satisfy

(1) $k \in C[0, a] \cap C^2(0, a)$ for some $a > 0$, $k(t) > 0$, $k'(t) > 0$, $\forall t \in (0, a)$ and $k(0) = 0$;

(2) $k \in \Lambda$ with $C_k > 0$;

(3) $\lim_{t \to 0^+} \frac{d^2}{dt^2} \left( \frac{k(t)}{k'(t)} \right) = 0$.

The following two results hold.

(i) When $p + 1 > 2p_1$, then, on a sufficiently small neighborhood of $\partial \Omega$, for any solution $u$ of problem (1.1),

$$u(x) = c_1 \left( K(d(x)) \right)^{-2/(p-1)} \left[ 1 + c_2(N - 1)H(\bar{x}) \frac{K(d(x))}{k(d(x))} + o(d(x)) \right].$$  

where

$$c_1 = \left( \frac{2(2 + (p - 1)C_k)}{(p - 1)^2} \right)^{1/(p-1)}$$  

and

$$c_2 = \frac{1}{2 + (p + 1)C_k}.$$  

(ii) When $p + 1 \leq 2p_1$ and $k(t) = t^{\sigma/2}$ with $\sigma > 0$, $\frac{\sigma}{2 + \sigma} > \frac{2p_1 - p_1 - 1}{p + 1}$, then (i) still holds.
Remark 1.2. Some basic examples of $k$ which satisfies (k1)–(k3) are

(i) $k(t) = t^{\alpha/2}$, $\alpha > 0$, where $C_k = 2/(2+\alpha)$;
(ii) $k(t) = e^{\alpha t} - 1$, $\alpha > 1$, where $C_k = \frac{1}{1-\alpha}$;
(iii) $k(t) = \ln(1+t^{\alpha})$, $\alpha > 1$, where $C_k = \frac{1}{1+\alpha}$.

Remark 1.3. When $k(t) = (\ln(1+t))^{\alpha}$, $\alpha > 0$, then
$$
\lim_{t \to 0^+} \frac{d^2}{dt^2} \left( \frac{K(t)}{k(t)} \right) = \frac{\alpha}{2(1+\alpha)(2+\alpha)}.
$$
In this case, $k(t)$ does not satisfy (k3).

The outline of this paper is as follows. In Section 2, we give preliminary considerations. The proofs of Theorems 1.1–1.2 are in Section 3. Finally, in Appendices A and B, we give the existence and uniqueness of solutions to problems (1.1).

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic process (see Seneta [31], Resnick [30], Maric [28]), and has been applied to study the asymptotic behavior of solutions to differential equations and problem (1.1) (see Maric [28], Cîrstea and Rădulescu [8], Cîrstea [9], the author [37]).

In this section, we present some bases of the theory which come from Seneta [31], Preliminaries in Resnick [30], Introductions and the appendix in Maric [28].

Definition 2.1. A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity with index $\rho$, written as $f \in RV_{\rho}$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$
\lim_{s \to \infty} \frac{f(\xi s)}{f(s)} = \xi^\rho.
$$

In particular, when $\rho = 0$, $f$ is called slowly varying at infinity.

Definition 2.2. A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called rapidly varying at infinity if for each $p > 1$

$$
\lim_{s \to \infty} \frac{f(s)}{s^p} = \infty.
$$

Clearly, if $f \in RV_{\rho}$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are

1. every measurable function on $[a, \infty)$ which has a positive limit at infinity;
2. $(\ln s)^\beta$ and $(\ln(\ln s))^\beta$, $\beta \in \mathbb{R}$;
3. $e^{(\ln s)^p}$, $0 < p < 1$,

and some basic examples of rapidly varying functions at infinity are

1. $e^t$ and $e^{e^t}$;
2. $e^{(\ln s)^p}$, $e^{s^p}$ and $e^{e^p}$, $p > 0$;
3. $s^\beta e^{(\ln s)^p}$ and $(\ln s)^\beta e^{(\ln s)^p}$, $p > 1$, $\beta \in \mathbb{R}$;
4. $(\ln s)^\beta e^{s^p}$ and $s^\beta e^{s^p}$, $p > 0$, $\beta \in \mathbb{R}$.
We also see that a positive measurable function $g$ defined on $(0, a)$ for some $a > 0$, is **regularly varying at zero** with index $\sigma$ (written as $g \in RVZ_\sigma$) if $t \to g(1/t)$ belongs to $RV_{-\sigma}$. Similarly, $g$ is called **rapidly varying at zero** if $t \to g(1/t)$ is rapidly varying at infinity.

**Proposition 2.1** (Uniform convergence theorem). If $f \in RV_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form $(a_1, \infty)$ with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(0, a_1]$ provided $f$ is bounded on $(0, a_1]$ for all $a_1 > 0$.

**Proposition 2.2** (Representation theorem). A function $L$ is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \varphi(s) \exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$

(2.3)

for some $a_1 \geq a$, where the functions $\varphi$ and $y$ are measurable and for $s \to \infty$, $y(s) \to 0$ and $\varphi(s) \to c_0$, with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0 \exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$

(2.4)

is **normalized** slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geq a_1,$$

(2.5)

is **normalized** regularly varying at infinity with index $\rho$ (and written as $f \in NRV_\rho$).

Similarly, $g$ is called **normalized** regularly varying at zero with index $\sigma$, written as $g \in NRVZ_\sigma$ if $t \to g(1/t)$ belongs to $NRV_{-\sigma}$.

A function $f \in RV_\rho$ belongs to $NRV_\rho$ if and only if

$$f \in C^1[a_1, \infty) \quad \text{for some } a_1 > 0 \quad \text{and} \quad \lim_{s \to \infty} \frac{sf'(s)}{f(s)} = \rho,$$

(2.6)

**Proposition 2.3.** If functions $L, L_1$ are slowly varying at infinity, then

(i) $L^\sigma$ for every $\sigma \in \mathbb{R}$, $c_1 L + c_2 L_1 \quad (c_1 \geq 0, c_2 \geq 0 \text{ with } c_1 + c_2 > 0)$, $L \circ L_1 \quad (\text{if } L_1(t) \to +\infty \text{ as } t \to +\infty)$, are also slowly varying at infinity;

(ii) for every $\theta > 0$ and $t \to +\infty$, $t^\theta L(t) \to +\infty$ and $t^{-\theta} L(t) \to 0$;

(iii) for $\rho \in \mathbb{R}$ and $t \to +\infty$, $\frac{\ln(L(t))}{\ln(t)} \to 0$ and $\frac{\ln^{n+1}(L(t))}{\ln(t)} \to \rho$.

**Proposition 2.4** (Asymptotic behavior). If a function $L$ is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$,

(i) $\int_t^\infty s^\beta L(s) ds \equiv (\beta + 1)^{-1} t^{1+\beta} L(t)$, for $\beta > -1$;

(ii) $\int_t^\infty s^\beta L(s) ds \equiv (-\beta - 1)^{-1} t^{1+\beta} L(t)$, for $\beta < -1$.

**Proposition 2.5** (Asymptotic behavior). If a function $H$ is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,

(i) $\int_0^t s^\beta H(s) ds \equiv (\beta + 1)^{-1} t^{1+\beta} H(t)$, for $\beta > -1$;

(ii) $\int_0^a s^\beta H(s) ds \equiv (-\beta - 1)^{-1} t^{1+\beta} H(t)$, for $\beta < -1$. 
Our results in this section are summarized in the following.

**Lemma 2.1.** If \( f \) satisfies (f_1) (or (f_{01})), (f_2) and (f_3), then

(i) \( C_f \in [1, \infty) \);

(ii) there exists \( S_0 > 0 \) such that \( f(s)/s^q \) is increasing in \([S_0, \infty)\), where \( q \in (1, \frac{C_f}{C_f-1}) \) for \( C_f > 1 \) and \( q \in (1, \infty) \) for \( C_f = 1 \);

(iii) \( f \) satisfies the Keller–Osserman condition (1.2);

(iv) (f_3) holds for \( C_f > 1 \) if and only if \( f \in NRV_{C_f/(C_f-1)} \);

(v) \( C_f = 1 \), \( f \) is rapidly varying at infinity.

**Proof.**

(i) Let

\[
J(s) = f'(s) \int_s^\infty \frac{dv}{f(v)}, \quad \forall s > 0.
\]

Integrating \( J(s) \) from \( a \) \((a > 0)\) to \( t \) and integrate by parts, we obtain

\[
\int_a^t J(s) \, ds = f(t) \int_t^\infty \frac{dv}{f(v)} - f(a) \int_a^t \frac{dv}{f(v)} + t - a, \quad \forall t > a.
\]

It follows from the l'Hospital's rule that

\[
0 \leq \lim_{t \to \infty} \frac{f(t) \int_t^\infty \frac{dv}{f(v)}}{t} = \lim_{t \to \infty} \frac{1}{t} \int_a^t J(s) \, ds - 1 = \lim_{t \to \infty} J(t) - 1 = C_f - 1,
\]

i.e., \( C_f \geq 1 \).

(ii) By the choice of \( q \) and (i), one can see that

\[
\lim_{s \to \infty} \left( f'(s) - q \frac{f(s)}{s} \right) \int_s^\infty \frac{dv}{f(v)} = C_f - q(C_f - 1) > 0.
\]

Then there exists \( S_0 > 0 \) such that \( (\frac{f(s)}{s^q})' = s^{-q}(f'(s) - q \frac{f(s)}{s}) > 0 \), \( \forall s \geq S_0 \), i.e., \( f(s)/s^q \) is increasing on \([S_0, \infty)\).

(iii) It follows by (ii) that there exists \( C_q \in (0, \infty] \) such that

\[
\lim_{s \to \infty} \frac{f(s)}{s^q} = C_q.
\]

Consequently, there exist \( S_1 > 0 \) and \( c_q \in (0, C_q) \) such that

\[
\frac{f(s)}{s^q} > c_q, \quad \forall s \geq S_1.
\]

Then, there exists \( S_2 > S_1 \) such that

\[
F(s) \geq c_q s^{q+1}/2, \quad \forall s \geq S_2,
\]

i.e., (iii) holds.
(iv) Necessity. By (i), we see that
\[
\lim_{s \to +\infty} \frac{f(s)}{sf'(s)} = \lim_{s \to +\infty} \frac{f(s) \int_s^\infty \frac{dv}{f(v)}}{sf'(s) \int_s^\infty \frac{dv}{f(v)}} = \frac{1}{C_f} \lim_{s \to +\infty} \frac{f(s) \int_s^\infty \frac{dv}{f(v)}}{s} = \frac{C_f - 1}{C_f},
\]
i.e., \(f \in NRV_{C_f/(C_f-1)}\) for \(C_f > 1\).

Sufficiency. When \(f \in NRV_p\) with \(p > 1\), i.e., \(\lim_{s \to +\infty} \frac{sf'(s)}{f(s)} = p\) and \(f(s) = c_0 s^p \hat{L}(s)\) for sufficiently large \(s\), where \(\hat{L}\) is normalized slowly varying at infinity and \(c_0 > 0\). It follows from Propositions 2.3(i) and 2.4(ii) that
\[
\lim_{s \to +\infty} \frac{f(s) \int_s^\infty \frac{dv}{f(v)}}{s} = \lim_{s \to +\infty} \frac{sf'(s) \int_s^\infty \frac{dv}{f(v)}}{f(s)} = \frac{p}{p-1} = C_f.
\]
(v) When \(C_f = 1\), we see by the proof of (iv) that
\[
\lim_{s \to +\infty} \frac{f(s)}{sf'(s)} = 0.
\]
Consequently, for arbitrary \(p > 1\), there exists \(S_0 > 0\) such that
\[
\frac{f'(s)}{f(s)} > (p + 1)s^{-1}, \quad \forall s \geq S_0.
\]
Integrating the above inequality from \(S_0\) to \(s\), we obtain
\[
\ln(f(s)) - \ln(f(S_0)) > (p + 1)(\ln s - \ln S_0), \quad \forall s > S_0.
\]
i.e.,
\[
\frac{f(s)}{s^p} > \frac{f(S_0)s}{s_0^{p+1}}, \quad \forall s > S_0.
\]
Letting \(s \to +\infty\), we see by Definition 2.2 that \(f\) is rapidly varying at infinity. \(\Box\)

**Lemma 2.2.** Let \(f\) satisfy \((f_1)\) (or \((f_{01})\))–\((f_3)\) and let \(\psi\) be the solution to the problem
\[
\int_{\psi(t)}^\infty \frac{ds}{f(s)} = t, \quad \forall t > 0.
\]
Then

(i) \( -\psi'(t) = f(\psi(t)), \psi(t) > 0, t > 0, \psi(0) := \lim_{t \to 0^+} \psi(t) = +\infty \) and \( \psi''(t) = f(\psi(t))f'(\psi(t)), t > 0; \)

(ii) \( \psi \in NRVZ_{-C_f-1}; \)

(iii) \( -\psi' = f \circ \psi \in NRVZ_{-C_f}; \)

(iv) \( \lim_{t \to 0^+} \frac{\ln(\psi(t))}{\ln(t)} = C_f - 1 \) and \( \lim_{t \to 0^+} \frac{\ln(f(\psi(t)))}{\ln(t)} = C_f. \)

**Proof.** By the definition of \( \psi \) and a direct calculation, we show that (i) holds.

(ii) It follows from the proof of Lemma 2.1 that

\[
\lim_{t \to 0^+} \frac{t\psi'(t)}{\psi(t)} = - \lim_{t \to 0^+} \frac{tf(\psi(t))}{\psi(t)} = - \lim_{s \to +\infty} \frac{f(s)\int_s^\infty \frac{dv}{f(v)}}{s} = -(C_f - 1),
\]

i.e., \( \psi \in NRVZ_{-(C_f-1)} \).

(iii) (f3) implies

\[
\lim_{t \to 0^+} \frac{t\psi''(t)}{\psi(t)} = - \lim_{t \to 0^+} tf'(\psi(t)) = - \lim_{s \to +\infty} f'(s) \int_s^\infty \frac{dv}{f(v)} = -C_f,
\]

i.e., \( f \circ \psi \in NRVZ_{-C_f} \).

The last result (iv) follows from (ii)-(iii) and Proposition 2.3(iii). \( \square \)

**Lemma 2.3.** (i) \( k \in \Lambda \) implies:

(i) \( \lim_{t \to 0^+} \frac{K(t)}{K(t)} = 0; \)

(ii) \( C_k \in [0, 1] \) and \( \lim_{t \to 0^+} \frac{K(t)K'(t)}{K^2(t)} = 1 - C_k. \)

(ii) (k1)-(k3) imply

(iii) \( \lim_{t \to 0^+} \left( \frac{K(t)K'(t)}{K^2(t)} - (1 - C_k) \right) \frac{K(t)}{K(t)} = 0. \)

**Proof.** We only prove (iii). By the l'Hospital's rule and (i)-(ii), we have

\[
\lim_{t \to 0^+} \left( \frac{K(t)K'(t)}{K^2(t)} - (1 - C_k) \right) \frac{K(t)}{K(t)} = \lim_{t \to 0^+} -\frac{d}{dt} \frac{K(t)}{K(t)} + C_k = \lim_{t \to 0^+} -\frac{d^2}{dt^2} \frac{K(t)}{K(t)} = 0. \quad \square
\]

**Lemma 2.4.** Let \( k \in \Lambda, f_1 \) be as in Theorem 1.2 and

\[
\Phi(t) = \left( K(t) \right)^{-2/(p-1)} (1 + h(t))
\]

with \( \lim_{t \to 0} h(t) = 0. \) Then

(i) when \( p + 1 > 2p_1, k(t)(K(t))^{(p-1)/(p-1)} f_1(\Phi(t)) \to 0 \) as \( t \to 0; \)

(ii) when \( p + 1 \leq 2p_1 \) and \( k(t) = t^\sigma/2 \) with \( \sigma > 0, \frac{\sigma}{2+\sigma} > \frac{2p_1-p-1}{p+1}, \) (i) still holds.
Proof. Note that (f₀) means that \( f_1 \in NRV_{p_1} \) with \( p_1 \in (0, p) \) and \( f_1(s) = c_0 s^{p_1} \hat{L}(s) \) for sufficiently large \( s \), where \( \hat{L} \) is normalized slowly varying at infinity and \( c_0 > 0 \).

Let

\[
\Phi_1(t) = (K(t))^{-2/(p-1)},
\]

we see that \( \hat{L}(\Phi_1(t)) \) is also normalized slowly varying at zero and, by a similar argument as in Propositions 2.1 and 2.3(ii), for every \( \beta > 0 \) and \( t \to 0^+ \), we have

\[
(\Phi_1(t))^{\beta} \hat{L}(\Phi_1(t)) \to 0 \quad \text{and} \quad \hat{L}(\Phi(t))(\hat{L}(\Phi_1(t)))^{-1} (1 + h(t))^{p_1} \to 0. \tag{2.8}
\]

(i) When \( p + 1 > 2p_1 \), let \( 2\beta \in (0, p + 1 - 2p_1) \), we have

\[
k(t)(K(t))^{(p+1)/(p-1)} f_1(\Phi(t))
= c_0 k(t)(K(t))^{(p+1-2p_1-2\beta)/(p-1)} (\Phi_1(t))^{\beta} \hat{L}(\Phi_1(t))
\times \hat{L}(\Phi(t))(\hat{L}(\Phi_1(t)))^{-1} (1 + h(t))^{p_1} \to 0 \quad \text{as } t \to 0,
\]

by view of (2.8).

(ii) When \( p + 1 \leq 2p_1 \) and \( k(t) = t^{\sigma/2} \) with \( \sigma > 0, \frac{\sigma}{2+\sigma} > \frac{2p_1-p-1}{p+1} \), let

\[
\frac{2\beta}{p-1} < \left( 0, \frac{\sigma}{2+\sigma} - \frac{2p_1-p-1}{p+1} \right),
\]

then,

\[
k(t)(K(t))^{(p+1)/(p-1)} f_1(\Phi(t))
= c t^\theta (\Phi_1(t))^{\beta} \hat{L}(\Phi_1(t)) \hat{L}(\Phi(t))(\hat{L}(\Phi_1(t)))^{-1} (1 + h(t))^{p_1} \to 0 \quad \text{as } t \to 0.
\]

where

\[
c = c_0 \left( \frac{2+\sigma}{2} \right)^{(2p_1+2\beta-p-1)/(p-1)},
\]

\[
\theta = \frac{\sigma(p-1) - (2+\sigma)(2p_1-p-1+2\beta)}{2(p-1)} > 0. \quad \square
\]

3. Boundary behavior

In this section, we prove Theorems 1.1–1.2.

First, in the same proof of Lemma 2.4 in [12], we have the following result.

Lemma 3.1 (The comparison principle). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain, \( f \) be an increasing function and let \( b \) satisfy (b₁). Assume that \( u_1, u_2 \in C^2(\Omega) \) satisfy \( \Delta u_1 \geq b(x) f(u_1) \) and \( \Delta u_2 \leq b(x) f(u_2) \) in \( \Omega \).

If \( \liminf_{x \to \partial \Omega} (u_2 - u_1)(x) \geq 0 \), then \( u_2 \geq u_1 \) in \( \Omega \).

Now let \( v_0 \in C^{2+\sigma}(\Omega) \cap C^1(\hat{\Omega}) \) be the unique solution of the problem

\[-\Delta v_0 = 1, \quad v_0 > 0, \quad x \in \Omega, \quad v_0|_{\partial \Omega} = 0. \tag{3.1}\]
By the Höpf maximum principle in [18], we see that
\[
\nabla v_0(x) \neq 0, \quad \forall x \in \partial \Omega \quad \text{and} \quad c_1 d(x) \leq v_0(x) \leq c_2 d(x), \quad \forall x \in \Omega, \quad (3.2)
\]
where \(c_1, c_2\) are positive constants.

For any \(\delta > 0\), we define
\[
\Omega_\delta = \{ x \in \Omega : 0 < d(x) < \delta \}.
\]
Since \(\Omega\) is smooth, there exists \(\delta_0 > 0\) such that \(d \in C^2(\Omega_{\delta_0})\) and
\[
|\nabla d(x)| = 1 \quad \text{and} \quad \Delta d(x) = -(N - 1)H(\bar{x}) + o(1), \quad \forall x \in \Omega_{\delta_0}. \quad (3.3)
\]

**Proof of Theorem 1.1.** Let \(\varepsilon \in (0, b_0/4)\) and
\[
\tau_1 = \tau_0 - 2\varepsilon \tau_0/b_0, \quad \tau_2 = \tau_0 + 2\varepsilon \tau_0/b_0.
\]

It follows that
\[
\tau_0/2 < \tau_1 < \tau_0 < \tau_2 < 2\tau_0.
\]

By (b1), (b2) and Lemmas 2.1–2.3, we see that there is \(\delta_\varepsilon \in (0, \delta_0/2)\) (which is corresponding to \(\varepsilon\)) sufficiently small such that

1. \(b_0 - \varepsilon)k^2(d(x) - \rho) \leq (b_0 - \varepsilon)k^2(d(x)) < b(x), \quad x \in D^-_\rho = \Omega_{2\delta_\varepsilon}/\Omega_{\delta_\varepsilon}^-; \quad \text{b}(x) < (b_0 + \varepsilon)k^2(d(x)) \leq (b_0 + \varepsilon)k^2(d(x) + \rho), \quad x \in D^+_\rho = \Omega_{2\delta_\varepsilon} - \rho, \quad \text{where} \quad \rho \in (0, \delta_\varepsilon).

2. For \(i = 1, 2,\)
\[
8\tau_0 |\tau_i K^2(t) f'(|\psi(\tau_i K^2(t))| - C_f) + 4\tau_0 \left| \frac{k'(t) K(t)}{k^2(t)} - (1 - C_k) \right| + 4\tau_0 \frac{K(t)}{k(t)} |\Delta d(x)| < \varepsilon, \quad \forall (x, t) \in \Omega_{2\delta_\varepsilon} \times (0, 2\delta_\varepsilon).
\]

Let
\[
d_1(x) = d(x) - \rho, \quad d_2(x) = d(x) + \rho, \quad (3.4)
\]
\[
\bar{u}_\varepsilon = \psi(\tau_1 K^2(d_1(x))), \quad x \in D^-_\rho \quad \text{and} \quad \bar{u}_\varepsilon = \psi(\tau_2 K^2(d_2(x))), \quad x \in D^+_\rho. \quad (3.5)
\]

It follows that, for \(x \in D^-_\rho\)
\[
\Delta \bar{u}_\varepsilon (x) - b(x) f(\bar{u}_\varepsilon(x))
\]
\[
= \psi''(\tau_1 K^2(d_1(x))) (2\tau_1 K(d_1(x)) k(d_1(x)))^2 + 2\tau_1 \psi'(|\psi(\tau_1 K^2(d_1(x)))|) \Delta d(x)
\]
\[
+ K(d_1(x)) k'(d_1(x)) + K(d_1(x)) k(d_1(x)) \Delta d(x) - b(x) f(\psi(|\psi(\tau_1 K^2(d_1(x)))|))
\]
\[
= f(\psi(|\psi(\tau_1 K^2(d_1(x)))|)) k^2(d_1(x)) \left[ 4\tau_1 \tau_1 K^2(d_1(x)) f'(|\psi(\tau_1 K^2(d_1(x)))|) - C_f \right]
\]
Thus by letting \( \Gamma_{\rho} \), we see that

\[
\Gamma_{\rho} \leq \left\{ x \in \Omega : d(x) = 2 \delta \right\}.
\]

By \((f_1)\) or \((f_01)\), we see that \( u_{\varepsilon} \) is a subsolution of Eq. (1.1) in \( D_{\rho}^\varepsilon \). Since \( u < \bar{u}_\varepsilon \) on \( \Gamma_\rho := \left\{ x \in \Omega : d(x) = \rho \right\} \), (3.6) follows by Lemma 3.1.

In a similar way, we can show (3.7).

Hence, \( x \in D_{\rho}^\varepsilon \cap D_{\rho}^\varepsilon \), by letting \( \rho \to 0 \), we have

\[
1 - \frac{Mv_0(x)}{\psi(\tau_2 K^2(d(x)))} \leq \frac{u(x)}{\psi(\tau_2 K^2(d(x)))} \quad \text{and} \quad \frac{u(x)}{\psi(\tau_1 K^2(d(x)))} \leq 1 + \frac{Mv_0(x)}{\psi(\tau_1 K^2(d(x)))}.
\]

Consequently,

\[
1 \leq \lim_{d(x) \to 0} \inf \frac{u(x)}{\psi(\tau_2 K^2(d(x)))} \quad \text{and} \quad \lim_{d(x) \to 0} \sup \frac{u(x)}{\psi(\tau_1 K^2(d(x)))} \leq 1.
\]

Thus by letting \( \varepsilon \to 0 \), we have

\[
\lim_{d(x) \to 0} \frac{u(x)}{\psi(\tau_0 K^2(d(x)))} = 1.
\]

The proof is finished. \( \square \)

**Proof of Theorem 1.2.** Let \( \varepsilon \in (0, 1) \). First we note by Lemmas 2.3–2.4 that there exists \( \delta_\varepsilon \in (0, \delta/2) \) sufficiently small such that for \( (x, t) \in \Omega_{2\delta_\varepsilon} \times (0, 2\delta) \).

\[
\frac{2c_1}{p - 1} K(t) \left[ 2 \left| \frac{K(t)k(t)}{K(t)} \right| - \frac{d^2}{dt^2} \left( \frac{K(t)}{k(t)} \right) \right] + \frac{4}{p - 1} \left| \frac{K(t)}{k(t)} \right| \left[ \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) \right] + \frac{tK(t)}{k(t)} \left| \Delta d(x) \right| \leq 0.
\]

i.e., \( \bar{u}_\varepsilon \) is a supersolution of Eq. (1.1) in \( D_{\rho}^{-} \).

In a similar way, we can show that \( u_{\varepsilon} \) is a subsolution of Eq. (1.1) in \( D_{\rho}^{+} \).

Now let \( u \) be an arbitrary solution of problem (1.1). We assert that there exists a positive constant \( M \) such that

\[
u(x) \leq Mv_0(x) + u_{\varepsilon}, \quad x \in D_{\rho}^{-}, \tag{3.6}\]

\[
u(x) \leq u + Mv_0(x), \quad x \in D_{\rho}^{+}, \tag{3.7}\]

where \( v_0 \) is the solution of problem (3.1).

In fact, we may choose a large \( M \) such that

\[
u(x) \leq Mv_0(x) + \bar{u}_\varepsilon \quad \text{on} \quad \Gamma_{2\delta_\varepsilon} := \left\{ x \in \Omega : d(x) = 2 \delta \right\}.
\]

By \((f_1)\) or \((f_01)\), we see that \( u_{\varepsilon} + Mv_0 \) is also a supersolution of Eq. (1.1) in \( D_{\rho}^{-} \). Since \( u < \bar{u}_\varepsilon \) on \( \Gamma_\rho := \left\{ x \in \Omega : d(x) = \rho \right\} \), (3.6) follows by Lemma 3.1.

In a similar way, we can show (3.7).

Hence, \( x \in D_{\rho}^{-} \cap D_{\rho}^{+} \), by letting \( \rho \to 0 \), we have

\[
1 - \frac{Mv_0(x)}{\psi(\tau_2 K^2(d(x)))} \leq \frac{u(x)}{\psi(\tau_2 K^2(d(x)))} \quad \text{and} \quad \frac{u(x)}{\psi(\tau_1 K^2(d(x)))} \leq 1 + \frac{Mv_0(x)}{\psi(\tau_1 K^2(d(x)))}.
\]

Consequently,

\[
1 \leq \lim_{d(x) \to 0} \inf \frac{u(x)}{\psi(\tau_2 K^2(d(x)))} \quad \text{and} \quad \lim_{d(x) \to 0} \sup \frac{u(x)}{\psi(\tau_1 K^2(d(x)))} \leq 1.
\]

Thus by letting \( \varepsilon \to 0 \), we have

\[
\lim_{d(x) \to 0} \frac{u(x)}{\psi(\tau_0 K^2(d(x)))} = 1.
\]

The proof is finished. \( \square \)
\[ + c_0 p_1 k(t) (K(t))^{(p+1-2p)/2} \hat{L}(\Phi(t)) + \left| 1 + c_2 (N-1) (|H(\bar{x})|+1) \frac{K(d(\bar{x}))}{k(d(\bar{x}))} \right|^{p_1} \]

\[ \leq \frac{c_1 (N-1)}{p-1} \epsilon, \]

where \( \Phi(t) \) is as in Lemma 2.4 with \( h(t) = \pm c_2 (N-1) (|H(\bar{x})|+1) \frac{K(t)}{k(t)}. \)

Let

\[ \hat{u}_\epsilon(x) = c_1 (K(d_1(x)))^{-2/(p-1)} \left( 1 + c_2 (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} \right), \quad x \in D_{\rho}^- \]

and

\[ u_\epsilon(x) = c_1 (K(d_2(x)))^{-2/(p-1)} \left( 1 + c_2 (N-1) (H(\bar{x}) - \epsilon) \frac{K(d_2(x))}{k(d_2(x))} \right), \quad x \in D_{\rho}^+. \]

By using Lemma 2.4 and by a direct calculation, we see that for \( x \in D_{\rho}^- \)

\[ k^2(d(\bar{x})) f(\hat{u}_\epsilon(x)) = k^2(d(x)) (\hat{u}_\epsilon^p(x) \pm c_0 \hat{u}_\epsilon^{p_1}(x) \hat{L}(\hat{u}_\epsilon(x))) \]

\[ \geq k^2(d_1(x)) \left[ c_1^p (K(d_1(x)))^{-2p/(p-1)} \right. \]

\[ \times \left( 1 + p c_2 (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} + o \left( \frac{K(d_1(x))}{k(d_1(x))} \right) \right) \]

\[ \pm c_0 c_1^{p_1} (K(d_1(x)))^{-2p_1/(p-1)} \hat{L}(\hat{u}_\epsilon(x)) \left( 1 + c_2 (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} \right)^{p_1} \]

\[ \geq k(d_1(x)) (K(d_1(x)))^{-(p+1)/(p-1)} \]

\[ \times \left[ c_1^p \frac{K(d_1(x))}{k(d_1(x))} + p c_2 c_1^p (N-1) (H(\bar{x}) + \epsilon) + o(1) \right] \]

and

\[ \Delta \hat{u}_\epsilon(x) = k^2(d_1(x)) (K(d_1(x)))^{-2p/(p-1)} \]

\[ \times \left[ \frac{2(p+1)c_1}{(p-1)^2} + \frac{2(p+1)c_1 c_2}{(p-1)^2} (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} \right. \]

\[ - \frac{2c_1 K(d_1(x))}{p-1} \frac{K(d_1(x))}{k^2(d_1(x))} \left( 1 + c_2 (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} \right) \]

\[ - \frac{4c_1 c_2 K(d_1(x))}{p-1} (N-1) (H(\bar{x}) + \epsilon) \frac{dK(t)}{dt} \left|_{t=d_1(x)} \right. \]

\[ - \frac{2c_1 K(d_1(x))}{p-1} \left( 1 + C_4 (N-1) (H(\bar{x}) + \epsilon) \frac{K(d_1(x))}{k(d_1(x))} \right) \Delta d(x) \]

\[ + c_1 c_2 (N-1) (H(\bar{x}) + \epsilon) (d_1(x)) \frac{K^2(d_1(x))}{k^2(d_1(x))} \frac{d^2}{dt^2} \left. \left( \frac{K(t)}{k(t)} \right) \right|_{t=d_1(x)} \]
\[ + c_1 c_2 (N - 1)(H(\bar{x}) + \varepsilon)(d_1(x)) \frac{K^2(d_1(x))}{k^2(d_1(x))} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} \Delta d(x) \]

\[ = k^2(d_1(x)) K\left( d_1(x) \right) - 2 \frac{p}{p - 1} \]

\[ \times \left[ \frac{2(p + 1)c_1}{(p - 1)^2} + \frac{2(p + 1)c_1 c_2}{(p - 1)^2} (N - 1)(H(\bar{x}) + \varepsilon) \frac{K(d_1(x))}{k(d_1(x))} \right. \]

\[ - \frac{2c_1}{p - 1} \left( \frac{k'(d_1(x)) K(d_1(x))}{k^2(d_1(x))} - (1 - C_k) + (1 - C_k) \right) \]

\[ \times \left( 1 + c_2(N - 1)(H(\bar{x}) + \varepsilon) \frac{K(d_1(x))}{k(d_1(x))} \right) \]

\[ - \frac{4c_1 c_2}{p - 1} \left( \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} - C_k + C_k \right) (N - 1)(H(\bar{x}) + \varepsilon) \frac{K(d_1(x))}{k(d_1(x))} \]

\[ - \frac{2c_1}{p - 1} K\left( d_1(x) \right) \left( 1 + c_2(N - 1)(H(\bar{x}) + \varepsilon) \frac{K(d_1(x))}{k(d_1(x))} \right) \Delta d(x) \]

\[ + c_1 c_2 (N - 1)(H(\bar{x}) + \varepsilon)(d_1(x)) \frac{K^2(d_1(x))}{k^2(d_1(x))} \frac{d^2}{dt^2} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} \Delta d(x) \]

\[ = k(d_1(x)) K\left( d_1(x) \right) - \frac{p + 1}{p - 1} \]

\[ \times \left[ \frac{k(d_1(x))}{k(d_1(x))} \left( \frac{2(p + 1)c_1}{(p - 1)^2} - \frac{2c_1(1 - C_k)}{p - 1} \right) \right. \]

\[ + c_1 c_2 (N - 1)(H(\bar{x}) + \varepsilon) \left[ \frac{2(p + 1)}{(p - 1)^2} - \frac{2(1 - C_k)}{p - 1} - \frac{4C_k}{p - 1} \right] - \frac{2c_1}{p - 1} \Delta d(x) \]

\[ - \frac{2c_1}{p - 1} \left( \frac{k'(d_1(x)) K(d_1(x))}{k^2(d_1(x))} - (1 - C_k) \right) \frac{k(d_1(x))}{K(d_1(x))} \]

\[ - \frac{2c_1 c_2}{p - 1} \left( \frac{k'(d_1(x)) K(d_1(x))}{k^2(d_1(x))} - (1 - C_k) \right) (N - 1)(H(\bar{x}) + \varepsilon) \]

\[ - \frac{4c_1 c_2}{p - 1} \left( \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} - C_k \right) (N - 1)(H(\bar{x}) + \varepsilon) \]

\[ - \frac{2c_1 c_2}{p - 1} \frac{K(d_1(x))}{k(d_1(x))} (N - 1)(H(\bar{x}) + \varepsilon) \Delta d(x) \]

\[ + c_1 c_2 (N - 1)(H(\bar{x}) + \varepsilon)(d_1(x)) \frac{K(d_1(x))}{k(d_1(x))} \frac{d^2}{dt^2} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} \Delta d(x) \]

\[ + c_1 c_2 (N - 1)(H(\bar{x}) + \varepsilon)(d_1(x)) \frac{K(d_1(x))}{k(d_1(x))} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) \bigg|_{t = d_1(x)} \Delta d(x) \].

By (3.3) and

\[ \zeta_1^p = \frac{2(p + 1)c_1}{(p - 1)^2} - \frac{2(1 - C_k)c_1}{p - 1}; \]
\[ pc_1^pc_2 = \frac{2(p + 1)c_1c_2}{(p - 1)^2} - \frac{2(1 - C_k)c_1c_2}{p - 1} - \frac{4Ckc_1c_2}{p - 1} + \frac{2c_1}{p - 1}, \]

consequently, for \( x \in D_{\rho}^- \)
\[
\Delta \bar{u}_\varepsilon(x) - k^2(d(x)) f(\bar{u}_\varepsilon(x)) \leq k(d_1(x)) \left( K(d_1(x)) \right)^{(p+1)/(p-1)} \left( -\frac{2c_1(N - 1)}{p - 1} + \frac{c_1(N - 1)}{p - 1} \varepsilon \right) \leq 0,
\]
i.e., \( \bar{u}_\varepsilon(x) \) is a supersolution of Eq. (1.1) in \( D_{\rho}^- \).

In a similar way, we can show that \( u_\varepsilon \) is a subsolution of Eq. (1.1) in \( D_{\rho}^+ \).

By (3.6) and (3.7), letting \( \rho \to 0 \), we have that for \( x \in D_{\rho}^- \cap D_{\rho}^+ \),
\[
\begin{align*}
&c_1 \left( K(d(x)) \right)^{-2/(p-1)} \left( 1 + c_2(N - 1)(H(\bar{x}) + \varepsilon) \frac{K(d(x))}{k(d(x))} \right) + Mv_0(x) \geq u(x), \\
&c_1 \left( K(d(x)) \right)^{-2/(p-1)} \left( 1 + c_2(N - 1)(H(\bar{x}) - \varepsilon) \frac{K(d(x))}{k(d(x))} \right) - Mv_0(x) \leq u(x).
\end{align*}
\]

Thus Theorem 1.2 follows. The proof is finished. \( \Box \)

Acknowledgment

The authors are very grateful to the anonymous referees for the inspiring suggestions and comments which surely improved the quality of our paper.

Appendix A. The existence of solutions

In the first appendix, we give the existence of solutions to problems (1.1).

**Theorem A.1.** Let \( f \) satisfy (\( f_1 \)) (or (\( f_{01} \))) and the Keller–Osserman condition (\( 1.2 \)), and let \( b \) satisfy (\( b_1 \)). Then problem (1.1) has at least one solution \( u \in C^{2+\alpha}(\Omega) \) satisfying
\[
u(x) \geq \psi(\bar{v}(x)), \quad \forall x \in \Omega. \quad \text{(A.1)}
\]

Furthermore, if \( f \) satisfies \( \int_0^1 \frac{ds}{f(s)} = \infty \), then
\[
u > 0, \quad \forall x \in \Omega, \quad \text{(A.2)}
\]
where \( \psi \) is the solution of problem (1.7) and \( \bar{v} \in C^{2+\alpha}(\bar{\Omega}) \) is the unique solution of the problem
\[
-\Delta \bar{v} = b(x), \quad \bar{v}(x) > 0, \ x \in \Omega, \ \bar{v}|_{\partial \Omega} = 0. \quad \text{(A.3)}
\]

**Remark A.1.** By Lemma 2.1(iii), one can see that \( f \) satisfies the Keller–Osserman condition under our hypotheses on \( f \) in Theorem 1.1.

**Remark A.2.** In [20], Lair showed that (\( f_1 \)) (or (\( f_{01} \))) and the Keller–Osserman condition imply (\( f_2 \)).
Proof of Theorem A.1. Let

\[ v = \Psi(u) := \int_{u}^{\infty} \frac{dv}{f(v)}, \quad u > 0 \text{ (or } u \in \mathbb{R}). \]  \hspace{1cm} (A.4)

We see that problem (1.1) is equivalent to the following

\[ -\Delta v + g(v)|\nabla v|^2 = b(x), \quad v > 0, \ x \in \Omega, \ v|_{\partial \Omega} = 0, \]  \hspace{1cm} (A.5)

where \( g(v) = f'(\psi(v)) \) and \( \psi \) is also the inverse function of \( \Psi \).

Now letting \( v \in C^2(\Omega) \cap C(\bar{\Omega}) \) be any solution of problem (A.5), we claim that

\[ v(x) \leq \bar{v}(x), \ \forall x \in \Omega. \]  \hspace{1cm} (A.6)

In fact, we assume on the contrary that \( \{ x \in \Omega : v(x) > \bar{v}(x) \} \neq \emptyset \). Then, for its arbitrary connected component \( D \), we have \(-\Delta (v - \bar{v})(x) \leq 0, \ x \in D \) since \( g(v) \geq 0 \). It follows by \((v - \bar{v})|_{\partial D} = 0\) and the maximum principle that \( v(x) \leq \bar{v}(x), \ \forall x \in D \). This is a contradiction. Thus (A.6) holds, i.e., any classical solution \( u \) of problem (1.1) satisfies (A.1). Moreover, by the definition of \( \psi \) and the condition \( \int_{0}^{1} \frac{ds}{f(s)} = \infty \), we see that \( \psi(\bar{v}(x)) > 0, \ \forall x \in \Omega \) and (A.2) holds.

Next we consider the perturbed problem

\[ \Delta u = b(x)f(u), \quad x \in \Omega, \quad u|_{\partial \Omega} = m \in \mathbb{N}. \]  \hspace{1cm} (A.7)

By (b1) and (f1) (or (f01)), we see that \( \bar{u}_m = m \) is a supersolution of problem (A.7). To construct a subsolution \( u_1 \) of problem (A.7), we let \( \bar{v}_1 \in C^{2+\alpha}(\bar{\Omega}) \) be the unique solution of the problem

\[ -\Delta \bar{v}_1 = b(x), \quad \bar{v}_1(x) > 0, \ x \in \Omega, \ \bar{v}_1|_{\partial \Omega} = \int_{0}^{1} \frac{ds}{f(s)}, \]  \hspace{1cm} (A.8)

and let \( u_1 = \psi(\bar{v}_1) \). Then we see that \( u_1|_{\partial \Omega} = 1 \leq m \) and

\[ -\Delta \bar{v}_1 = \frac{\Delta u_1}{f(u_1)} - \frac{f'(u_1)}{f^2(u_1)}|\nabla u_1|^2 = b(x), \quad x \in \Omega, \] 

which yields

\[ \Delta u_1 \geq b(x)f(u_1), \quad x \in \Omega, \] 

i.e., \( u_1 \) is a subsolution of problem (A.7). Moreover, \( u_1 \leq 1 \leq m, \ x \in \Omega, \) thanks to the maximum principle. Thus problem (A.7) has one solution \( u_m \in C^{2+\alpha} \) in the order interval \([u_1, m] \) and, the maximum principle again yields that the map \( m \rightarrow u_m \) is increasing. On the other hand, the classical Keller-Osserman condition guaranteed that the problem

\[ \Delta u = b_0f(u), \quad x \in \Omega_0, \quad u|_{\partial \Omega_0} = \infty. \]  \hspace{1cm} (A.9)

has one solution \( u_{\Omega_0} \in C^2(\Omega_0) \) for each \( \Omega_0 \Subset \Omega \), where \( b_0 = \min_{x \in \bar{\Omega}_0} b(x) \). By the maximum principle we have \( u_m \leq u_{\Omega_0}(x), \ x \in \Omega_0 \) and \( u(x) := \lim_{m \rightarrow \infty} u_m(x) \) exists for \( x \in \Omega_0 \). Thus \( u \) is the desired solution of problem (1.1) by the standard bootstrap argument, the arbitrariness of \( \Omega_0 \) and (A.1). \( \square \)
Appendix B. The uniqueness of solutions

In this part, we give the uniqueness of solutions to problem (1.1). The method is similar to the proof of Theorem 2 in [17].

Theorem B.1. Under the hypotheses in Theorem 1.1, problem (1.1) admits a unique solution.

Proof. Since \( f(s) \) is increasing in \([S_0, \infty)\) for some \( q > 1 \) and \( S_0 \) large enough, thanks to Lemma 2.1(ii), we have

\[
\frac{f(s)}{s} \text{ is also increasing in } [S_0, \infty).
\]

(B.1)

Let \( u_0 \) be the minimal solution of problem (1.1), and let \( u \) be any other solution to problem (1.1). We prove \( u = u_0 \) in \( \Omega \). In fact, by the maximum principle, we have

\[
u_0 \leq u \text{ in } \Omega.
\]

(B.2)

Moreover, by the asymptotic behavior (1.11) we deduce that

\[
\lim_{d(x) \to 0} \frac{u_0(x)}{u(x)} = 1.
\]

(B.3)

For \( \varepsilon > 0 \) arbitrary, setting \( w := (1 + \varepsilon)u_0 \), we have

\[
\lim_{d(x) \to 0} \left( w(x) - u(x) \right) = \lim_{d(x) \to 0} u(x) \left( \frac{(1 + \varepsilon)u_0(x)}{u(x)} - 1 \right) = +\infty.
\]

(B.4)

Now, for small \( \varepsilon > 0 \), we define the (open) set

\[
D_\varepsilon := \{ x \in \Omega : w(x) < u(x) \}.
\]

(B.5)

We may assume that \( D_\varepsilon \) is nonempty for \( \varepsilon \) small enough, for otherwise there is nothing to prove. Indeed, notice that \( D_\varepsilon \) monotonically increases as \( \varepsilon \downarrow 0 \). Moreover, we may also assume that \( D_\varepsilon \to \Omega \) as \( \varepsilon \to 0 \), for if there exists \( x_0 \in \Omega \) and a sequence \( \varepsilon_n \to 0 \) such that \( x_0 \notin D_{\varepsilon_n} \) for all \( n \), we have \( (1 + \varepsilon_n)u_0(x_0) \geq u(x_0) \), and hence \( u_0(x_0) = u(x_0) \). The strong maximum principle then yields \( u \equiv u_0 \) in \( \Omega \). Finally, we have \( D_\varepsilon \subset \Omega \) by (B.4).

Next we choose \( \eta > 0 \) so that \( u_0 \geq S_0 \) in \( \Omega_\eta \) and define \( D_{\varepsilon,\eta} = D_\varepsilon \cap \Omega_\eta \). Notice that \( D_{\varepsilon,\eta} \) is a nonempty open set for small \( \varepsilon \). Moreover, we have by (B.1) that

\[
\Delta w = (1 + \varepsilon)b(x)f(u_0) \leq b(x)f(w), \quad x \in D_{\varepsilon,\eta}.
\]

(B.6)

It follows by \( (f_1) \) (or \( (f_{01}) \)) that

\[
\Delta (u - w) \geq b(x)(f(u) - f(w)) \geq 0, \quad x \in D_{\varepsilon,\eta}.
\]

(B.7)

Thus, there is

\[
u(x) - w(x) \leq \max(u - w), \quad x \in D_{\varepsilon,\eta},
\]

(B.8)

by view of the maximum principle.
Since \( \partial D_{\varepsilon, \eta} = (\partial D_\varepsilon \cap \partial D_\eta) \cup (D_\varepsilon \cap \partial D_\eta), \) \( D_\varepsilon \cap \partial \Omega = \emptyset \) and \( (u - w)|_{\partial D_\varepsilon} = 0, \) we see that the maximum of \( u - w \) is achieved on \( D_\varepsilon \cap \partial D_\eta = D_\varepsilon \cap \{ x : d(x) = \eta \}. \) Hence

\[
  u(x) - w(x) \leq \max_{D_\varepsilon \cap \{ x : d(x) = \eta \}} (u - w), \quad x \in D_{\varepsilon, \eta}.
\]

Letting \( \varepsilon \to 0 \) in (B.9) we obtain

\[
  u - u_0 \leq \max_{d(x) = \eta} (u - u_0) := \theta \quad \text{in } \Omega_\eta.
\]

On the other hand, by (B.2) and \((f_1) \) (or \((f_{01}) \)), we have

\[
  \Delta (u - u_0) = b(x)(f(u) - f(u_0)) \geq 0, \quad x \in \Omega_\eta := \{ x \in \Omega : d(x) > \eta \}.
\]

The maximum principle implies that \( u - u_0 \leq \theta \) in \( \Omega_\eta, \) and hence \( u - u_0 \leq \theta \) in the whole \( \Omega. \) Then the strong maximum principle gives \( u - u_0 \equiv \theta. \) We obtain that \( f(u) = f(u + \theta) \) in \( \Omega, \) which can only hold if \( \theta = 0. \) Thus \( u \equiv u_0, \) and this shows the uniqueness. \( \square \)

References


[34] Z. Xie, Uniqueness and blow-up rate of large solutions for elliptic equation \(-\Delta u = \lambda u - b(x)h(u)\), J. Differential Equations 247 (2009) 344–363.

