JOURNAL OF DIFFERENTIAL EQUATIONS **150**, 317–348 (1998)
ARTICLE NO. DE983489

Dirichlet Problem for the Diffusive Nicholson's Blowflies Equation

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The object of this paper is to consider the asymptotic behavior of solutions of the diffusive Nicholson's blowflies equation under Dirichlet boundary condition. A new approach is developed to study the global attractivity of the positive steady state for a reaction diffusion equation with time delay, when monotone or quasi–monotone conditions fail to apply. This approach should be applicable to other Dirichlet problems as well, even though our analysis is tailored to the diffusive Nicholson's blowflies equation. © 1998 Academic Press

1. INTRODUCTION

Nicholson's blowflies [19] was modeled and studied numerically by Gurney, Blythe and Nisbet [9]. This model was further analyzed in Kulenovic and Ladas [17], Karakostas, Philos and Sficas [15], So and Yu [25] and Smith [24]. Some revised/generalized version of such a model was also studied by Kuang [16]. More recently, So and Yang [30] studied the dynamics for the diffusive Nicholson's blowflies equation under Neumann boundary condition. In particular, criteria for the global attractivity of the nonnegative equilibria were obtained. In addition, the existence and stability of periodic solutions were studied by means a Hopf bifurcation analysis.

The object of this paper is to consider the asymptotic behavior of the diffusive Nicholson's blowflies equation under Dirichlet boundary condition. Apparently, there are only a handful of papers treating the long time behavior

0022-0396/98 $25.00

of solutions for a reaction diffusion equation with delay under Dirichlet boundary condition. Among those that we can find the nonlinear term containing the delay is often assumed to satisfy a monotonicity or quasi-monotonicity condition. Unfortunately, this is not the case here. Friesecke's [7] results require severe restriction on the delay due to the use of a Lyapunov function for a corresponding reaction–diffusion equation (without delay). Inoue, Miyakawa, and Yoshida's [14] approach, on the other hand, can only give local attractivity. Although convergence results could be found for a large number of semilinear parabolic Volterra integro-differential equations (c.f. Engler [3], Schiaffina and Tesei [23], Heard and Rankin [10], Yamada [29], and the references therein), these approaches cannot be applied to our equation either. One should also mention Cooke and Huang [2], who investigated the global dynamics of the generalized diffusive Hutchinson's equation with the Dirichlet boundary condition. But the idea in their paper is essentially similar to that of Yamada [29]. In this paper, we will develop a new approach to study the global attractivity of the positive steady state for a reaction diffusion equation with time delays under Dirichlet boundary condition. Roughly speaking, the idea is to divide the spatial domain according to the information given by the positive steady state and treat the subdomains separately. Our approach should be applicable to other Dirichlet problems as well, although our analysis is specialized to the diffusive Nicholson's blowflies equation.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results on the solutions of the diffusive Nicholson's blowflies equation, followed by the existence and uniqueness of the positive steady state. Condition guaranteeing the global attractivity of the zero solution is presented in Section 3. In Section 4, the local stability of the positive steady state is studied by analyzing the spectrum of an associated linear operator, a procedure used in Green and Stech [8] and Huang [13]. Finally, in Section 5, we discuss the global attractivity of the positive steady state. Here, a new approach is introduced and a sufficient condition is obtained. At the end of this section, the pointwise and $L^2(\Omega)$ attractivity results will be improved to that of $C^1(\Omega)$ by using an interpolation inequality (the Nirenberg–Gagliardo inequality) and an a priori estimate.

2. PRELIMINARIES

We consider the diffusive Nicholson's blowflies equation

$$\frac{\partial u(t, x)}{\partial t} = d\Delta\, u(t, x) - \tau u(t, x) + \beta\tau u(t-1, x)\, e^{-u(t-1, x)}, \tag{2.1}$$

where $(t, x) \in D \equiv (0, \infty) \times \Omega$, with (homogeneous) Dirichlet boundary condition

$$u = 0, \qquad \text{on} \quad \Gamma \equiv (0, \infty) \times \partial\Omega \tag{2.2}$$

and initial condition

$$u(\theta, x) = u_0(\theta, x) \geqslant 0, \qquad \text{in} \quad D_1 \equiv [-1, 0] \times \bar{\Omega}, \tag{2.3}$$

where $\Omega \subset \mathbb{R}^n (n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$. Here β, τ, d are positive constants. The steady states ϕ of (2.1)–(2.2) satisfy:

$$\begin{aligned} d\Delta\phi(x) - \tau\phi(x) + \beta\tau\phi(x)\, e^{-\phi(x)} &= 0, \qquad \text{for} \quad x \in \Omega \\ \phi(x) &= 0, \qquad \text{for} \quad x \in \partial\Omega \end{aligned} \tag{2.4}$$

Let $n < p < \infty$ and set $X = L^p(\Omega)$. Let $\mathscr{C} := C([-1, 0]; X)$ and define a linear operator A: $D(A) \to X$ by

$$Aw = -d\Delta w + \tau w,$$

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

It is well-known that $-A$ generates an analytic, compact semigroup $T(t)$ $(t \geqslant 0)$ on X. For any $\alpha > 0$, we define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At}\, dt$$

and let $A^\alpha = (A^{-\alpha})^{-1}$. Let $X_1 = D(A)$ and $X_\alpha = D(A^\alpha)$, where $\frac{1}{2} + (n/2p) < \alpha < 1$, and equip these spaces with their corresponding graph norms. Then $X_1 \subset X_\alpha \subset C^1(\bar{\Omega})$. Furthermore,

$$\|A^\alpha T(t)\| \leqslant \frac{K_1}{t^\alpha}\, e^{-\omega t} \tag{2.5}$$

for some positive constants K_1 and ω. For details, see Pazy [20, p. 243], Henry [11, p. 39] and Friedman [6, p. 160]. Moreover, let us define F: $\mathscr{C} \to X$ by

$$[F(u_0)](x) = \beta\tau u_0(-1, x)\, e^{-u_0(-1, x)}.$$

Then (2.1)–(2.3) can be written in an integral form (the variation of constants formula)

$$u(t) = T(t)\, u_0(0) + \int_0^t T(t-s)\, F(u_s)\, ds. \tag{2.6}$$

It is clear that F is Lipschitz continuous and hence the existence and uniqueness of the solution of (2.6) (called “mild solution” of (2.1)–(2.3)) follow from Travis and Webb [26, 27]. Furthermore, global continuation of solutions of (2.6) is due to the following proposition (here F can be more general and depends on t).

PROPOSITION 2.1. *Assume that there exist locally integrable functions* k_1 *and* k_2 *such that* $|F(t, u_0)| \leqslant k_1(t)\,|u_0| + k_2(t)$ *for* $u_0 \in \mathscr{C}$ *and* $t \geqslant 0$. *Then solutions of equation* (2.6) *are defined for all* $t \geqslant 0$.

Proof. See Wu [28, pp. 49–50].

One should also note that according to Fitzgibbon [4] and Martin and Smith [18], every mild solution is a classical solution of (2.1)–(2.3) for $t > 1$ since $T(t)$ $(t \geqslant 0)$ is analytic. Furthermore, one has

PROPOSITION 2.2. *Let* $u(t)$ *be a nonnegative solution of* (2.6) *with* $u_0(0, \cdot) \in L^p(\Omega)$. *Then there exists a constant K independent of t such that*

$$\|u(t, \cdot)\|_{C^{1+\mu}(\Omega)} \leqslant K \qquad \text{for all} \quad t \geqslant 1, \tag{2.7}$$

where $0 < \mu < 1$.

Proof. Multiplying (2.6) by A^α and using (2.5), one has

$$\begin{aligned} \|A^\alpha u(t, \cdot)\|_{L^p(\Omega)} &\leqslant \|A^\alpha T(t)\, u_0(0, \cdot)\|_{L^p(\Omega)} \\ &\quad + \beta\tau \int_0^t \|A^\alpha T(t-s)\, u(s-1, \cdot)\, e^{-u(s-1, \cdot)}\|_{L^p(\Omega)}\, ds \\ &\leqslant \frac{K_1 e^{-\omega t}}{t^\alpha} \|u_0(0, \cdot)\|_{L^p(\Omega)} \\ &\quad + \beta\tau \int_0^t \frac{K_1 e^{-\omega(t-s)}}{(t-s)^\alpha} \|u(s-1, \cdot)\, e^{-u(s-1, \cdot)}\|_{L^p(\Omega)}\, ds \\ &\leqslant \frac{K_1 e^{-\omega t}}{t^\alpha} \|u_0(0, \cdot)\|_{L^p(\Omega)} + K_1 \beta\tau e^{-1}\, |\Omega|^{1/p}\, \omega^{\alpha-1} \Gamma(1-\alpha). \end{aligned} \tag{2.8}$$

Now recall (cf. Amann [1]) that for $p > n$ and $0 < \mu < 1 - (n/p)$, there exists a constant K_2 independent of u and t such that

$$\|u(t, \cdot)\|_{C^{1+\mu}(\Omega)} \leqslant K_2\, \|A^\alpha u(t, \cdot)\|_{L^p(\Omega)} \tag{2.9}$$

for all $u \in X_\alpha$, where, $\frac{1}{2} + (\mu/2) + (n/2p) < \alpha < 1$. Therefore, one obtains from (2.8) a constant K_3 independent of t such that

$$\|A^\alpha u(t, \cdot)\|_{L^p(\Omega)} \leqslant K_3 \qquad \text{for all} \quad t \geqslant 1. \tag{2.10}$$

Thus, (2.7) follows by substituting (2.10) into (2.9) with $K = K_2 K_3$. This completes the proof.

The following existence and uniqueness result on the positive solution of (2.4) is an immediate consequence of a theorem in Hess [12].

PROPOSITION 2.3. *The boundary value problem* (2.4) *possesses a unique positive solution if and only if*

$$(\beta - 1)\tau > d\lambda_1, \tag{2.11}$$

where λ_1 *is the principal eigenvalue of* $-\Delta$ *with Dirichlet boundary condition.*

Proof. See Hess's [12] theorem and the remark following it.

Remark. 2.4. One easily shows, by means of the maximum principle, that $\|\phi\|_\infty \leqslant \ln \beta$ for any positive solution ϕ of (2.4). Indeed, suppose there exists $x^* \in \Omega$ such that $\ln \beta < \phi(x^*) = \max\{\phi(x) : x \in \Omega\}$. Then $\Delta\phi(x^*) \leqslant 0$ but $1 - \beta e^{-\phi(x^*)} > 0$, which is a contradiction.

3. GLOBAL ATTRACTIVITY OF THE ZERO SOLUTION

In this section, we will consider the global attractivity of the trivial solution. First we have

LEMMA 3.1. *Suppose* $y(t) \geqslant 0$ *satisfy the differential inequality*

$$\dot{y}(t) \leqslant -\alpha y(t) + \gamma y(t-1).$$

If $\alpha > \gamma \geqslant 0$, *then* $\lim_{t \to \infty} y(t) = 0$.

Proof. Let $z(t)$ be a solution of the linear equation

$$\dot{z}(t) = -\alpha z(t) + \gamma z(t-1).$$

with $z(\theta) > y(\theta)$, $\theta \in [-1, 0]$, then by comparison, it is obvious that $z(t) > y(t)$ for all $t \geqslant 0$. Since it is well known that $\lim_{t \to \infty} z(t) = 0$ if $\alpha > \gamma \geqslant 0$, this implies that

$$0 \leqslant y(t) < z(t) \to 0 \qquad \text{as} \quad t \to \infty.$$

The proof is complete.

We have the following theorem.

THEOREM 3.2. *Suppose*

$$(\beta-1)\tau < d\lambda_1. \tag{3.1}$$

Then solutions of (2.1)–(2.2) *satisfy* $\|u(t,\cdot)\|_{L^2(\Omega)} \to 0$ *as* $t \to \infty$.

Proof. We multiply (2.1) by $u(t,x)$ and integrate it over Ω. Using integration by parts, the Poincaré inequality and the Hölder inequality, one obtains

$$\frac{d}{t}\|u(t,\cdot)\|_{L^2(\Omega)} \leqslant -(d\lambda_1+\tau)\|u(t,\cdot)\|_{L^2(\Omega)} + \beta\tau\|u(t-1,\cdot)\|_{L^2(\Omega)}.$$

The conclusion then follows immediately from Lemma 3.1 and the proof is complete.

4. LOCAL STABILITY OF THE POSITIVE STEADY STATE

From now on, we assume that (2.11) holds. Hence there exists a unique positive steady state $\phi(x)$ according to Proposition 2.3. Linearizing (2.1) about this steady state, we get

$$\begin{aligned} &\frac{\partial v(t,x)}{\partial t} = d\Delta v(t,x) - \tau v(t,x) + \beta\tau e^{-\phi(x)}[1-\phi(x)]\, v(t-1,x) \qquad \text{in } D \\ &v(t,x) = 0 \qquad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

The corresponding eigenvalue problem is

$$\begin{aligned} -d\Delta\psi + (\tau+\lambda-\beta\tau e^{-\phi(x)}[1-\phi(x)]\, e^{-\lambda})\psi &= 0 \qquad \text{in } \Omega \\ \psi &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The following lemma is an analogue of the Sturm comparison theorem in one dimension.

LEMMA 4.1. *Let*

$$\begin{aligned} -d\Delta\psi + P(x)\psi &= 0 \qquad \textit{in } \Omega \\ \psi &= 0 \qquad \textit{on } \partial\Omega, \end{aligned}$$

and

$$-d\Delta\phi + Q(x)\,\phi = 0 \qquad \text{in } \Omega$$
$$\phi = 0 \qquad \text{on } \partial\Omega.$$

Suppose $\phi > 0$ in Ω and $P(x) > Q(x)$ in Ω. Then $\psi \equiv 0$.

Proof. Suppose $\Omega_+ = \psi^{-1}(0, \infty)$ is non-empty and let Ω_1 be a connected component of Ω_+. Multiplying the first differential equation by ϕ and the second by ψ, subtracting and integrating over Ω_1, we get

$$-d\int_{\partial\Omega_1}\left(\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}\right) + \int_{\Omega_1}(P - Q)\,\phi\psi = 0.$$

This contradicts the fact that $\psi = 0$ and $\partial\psi/\partial n \leqslant 0$ on $\partial\Omega_1$, and hence the proof is complete.

Let us compare

$$-d\Delta\psi + (\tau - \beta\tau e^{-\phi(x)}[1 - \phi(x)])\,\psi = 0 \qquad \text{in } \Omega$$
$$\psi = 0 \qquad \text{on } \partial\Omega,$$

with

$$-d\Delta\phi + (\tau - \beta\tau e^{-\phi(x)})\,\phi = 0 \qquad \text{in } \Omega$$
$$\phi = 0 \qquad \text{on } \partial\Omega.$$

We introduce the notation

$$\tilde{\Delta}(\lambda, \psi) := d\Delta\psi + (-\tau - \lambda + \beta\tau e^{-\phi(x)}[1 - \phi(x)]\, e^{-\lambda})\,\psi.$$

Since $\tau - \beta\tau e^{-\phi(x)}[1 - \phi(x)] > \tau - \beta\tau e^{-\phi(x)}$, by Lemma 4.1, we have, $0 \notin \sigma(\tilde{\Delta})$, where

$$\sigma(\tilde{\Delta}) := \{\lambda \in \mathbb{C}\text{: there exists } \psi \not\equiv 0 \text{ with } \psi = 0 \text{ on } \partial\Omega \text{ such that } \tilde{\Delta}(\lambda, \psi) = 0\}.$$

Let $\mathscr{L} := d\Delta - \tau + \beta\tau e^{-\phi(x)}$. Since $\mathscr{L}$ is (formally) self-adjoint, the eigenvalues of $\mathscr{L}$ are real. Since

$$-\tau + \beta\tau e^{-\phi(x)} > -\lambda - \tau + \beta\tau e^{-\phi(x)} \qquad \text{for} \quad \lambda > 0,$$

it follows from Lemma 4.1 that all the eigenvalues of $\mathscr{L}$ are non-positive. Therefore $(\mathscr{L}\psi, \psi) \leqslant 0$ for all $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$. Let ψ be a solution of (4.2). Multiplying (4.2) by $\bar{\psi}$ and integrating the result over Ω, we get

$$-(\mathscr{L}\psi, \psi) + \int_\Omega (\lambda - \beta\tau e^{-\phi(x)}[1-\phi(x)]\, e^{-\lambda} + \beta\tau e^{-\phi(x)})\, |\psi|^2 = 0. \tag{4.3}$$

THEOREM 4.2. *Suppose* $1 < \beta \leqslant e^2$. *Then all the eigenvalues of* (4.2) *have negative real parts.*

Proof. Let $\lambda = a + bi$, where a, b are real, and let ψ be a non-trivial solution of (4.2). Then (4.3) can be rewritten as

$$-(\mathscr{L}\psi, \psi) + \int_\Omega (a - \beta\tau e^{-\phi(x)}[1-\phi(x)]\, e^{-a}\cos b + \beta\tau e^{-\phi(x)})\, |\psi|^2 = 0, \tag{4.4}$$

and

$$\int_\Omega (b + \beta\tau e^{-\phi(x)}[1-\phi(x)]\, e^{-a}\sin b)\, |\psi|^2 = 0. \tag{4.5}$$

Note that $|1-\phi(x)| \leqslant 1$ for all $x \in \bar{\Omega}$, since $0 \leqslant \phi(x) \leqslant \ln\beta \leqslant 2$, according to Remark 2.4. We will now show that $a \leqslant 0$. Suppose $a > 0$. Then

$$\begin{aligned} &a - \beta\tau e^{-\phi(x)}[1-\phi(x)]\, e^{-a}\cos b + \beta\tau e^{-\phi(x)} \\ &\quad\geqslant a - \beta\tau e^{-\phi(x)}\, |[1-\phi(x)]\, e^{-a}\cos b| + \beta\tau e^{-\phi(x)} \\ &\quad\geqslant a - \beta\tau e^{-\phi(x)} + \beta\tau e^{-\phi(x)} = a > 0. \end{aligned}$$

This contradicts (4.4) since $-(\mathscr{L}\psi, \psi) \geqslant 0$. Next we will show that $a \neq 0$. Suppose $a = 0$. Then $b \neq 0$, since $0 \notin \sigma(\tilde{\Delta})$. Equality (4.5) implies that b cannot be an integer multiple of π and hence $|\cos b| < 1$. Moreover, by (4.4)

$$\begin{aligned} 0 &= -(\mathscr{L}\psi, \psi) + \int_\Omega (-\beta\tau e^{-\phi(x)}[1-\phi(x)]\cos b + \beta\tau e^{-\phi(x)})\, |\psi|^2 \\ &\geqslant \beta\tau(1 - |\cos b|) \int_\Omega e^{-\phi(x)}\, |\psi|^2 > 0, \end{aligned}$$

which is a contradiction. This completes the proof.

It follows from Theorem 4.2 that the positive steady state is locally stable without any restriction on the time delay in the case where $1 < \beta \leqslant e^2$. Thus, the time delay is *harmless* in this case. When $\beta > e^2$, however, the local stability of the positive steady state is only guaranteed for small delays. To prove that, we need the following lemma.

Lemma 4.3. *Suppose $\beta > e^2$ and $\tau \in [0, \tau_s]$, where*

$$\tau_s := \frac{\pi - \arccos(1/\ln \beta - 1)}{\beta \sqrt{e^{-1}(\ln \beta - 2)}}. \tag{4.6}$$

Let $a \geqslant 0$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$, with $\|\psi\|_{L^2(\Omega)} = 1$. Assume that $a + \beta\tau I_2(\psi) \leqslant \beta\tau e^{-a} I_1(\psi)$, where

$$I_1(\psi) = -\int_\Omega e^{-\phi(x)}(1 - \phi(x))\, |\psi|^2 \qquad \text{and} \qquad I_2(\psi) = \int_\Omega e^{-\phi(x)}\, |\psi|^2.$$

Then $F(\psi, a) < \pi$, where,

$$F(\psi, a) := \sqrt{(\beta\tau e^{-a} I_1(\psi))^2 - (a + \beta\tau I_2(\psi))^2} + \arccos \frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}.$$

Proof. Note that, under the above assumptions on ψ, $I_2(\psi) \leqslant I_1(\psi)$. By differentiating F with respect to a, it is easy to show that $F(\psi, a)$ is decreasing in a. Therefore,

$$F(\psi, a) \leqslant F(\psi, 0) = \beta\tau \sqrt{I_1^2(\psi) - I_2^2(\psi)} + \arccos \frac{I_2(\psi)}{I_1(\psi)}.$$

Now, $\phi(x) \leqslant \ln \beta$ and $I_1(\psi) > 0$, therefore

$$\frac{I_2(\psi)}{I_1(\psi)} = \frac{\int_\Omega e^{-\phi}\, |\psi|^2}{\int_\Omega e^{-\phi}(\phi - 1)\, |\psi|^2} \geqslant \frac{\int_\Omega e^{-\phi}\, |\psi|^2}{(\ln \beta - 1) \int_\Omega e^{-\phi}\, |\psi|^2} = \frac{1}{\ln \beta - 1}.$$

Also

$$\begin{aligned} I_1^2(\psi) - I_2^2(\psi) &= (I_1(\psi) + I_2(\psi))(I_1(\psi) - I_2(\psi)) \\ &\leqslant \left(\int_\Omega e^{-\phi}\phi\, |\psi|^2\right)\left(\int_\Omega e^{-\phi}(\phi - 2)\, |\psi|^2\right). \end{aligned}$$

Thus

$$F(\psi, a) \leqslant \beta\tau \sqrt{e^{-1}(\ln \beta - 2)} + \arccos \frac{1}{\ln \beta - 1} < \pi,$$

since $0 \leqslant \int_\Omega e^{-\phi}\phi\, |\psi|^2 \leqslant \int_\Omega e^{-1}\, |\psi|^2 = e^{-1}$ and $\tau \leqslant \tau_s$. The proof is complete.

We have the following theorem.

Theorem 4.4. *Suppose $\beta > e^2$. Then all the eigenvalues of* (4.2) *have negative real parts provided $\tau \in [0, \tau_s]$.*

Proof. Let $\lambda = a + bi$, where a is real and $b \geqslant 0$, be an eigenvalue of (4.2) with a corresponding eigenfunction ψ such that $\|\psi\|_{L^2(\Omega)} = 1$. Suppose $a \geqslant 0$. There are two possibilities to consider.

(i) $\cos b \geqslant 0$. By (4.4),

$$0 = -(\mathscr{L}\psi, \psi) + \int_\Omega [(a + \beta\tau e^{-\phi}\phi e^{-a}\cos b) + \beta\tau e^{-\phi}(1 - e^{-a}\cos b)]\,|\psi|^2 > 0,$$

which is a contradiction.

(ii) $\cos b < 0$ and $\sin b \leqslant 0$. By (4.5), $\int_\Omega e^{-\phi}(1-\phi)\,|\psi|^2 > 0$ so that by (4.4)

$$0 = -(\mathscr{L}\psi, \psi) + \int_\Omega (a + \beta\tau e^{-\phi})\,|\psi|^2 - \beta\tau\cos b e^{-a}\int_\Omega e^{-\phi}(1-\phi)\,|\psi|^2 > 0,$$

which is also a contradiction.

Hence $\cos b < 0$ and $\sin b > 0$. Recall the definitions of $I_1(\psi)$ and $I_2(\psi)$ from Lemma 4.3. Clearly $I_2(\psi) > 0$. By (4.5) and the fact that b is a second quadrant angle, we have $I_1(\psi) > 0$. Then by (4.4), we have

$$0 < \frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)} \leqslant 1.$$

Also, by (4.4), we have

$$\cos b \leqslant -\frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)},$$

which leads to

$$b \geqslant \pi - \arccos \frac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}.$$

Since the function $\eta \mapsto \eta/\sin\eta$ is increasing for $\pi/2 < \eta < \pi$, therefore

$$\frac{b}{\sin b} \geqslant \frac{\pi - \arccos \dfrac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}}{\sqrt{1 - \left(\dfrac{a + \beta\tau I_2(\psi)}{\beta\tau e^{-a} I_1(\psi)}\right)^2}}. \tag{4.7}$$

It follows immediately from (4.7) and (4.5) that

$$\sqrt{(\beta\tau e^{-a}I_1(\psi))^2-(a+\beta\tau I_2(\psi))^2}+\arccos\frac{a+\beta\tau I_2(\psi)}{\beta\tau e^{-a}I_1(\psi)}\geqslant\pi.$$

However, Lemma 4.3 shows that this is impossible. Hence $a<0$ and the proof is complete.

5. GLOBAL ATTRACTIVITY OF THE POSITIVE STEADY STATE

In this section, we will consider the global dynamics of the diffusive blowflies equation. For the case $1<\beta<e$, the well-known monotone method is applicable. We will develop a new approach to handle the case where $e\leqslant\beta<e^2$. Our first lemma provides appropriate bounds for solutions to (2.1)–(2.2).

LEMMA 5.1. *Let $u(t,x)$ be the solution of* (2.1)–(2.3). *Then $u(t,x)\geqslant 0$ for all $x\in\bar{\Omega}$ and $t>0$. Moreover, $u(t,x)>0$ for all $x\in\Omega$ and $t>1$ if $u_0\neq 0$. Furthermore*, $\limsup_{t\to\infty} u(t,x)\leqslant\beta e^{-1}$.

Proof. It is easy to show that $u(t,x)\geqslant 0$ for all $x\in\bar{\Omega}$ and $t>0$. Since $u_0\neq 0$, we have

$$\{t\geqslant 0\colon u(t,x)=0,\ \forall x\in\Omega\}\not\supseteq[0,1].$$

Therefore there exists $t_0\in[0,1)$ such that for any given $t>t_0$, we can find $x\in\Omega$ satisfying $u(t,x)\neq 0$. Moreover, according to the minimum principle and the strong minimum principle (c.f. [21]), we have $u(t,x)>0$ for $(t,x)\in(t_0,\infty)\times\Omega$, and $\partial u/\partial n|_{\partial\Omega}<0$ for $t>t_0$. Let $w(t,x)=u(t,x)-\beta e^{-1}$, then

$$\frac{\partial w}{\partial t}\leqslant d\Delta w-\tau w.$$

Therefore, w is a lower solution of the parabolic equation

$$\frac{\partial v}{\partial t}=d\Delta v-\tau v$$

together with Dirichlet boundary condition and an initial data which dominate those of w. By the comparison theorem, we have

$$w(t,x)\leqslant v(t,x).$$

It follows from Friedman [5, p. 158 Theorem 1] that $\lim_{t\to\infty} v(t,x)=0$ uniformly in Ω. Consequently, $\limsup_{t\to\infty} w(t,x) \leqslant \lim_{t\to\infty} v(t,x)=0$. This completes the proof.

One has the following convergence theorem whose proof will be carried out using the monotone method which was originally used by Sattinger [22] for reaction diffusion equations (without time delay). Of course, the appropriate modification is needed in order to apply this method to the case with time delay.

THEOREM 5.2. *Suppose* $1<\beta<e$. *Then the solutions of* (2.1)–(2.2) *converge to the positive solution of* (2.4).

Proof. Lemma 5.1 implies that for sufficiently large t, $0<u(t,x)\leqslant 1$ for all $x\in\Omega$. On the other hand, since the function $u\mapsto ue^{-u}$ is increasing for $0\leqslant u\leqslant 1$, monotone method can be applied. Consider the eigenvalue problem in $H_0^1(\Omega)\cap H^2(\Omega)$:

$$d\Delta\phi-\tau\phi+\beta\tau\phi=\lambda\phi$$

which has positive solution (λ^*,ϕ^*) since $(\beta-1)\tau>d\lambda_1$. Then for ε small enough such that $e^{-\varepsilon\phi^*}>1-(\lambda^*/\beta\tau)$, $\varepsilon\phi^*$ is a lower solution of (2.4). Let $\underline{u}(t,x)$ be the solution of (2.1)–(2.2) with initial data $\varepsilon\phi^*$. Claim: $\partial\underline{u}/\partial t\geqslant 0$. Consider $S=\{t\geqslant 0\colon \partial\underline{u}/\partial t\geqslant 0,\ \forall x\in\Omega\}$. Clearly, $0\in S$ since

$$\lim_{t\to 0^+}\frac{\partial\underline{u}}{\partial t}=d\Delta(\varepsilon\phi^*)-\tau(\varepsilon\phi^*)+\beta\tau(\varepsilon\phi^*)\,e^{-(\varepsilon\phi^*)}>0.$$

We will show $(0,1)\subset S$. For $t\in(0,1)$, let $w_h(t,x)=\underline{u}(t+h,x)-\underline{u}(t,x)$, where h is sufficiently small such that $t+h\in(0,1]$ and $\underline{u}(h,x)-\underline{u}(0,x)\geqslant 0$. Then we have

$$\begin{aligned}\frac{\partial w_h}{\partial t}&=d\Delta w_h-\tau w_h+\beta\tau u(t+h-1)\,e^{-u(t+h-1)}-\beta\tau u(t-1)\,e^{-u(t-1)}\\&=d\Delta w_h-\tau w_h,\end{aligned}$$

and

$$w_h(0,x)=\underline{u}(h,x)-\underline{u}(0,x)\geqslant 0.$$

The maximum principle implies that $w_h(t,x)\geqslant 0$ and hence $\partial\underline{u}/\partial t\geqslant 0$. Therefore $[0,1)\subset S$ holds. Moreover since S is a closed set, $[0,1]\subset S$ holds as well. Noting that the nonlinear term (delay term) is a monotone increasing function for $0<u\leqslant 1$, we obtain by induction $[0,n]\subset S$ for any integer $n\geqslant 0$. Hence $[0,\infty)=S$, that is $\partial\underline{u}/\partial t\geqslant 0$ for all $t\geqslant 0$. Therefore

$\underline{u}(t, x) \to \phi(x)$ as $t \to \infty$. Similarly, we can show that $\bar{\phi} = 1$ is an upper solution of (2.4). Let $\bar{u}(t, x)$ be the solution of (2.1)–(2.2) with initial data $\bar{\phi}$. Then we use the same approach as above, with a slight modification if necessary, to obtain $\partial \bar{u}/\partial t \leqslant 0$. Hence $\bar{u}(t, x) \to \phi(x)$ as $t \to \infty$. This completes the proof.

Next, we will consider the case $e \leqslant \beta < e^2$. To prove a convergence theorem in this case, we propose a new approach. We expect that this new method is applicable to other non-monotone Dirichlet boundary problems as well. The idea is as follows.

The spatial domain Ω is decomposed into two parts, i.e.,

$$\Omega = \{x \in \Omega, \phi(x) \leqslant 1\} \cup \{x \in \Omega, \phi(x) > 1\},$$

where $\phi(x)$ is the unique positive solution of (2.4). Let $u(t, x)$ be a positive solution of (2.1)–(2.2). First we will prove that $u(t, x) \to \phi(x)$ for each $x \in \{x \in \Omega, \phi(x) \leqslant 1\}$. This can be done by the monotone method together with an extension trick (see Lemma 5.6). As for $x \in \{x \in \Omega, \phi(x) > 1\}$, the convergence will be shown by discussing the properties of the functions $M(t)$ and $M_\partial(t)$ defined in Lemma 5.7 and 5.8. Based on Lemma 5.4, Lemma 5.6 is an auxiliary result to Lemma 5.7. Finally, the combination of Lemmas 5.4–5.8 gives rise to the following global attractivity result.

THEOREM 5.3. *Suppose $e \leqslant \beta < e^2$. Let $u(t, x)$ be a nontrivial and non-negative solution of* (2.1)–(2.2) *and let $\phi(x)$ be the positive solution of* (2.4). *Then*

$$\lim_{t \to \infty} \|u(t, \cdot) - \phi(\cdot)\|_{C(\bar{\Omega})} = 0 \qquad \text{and} \qquad \lim_{t \to \infty} \|u(t, \cdot) - \phi(\cdot)\|_{L^2(\Omega)} = 0.$$

The proof of Theorem 5.3 will be carried out after a number of lemmas below. The following lemma will be used in the proof of Lemma 5.6. It is also the bases of our approach using spatial decomposition.

LEMMA 5.4. *Assume $e < \beta \leqslant e^2$. Let $u(t, x)$ be a solution of* (2.1)–(2.2) *and let Ω_1 be an open subset of Ω satisfying $\overline{\Omega_1} \subset \Omega$. Suppose that there exists $T_0 \geqslant 1$, for all $t > T_0$, there exists $(\xi(t), \eta(t)) \in [t-1, t] \times \Omega_1$, such that $u(\xi(t), \eta(t)) = \min_{(\xi, x) \in [t-1, t] \times \overline{\Omega_1}} u(\xi, x)$. Then there exists $T_c \geqslant T_0$ sufficiently large such that $u(t, x) \geqslant 1$ for $(t, x) \in [T_c, \infty) \times \overline{\Omega_1}$.*

Proof. We define

$$g(\varepsilon) = \beta(\beta e^{-1} + \varepsilon) - (1 + \varepsilon) e^{\beta e^{-1} + \varepsilon}.$$

Note that $g(0)=\beta^2 e^{-1}-e^{\beta e^{-1}}>0$ for $e<\beta\leqslant e^2$. Then there exists $\varepsilon_1>0$, such that $g(\varepsilon)\geqslant 0$ for all $0\leqslant\varepsilon\leqslant\varepsilon_1$. Hence,

$$\beta(\beta e^{-1}+\varepsilon)\, e^{-(\beta e^{-1}+\varepsilon)}\geqslant 1+\varepsilon, \qquad \text{for} \quad 0\leqslant\varepsilon\leqslant\varepsilon_1.$$

Now for any $0<\varepsilon\leqslant\varepsilon_1$, by Lemma 5.1, there exists $T_1\geqslant T_0$, such that

$$u(t-1, x)\leqslant\beta e^{-1}+\varepsilon, \qquad \text{for all} \quad t\geqslant T_1 \quad \text{and} \quad x\in\bar{\Omega}. \tag{5.1}$$

To complete our proof, we divide our discussion into three parts.

Part A. Suppose that for any given $t\geqslant T_1$, $\xi(t)>t-1$. Then, we have $(\partial u/\partial t)(\xi(t), \eta(t))\leqslant 0$ and $\Delta u(\xi(t), \eta(t))\geqslant 0$. By (2.1), this implies

$$u(\xi(t), \eta(t))\geqslant\beta u(\xi(t)-1, \eta(t))\, e^{-u(\xi(t)-1, \eta(t))}. \tag{5.2}$$

We will complete Part A by discussing the following two cases.

Case 1. There exists $T_2\geqslant T_1$ such that $u(\xi(T_2)-1, \eta(T_2))\geqslant 1$. Since $u\mapsto ue^{-u}$ is decreasing for $u\geqslant 1$, we use (5.1) and (5.2) to get

$$\begin{aligned} u(\xi(T_2), \eta(T_2)) &\geqslant\beta(\beta e^{-1}+\varepsilon)\, e^{-(\beta e^{-1}+\varepsilon)} \\ &\geqslant 1+\varepsilon>1. \end{aligned}$$

Consequently, $u(\xi, x)\geqslant u(\xi(T_2), \eta(T_2))\geqslant 1$ for all $(\xi, x)\in[T_2-1, T_2]\times\overline{\Omega_1}$. Therefore, by induction, we conclude that $u(t, x)\geqslant 1$ for all $(t, x)\in[T_2, \infty)\times\overline{\Omega_1}$.

Case 2. $u(\xi(t)-1, \eta(t))<1$ for all $t\geqslant T_1$. Denote

$$m(t):=\min_{(\xi, x)\in[t-1, t]\times\overline{\Omega_1}} u(\xi, x).$$

Then by (5.2) we have

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geqslant\beta u(\xi(t)-1, \eta(t))\, e^{-u(\xi(t)-1, \eta(t))} \\ &>\beta u(\xi(t)-1, \eta(t))\, e^{-1} \\ &>u(\xi(t)-1, \eta(t))\geqslant m(t-1). \end{aligned}$$

for all $t\geqslant T_1$. Next we show that the function $m(t)$ is monotone increasing for $t\geqslant T_1$. For $s\geqslant T_1$, suppose that $t-1\leqslant s\leqslant t$. Firstly, if $t-1\leqslant\xi(s)$ and $\xi(t)\leqslant s$, then clearly $m(s)=m(t)$. Secondly, if $s-1<\xi(s)\leqslant t-1$ and $t-1<\xi(t)\leqslant s$, one concludes that $m(s)\leqslant m(t)$ since $u(\xi(s), \eta(s))$ is the minimum of $u(\xi, x)$ on $[s-1, s]\times\overline{\Omega_1}$. Thirdly, if $s<\xi(t)\leqslant t$, then $s-1<\xi(t)-1\leqslant t-1\leqslant s$ and hence

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geqslant \beta u(\xi(t)-1, \eta(t))\, e^{-u(\xi(t)-1,\, \eta(t))} \\ &\geqslant \beta u(\xi(s), \eta(s))\, e^{-u(\xi(s),\, \eta(s))} \\ &> \beta u(\xi(s), \eta(s))\, e^{-1} \\ &> u(\xi(s), \eta(s)) = m(s). \end{aligned}$$

On the other hand, suppose $T_1 \leqslant s < t-1$. Then there exists an integer $l \geqslant 1$ such that $s \in [t-l-1, t-l]$, so that we have

$$m(s) \leqslant m(t-l) \leqslant m(t).$$

Therefore we conclude that $m(t)$ is monotone increasing for $t \geqslant T_1$. Let

$$m_0 := \lim_{t \to \infty} m(t) > 0.$$

We will show $m_0 > 1$. In fact, since

$$\begin{aligned} m(t) &= u(\xi(t), \eta(t)) \\ &\geqslant \beta u(\xi(t)-1, \eta(t))\, e^{-u(\xi(t)-1,\, \eta(t))} \\ &\geqslant \beta m(t-1)\, e^{-m(t-1)}, \end{aligned}$$

we take the limit as $t \to \infty$ to obtain

$$m_0 \geqslant \beta m_0 e^{-m_0}.$$

This implies $e^{m_0} \geqslant \beta$ and hence $m_0 > 1$ since $\beta > e$. Therefore, there exists $T_2 > T_1$ such that $m(T_2) \geqslant 1$. Then one concludes that, by repeating Case 1, $u(t, x) \geqslant 1$ for all $(t, x) \in [T_2, \infty) \times \overline{\Omega_1}$. This completes the proof of Part A.

Part B. Suppose that for any given $t \geqslant T_1$, $\xi(t) = t-1$. Let $m(t)$ be defined as in Part A. Clearly, $m(t)$ is monotone increasing for $t \geqslant T_1$. Now, for $0 < |h| < 1$, we have

$$\begin{aligned} &\frac{m(t+1+h) - m(t+1)}{h} \\ &\quad = \frac{u(t+h, \eta(t+1+h)) - u(t, \eta(t+1))}{h} \\ &\quad = \frac{u(t+h, \eta(t+1+h)) - u(t+h, \eta(t+1))}{h} \\ &\qquad + \frac{u(t+h, \eta(t+1)) - u(t, \eta(t+1))}{h}. \end{aligned}$$

Noting that $u(t+h, \eta(t+1+h)) \leqslant u(t+h, \eta(t+1))$, we obtain

$$\frac{m(t+1+h)-m(t+1)}{h} \leqslant \frac{u(t+h, \eta(t+1))-u(t, \eta(t+1))}{h}, \quad \text{for} \quad 0<h<1.$$

and

$$\frac{m(t+1+h)-m(t+1)}{h} \geqslant \frac{u(t+h, \eta(t+1))-u(t, \eta(t+1))}{h}, \quad \text{for} \quad -1<h<0.$$

Therefore

$$D^{-}m(t+1) \geqslant D_{-}m(t+1) \geqslant \frac{\partial u(t, \eta(t+1))}{\partial t} \geqslant D^{+}m(t+1) \geqslant D_{+}m(t+1),$$

where $D^{-}, D_{-}, D^{+}, D_{+}$ are the Dini derivatives. Note that monotone function is differentiable almost everywhere. Therefore, we have

$$\frac{dm(t+1)}{\partial t} = \frac{\partial u(t, \eta(t+1))}{\partial t}, \qquad \text{a.e. for} \quad t \geqslant T_1. \tag{5.3}$$

Next, we will show that for any $0<\varepsilon \leqslant \varepsilon_1$, there exists a sequence $\{t_k\}$ satisfying

$$\begin{aligned} & t_k \geqslant T_1, \qquad 0<t_{k+1}-t_k<1, && \text{for all} \quad k \geqslant 1; \\ & t_k \to \infty, \qquad \text{as} \quad k \to \infty; && \text{and} \\ & 0 \leqslant \frac{dm(t_k+1)}{dt} < \varepsilon, && \text{for all} \quad k \geqslant 1. \end{aligned} \tag{5.4}$$

If this is not the case, then one can find $0<\varepsilon_0 \leqslant \varepsilon_1$, and a sequence of intervals $\{I_k := (a_k, b_k)\}_{k=1}^{\infty}$, on which $dm(t+1)/dt \geqslant \varepsilon_0$, where $a_k < b_k < a_{k+1}$, $b_k \to \infty$ as $k \to \infty$, and moreover, $|I_k| \geqslant 1$. Therefore, for any $k \geqslant 1$, we have

$$m(b_k+1)-m(a_1+1) \geqslant \int_{a_1}^{b_k} \frac{dm(t+1)}{dt} \geqslant \sum_{l=1}^{k} \int_{I_l} \frac{dm(t+1)}{dt} \geqslant k\varepsilon_0.$$

This contradicts the boundedness of the function $m(t)$. Hence the aforementioned sequence $\{t_k\}$ exists. Now, for $k \geqslant 1$, since $u(t_k, \eta(t_k+1))$ is the minimum of $u(\xi, x)$ on $[t_k, t_k+1] \times \overline{\Omega_1}$ and $\eta(t_k+1) \in \Omega_1$, we use (2.1) and (5.3) to get,

$$\frac{dm(t_k+1)}{dt} \geqslant -\tau m(t_k+1) + \beta\tau u(t_k-1, \eta(t_k+1))\, e^{-u(t_k-1, \eta(t_k+1))}. \tag{5.5}$$

Using (5.4) and (5.5) instead of (5.2), one can follow the proof of Part A to complete the proof of Part B.

Part C. This is the complement of Part A and Part B. Suppose that there exists an increasing sequence $\{t_k\}$, where $t_k \geqslant T_1$ for all $k \geqslant 1$ and $t_k \to \infty$ as $n \to \infty$, such that $\xi(t_k) > t_k - 1$ and $\eta(t_k) \in \Omega_1$. Therefore, we have

$$m(t_k) \geqslant \beta u(\xi(t_k)-1, \eta(t_k))\, e^{-u(\xi(t_k)-1, \eta(t_k))}, \qquad \text{for all} \quad k \geqslant 1. \tag{5.6}$$

Without loss of generality, we assume $t_{k+1} - t_k > 1$.

Claim. There exists $T_2 \geqslant t_1$, such that $m(T_2) \geqslant 1$.

Using the arguments similar to Case 1 of Part A, one can show that, if there exists $k_0 \geqslant 1$, satisfying $u(\xi(t_{k_0})-1, \eta(t_{k_0})) \geqslant 1$, then $m(t_{k_0}) > 1$. Therefore, the claim is true for this case.

Next, we assume that $m(t) < 1$ for all $t \geqslant t_1$, and that $u(\xi(t_k)-1, \eta(t_k)) < 1$ for all $k \geqslant 1$. We show that $\{m(t_k)\}_{k=1}^{\infty}$ is a monotone increasing sequence. For any $k \geqslant 1$, we choose an integer $l \geqslant 2$, such that $t_k \in [t_{k+1}-l, t_{k+1}-l+1]$. According to (5.6), we have

$$\begin{aligned} m(t_{k+1}) &\geqslant \beta u(\xi(t_{k+1})-1, \eta(t_{k+1}))\, e^{-u(\xi(t_{k+1})-1, \eta(t_{k+1}))} \\ &\geqslant \beta m(t_{k+1}-1)\, e^{-m(t_{k+1}-1)} \\ &\geqslant m(t_{k+1}-1) \end{aligned} \tag{5.7}$$

We now consider $[t_{k+1}-2, t_{k+1}-1] \times \overline{\Omega_1}$. Clearly, if $\xi(t_{k+1}-1) = t_{k+1}-2$, we have

$$m(t_{k+1}-1) \geqslant m(t_{k+1}-2) \tag{5.8}$$

On the other hand, suppose $\xi(t_{k+1}-1) > t_{k+1}-2$. According to (5.6), we have

$$m(t_{k+1}-1) \geqslant \beta u(\xi(t_{k+1}-1)-1, \eta(t_{k+1}-1))\, e^{-u(\xi(t_{k+1}-1)-1, \eta(t_{k+1}-1))},$$

Clearly, $u(\xi(t_{k+1}-1)-1, \eta(t_{k+1}-1)) < 1$. Otherwise, we can get $m(t_{k+1}-1) > 1$ by following the same arguments as Case 1 of Part A. This contradicts our assumption. Now, using the same discussion as Case 2 of Part A, we obtain (5.8) as well. Invoking the above arguments $(l-2)$ times, we eventually get

$$m(t_{k+1}-1) \geqslant m(t_{k+1}-l+1). \tag{5.9}$$

Subclaim. $m(t_{k+1}-l+1) \geqslant m(t_k)$.

Clearly, on $[t_{k+1}-l, t_{k+1}-l+1] \times \overline{\Omega_1}$, this is true if $\xi(t_{k+1}-l+1) \in [t_{k+1}-l, t_k]$. Now we consider the case where $\xi(t_{k+1}-l+1) > t_k$. According to (5.6), we have

$$\begin{aligned} & m(t_{k+1}-l+1) \\ & \quad \geqslant \beta u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1))\, e^{-u(\xi(t_{k+1}-l+1)-1,\, \eta(t_{k+1}-l+1))}. \end{aligned} \tag{5.10}$$

Obviously, $u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) < 1$. Otherwise we use the same discussion as Case 1 of Part A to get $m(t_{k+1}-l+1) > 1$. This contradicts our assumption. Noting that $\xi(t_{k+1}-l+1) \in (t_k, t_{k+1}-l+1]$, we have

$$m(t_k) \leqslant u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1)) < 1. \tag{5.11}$$

Since $u \mapsto ue^{-u}$ is monotone increasing for $u \leqslant 1$, we use (5.10) and (5.11) to obtain

$$\begin{aligned} & m(t_{k+1}-l+1) \\ & \quad \geqslant \beta u(\xi(t_{k+1}-l+1)-1, \eta(t_{k+1}-l+1))\, e^{-u(\xi(t_{k+1}-l+1)-1,\, \eta(t_{k+1}-l+1))} \\ & \quad \geqslant \beta m(t_k)\, e^{-m(t_k)} \\ & \quad \geqslant \beta e^{-1} m(t_k) \geqslant m(t_k). \end{aligned} \tag{5.12}$$

Hence, the subclaim holds.

Now we combine (5.7), (5.9), and (5.12) to get $m(t_{k+1}) \geqslant m(t_k)$, and moreover

$$m(t_{k+1}) \geqslant \beta m(t_k)\, e^{-m(t_k)}.$$

Then we take the limit as $k \to \infty$ to obtain

$$m_0 \geqslant \beta m_0 e^{-m_0},$$

where $m_0 := \lim_{k \to \infty} m(t_k) > 0$. Hence $m_0 > 1$ since $\beta > e$. Therefore, there exists $k_0 \geqslant 1$, such that $m(t_{k_0}) \geqslant 1$. Again, this contradicts our assumption. The proof of the Claim is complete.

Finally, one can easily show $u(t, x) \geqslant 1$ for $(t, x) \in [T_2, \infty) \times \overline{\Omega_1}$. For that we just need to consider $[T_2, T_2+1] \times \overline{\Omega_1}$. Obviously, if $\xi(T_2+1) = T_2$, we have $m(T_2+1) \geqslant m(T_2) \geqslant 1$. On the other hand, if $\xi(T_2+1) > T_2$, then we use the same discussion as Case 1 of Part A to get $m(T_2+1) > 1$. This completes the proof of Part C.

We introduce the following notations

$$\Omega^1_\infty = \{x \in \Omega, \phi(x) < 1\},$$
$$\tilde{\Omega}^1_\infty = \{x \in \Omega, \phi(x) > 1\},$$
$$\Omega^1_t(u) = \{x \in \Omega, u(t, x) < 1\}.$$

If $\beta = e$, it follows from Remark 2.4 that $\tilde{\Omega}^1_\infty$ is empty. Therefore, for this case, the global attractivity of the positive steady state can be concluded by the following lemma.

LEMMA 5.5. *Let $u(t, x)$ be a solution of* (2.1)–(2.2). *Then, for $x \in \overline{\Omega^1_\infty}$, $u(t, x) \to \phi(x)$ (pointwise) as $t \to \infty$.*

Proof. Without loss of generality, we assume that $u_0(\theta, x) > 0$ for all $x \in \Omega$ and $\theta \in [-1, 0]$. We also assume that $\partial u_0/\partial n|_{\partial\Omega} < 0$ for all $\theta \in [-1, 0]$, and that $u(t, x)$ satisfies (2.7) for all $t \geqslant 0$. Let $\underline{u}(t, x)$ be the solution of (2.1)–(2.2) with the initial function $\varepsilon\phi^*(x)$, where ε is chosen small enough such that $u_0(\theta, x) \geqslant \varepsilon\phi^*(x)$ for all $x \in \bar{\Omega}$ and $\theta \in [-1, 0]$, and $\phi^*(x)$ is defined as Theorem 5.2. Then one can use the same arguments as the proof of Theorem 5.2 to show $\underline{u}(t, x) \leqslant \phi(x)$ and $\partial\underline{u}/\partial t \geqslant 0$ for all $x \in \overline{\Omega^1_\infty}$ and $t > 0$. Therefore, we have

$$\lim_{t \to \infty} \underline{u}(t, x) = \phi(x) < 1, \qquad \text{for all} \quad x \in \overline{\Omega^1_\infty}.$$

Let $\bar{u}(t, x)$ be the solution of (2.1)–(2.2) with the initial function $\zeta\phi(x)$, where $\zeta > 1$ is large enough so that $\zeta\phi(x) \geqslant u_0(\theta, x)$ for all $x \in \bar{\Omega}$ and $\theta \in [-1, 0]$. Let

$$\Omega^1_{-1} = \{x \in \Omega, \zeta\phi(x) < 1\}.$$

As before, we can show that $\partial\bar{u}/\partial t \leqslant 0$ for all $x \in \Omega^1_{-1}$ and hence

$$\lim_{t \to \infty} \bar{u}(t, x) = \phi(x), \qquad \text{for} \quad x \in \Omega^1_{-1}.$$

We denote

$$\Omega^1_0(u) = \bigcap_{t \geqslant 0} \Omega^1_t(u).$$

It is easy to see that, according to (2.7) and the (homogeneous) Dirichlet boundary condition, $\Omega_0^1(u)$ is a nonempty open set. Moreover, $\Omega_0^1(u) \cap \Omega_\infty^1$ is nonempty and $u(t, x) \geqslant \underline{u}(t, x)$ for all $x \in \Omega_0^1(u) \cap \Omega_\infty^1$. Hence

$$\liminf_{t \to \infty} u(t, x) \geqslant \lim_{t \to \infty} \underline{u}(t, x) = \phi(x), \qquad \text{for all} \quad x \in \Omega_0^1(u) \cap \Omega_\infty^1.$$

On the other hand, for $x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1$, one has $\bar{u}(t, x) \geqslant u(t, x)$. This leads to

$$\limsup_{t \to \infty} u(t, x) \leqslant \lim_{t \to \infty} \bar{u}(t, x) = \phi(x), \qquad \text{for all} \quad x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Therefore,

$$\lim_{t \to \infty} u(t, x) = \phi(x), \qquad \text{for all} \quad x \in \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Next, we extend the region of convergence to the entire $\overline{\Omega_\infty^1}$. We denote

$$S_0 := \Omega_0^1(u) \cap \Omega_\infty^1 \cap \Omega_{-1}^1.$$

Suppose $S_0 \subsetneqq \Omega_\infty^1$. Let Ω_e be an open subset of Ω_∞^1 such that

$$S_0 \subset \Omega_e \subset \Omega_\infty^1.$$

For any given $\delta > 0$, we define a subset of $\overline{S_0}$ as follows:

$$S_\delta(\partial\Omega) := \{x \in \overline{S_0} \colon \operatorname{dist}(x, \partial\Omega) < \delta\},$$

where $\operatorname{dist}(x, \partial\Omega)$ means the distance from x to the boundary of Ω. Let δ be chosen small enough such that $S_\delta(\partial\Omega) \subsetneqq \overline{S_0}$. It is clear that $S_\delta(\partial\Omega) \supset \partial\Omega$. We denote

$$\bar{K} = \max_{x \in \overline{\Omega_e}} \phi(x), \qquad \underline{K} = \min_{x \in \overline{\Omega_e} \setminus S_\delta(\partial\Omega)} \phi(x).$$

Clearly, $0 < \underline{K} \leqslant \bar{K} < 1$. Now for any given $0 < \varepsilon < 1 - \bar{K}$, one can use the compactness of $\Omega_s := \overline{S_0} \setminus S_\delta(\partial\Omega)$ and the continuity of $\phi(x)$ to find $T_1 \geqslant 0$, only depending on ε, such that

$$u(t, x) \leqslant (1 + \varepsilon)\, \phi(x), \qquad \text{for all} \quad x \in \Omega_s \quad \text{and} \quad t \geqslant T_1. \tag{5.13}$$

Indeed, since $\lim_{t \to \infty} u(t, x) = \phi(x)$ for all $x \in \overline{S_0}$, we conclude that, for the above chosen ε and any $\tilde{x} \in \Omega_s$, there exists $\tilde{t}(\tilde{x}, \varepsilon) \geqslant 0$, such that

$$|u(t, \tilde{x}) - \phi(\tilde{x})| < \frac{\underline{K}\varepsilon}{3}, \qquad \text{for all} \quad t \geqslant \tilde{t}(\tilde{x}, \varepsilon). \tag{5.14}$$

Moreover, since $\phi(x)$ is continuous and since $u(t, x)$ satisfies (2.7), we can find an open neighborhood $\mathcal{N}(\tilde{x}, \varepsilon)$ of $\tilde{x}$ such that

$$|\phi(x) - \phi(\tilde{x})| < \frac{\underline{K}\varepsilon}{3}, \tag{5.15}$$

and

$$|u(t, x) - u(t, \tilde{x})| = \left| \sum_{j=1}^{n} (x^{(j)} - \tilde{x}^{(j)}) \frac{\partial u(t, \eta^{(j)})}{\partial x^{(j)}} \right| < \frac{\underline{K}\varepsilon}{3}, \tag{5.16}$$

for all $x \in \mathcal{N}(\tilde{x}, \varepsilon)$, where $\eta^{(j)}$, the jth component of η, is an intermediate value between $x^{(j)}$ and $\tilde{x}^{(j)}$. Since $\bigcup_{\tilde{x} \in \Omega_s} \mathcal{N}(\tilde{x}, \varepsilon) \supset \Omega_s$ and that Ω_s is compact, there exist $\mathcal{N}(\tilde{x}_i, \varepsilon)$, $i = 1, 2, ..., l < \infty$, such that $\bigcup_{i=1}^{l} \mathcal{N}(\tilde{x}_i, \varepsilon) \supset \Omega_s$, and for each $\tilde{x}_i$, $i = 1, 2, ..., l$, (5.14)–(5.16) hold. Then for any $x \in \Omega_s$, there exists $1 \leqslant i_0 \leqslant l$ such that $x \in \mathcal{N}(\tilde{x}_{i_0}, \varepsilon)$. Let

$$T_1 = \max_{1 \leqslant i \leqslant l} \{\tilde{t}(\tilde{x}_i, \varepsilon)\}.$$

Then, for all $t \geqslant T_1$, we have

$$\begin{aligned} u(t, x) - \phi(x) &\leqslant |u(t, x) - u(t, \tilde{x}_{i_0})| + |u(t, \tilde{x}_{i_0}) - \phi(\tilde{x}_{i_0})| + |\phi(\tilde{x}_{i_0}) - \phi(x)| \\ &\leqslant \underline{K}\varepsilon. \end{aligned}$$

Note $\underline{K} \leqslant \phi(x)$ for $x \in \Omega_s$. This implies (5.13).

For $B_s := \partial\Omega_s \cap (\Omega^1_\infty \backslash S_0)$, we can follow the same discussion as above to choose finite number of open balls, $B(\tilde{x}_{k,\partial})$, $k = 1, 2, ..., l'$, satisfying

$$\bigcup_{\tilde{x}_{k,\partial} \in B_s} B(\tilde{x}_{k,\partial}) \supset B_s.$$

We denote

$$S_1 := \left(\bigcup_{k=1}^{l'} B(\tilde{x}_{k,\partial}) \cap \Omega_e \right) \cup S_0.$$

Clearly, $S_0 \subsetneqq S_1$. Furthermore, for all $t \geqslant T_1$ and $x \in S_1 \backslash S_0$, we have

$$u(t, x) \leqslant (1 + \varepsilon)\, \phi(x).$$

Noting that $0 < \varepsilon < 1 - \bar{K}$ and that $\phi(x) < 1$ for $x \in S_1$, we get

$$\varepsilon\phi(x) < \varepsilon < 1 - \bar{K} < 1 - \phi(x).$$

Now, we choose $v(x) = (1+\varepsilon)\,\phi(x)$. Then, for all $x \in S_1$, $v(x)$ satisfies the following properties

$$u(t, x) \leqslant v(x) \leqslant 1, \qquad \text{for all} \quad t \in [T_1, T_1 + 1],$$
$$\text{and} \quad d\Delta v(x) - \tau v(x) + \beta\tau v(x)\, e^{-v(x)} \leqslant 0.$$

By redefining $\bar{u}(t, x)$ with the initial function $v(x)$ for $t \in [T_1, T_1+1]$, we can show as before $\lim_{t\to\infty} u(t, x) = \phi(x)$ for all $x \in S_1$. We can keep repeating the above extension to obtain a sequence of open sets $\{S_k\}$, $k = 1, 2, ...,$ satisfying

$$S_k \subset S_{k+1} \subset \Omega^1_\infty, \qquad \text{for all} \quad k \geqslant 0.$$

Obviously, $\lim_{k\to\infty} S_k = \Omega^1_\infty$. Therefore, we have $\lim_{t\to\infty} u(t, x) = \phi(x)$ for all $x \in \Omega^1_\infty$. The convergence of $u(t, x)$ to $\phi(x)$ on $\overline{\Omega^1_\infty}$ follows immediately from the continuity of $u(t, x)$ and $\phi(x)$. This completes the proof.

The following lemma implies that for any given $\varepsilon > 0$, there exists $\tilde{T}$ such that $u(t, x)$, a solution of (2.1)–(2.2), is bounded below from $1-\varepsilon$ for all $t \geqslant \tilde{T}$ and $x \in \overline{\tilde{\Omega}^1_\infty}$. Before proving the lemma, we recall that

$$\frac{\partial(u-\phi)}{\partial t} = d\Delta(u-\phi) - \tau(u-\phi) + \beta\tau[u(t-1)\, e^{-u(t-1)} - \phi e^{-\phi}]. \tag{5.17}$$

Lemma 5.6. *Suppose $e < \beta \leqslant e^2$ and $u(t, x)$ is a solution of* (2.1)–(2.2). *Then*

$$\liminf_{t\to\infty} u(t, x) \geqslant 1, \tag{5.18}$$

uniformly for $x \in \overline{\tilde{\Omega}^1_\infty}$.

Proof. We denote

$$m(t) := \min_{(\xi, x) \in [t-1, t] \times \overline{\tilde{\Omega}^1_\infty}} u(\xi, x),$$

and

$$m_\partial(t) := \min_{(\xi, x) \in [t-1, t] \times \partial\tilde{\Omega}^1_\infty} u(\xi, x).$$

Clearly, if there exists $T_0 > 1$ such that

$$m(t) < m_\partial(t),$$

for all $t \geqslant T_0$, then, by Lemma 5.4, we have $u(t, x) \geqslant 1$ for all $x \in \overline{\tilde{\Omega}^1_\infty}$ and all sufficiently large t. Hence, (5.18) holds. Next, we consider the case where there exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ satisfying

$$\begin{aligned} &1 \leqslant t_k < t_{k+1}, && \text{for all} \quad k \geqslant 1; \\ &t_k \to \infty, && \text{as} \quad k \to \infty; \\ &m(t_k) = m_\partial(t_k), && \text{for all} \quad k \geqslant 1. \end{aligned}$$

By Lemma 5.5, $\lim_{t \to \infty} u(t, x) = \phi(x) = 1$ for all $x \in \partial\tilde{\Omega}^1_\infty$. Hence, for any sufficiently small $\varepsilon > 0$, there exists $k_0 \geqslant 1$ such that

$$m_\partial(t) > 1 - \varepsilon, \qquad \text{for all} \quad t \geqslant t_{k_0}. \tag{5.19}$$

Claim. *For any integer* $l \geqslant 1$, *we have*

$$m(t_{k_0} + l) \geqslant \min\{m_\partial(t_{k_0} + l), m_\partial(t_{k_0} + l - 1), ..., m_\partial(t_{k_0}), 1\}. \tag{5.20}$$

Proof of the Claim. Clearly, (5.20) holds if $m(t_{k_0} + l) = m_\partial(t_{k_0} + l)$. Now suppose $m(t_{k_0} + l) < m_\partial(t_{k_0} + l)$. On $[t_{k_0} + l - 1, t_{k_0} + l] \times \tilde{\Omega}^1_\infty$, if the minimum of $u(\xi, x)$ is obtained at $t_{k_0} + l - 1$, then, $m(t_{k_0} + l) \geqslant m(t_{k_0} + l - 1)$. On the other hand, suppose the minimum of $u(\xi, x)$ is obtained in $(t_{k_0} + l - 1, t_{k_0} + l] \times \tilde{\Omega}^1_\infty$. We follow the proof of Part A of Lemma 5.4 to obtain

$$m(t_{k_0} + l) \geqslant \min\{m(t_{k_0} + l - 1), 1\}.$$

Note that $m(t_{k_0}) = m_\partial(t_{k_0})$. We get (5.20) after invoking the same procedure as above for at most $(l - 1)$ times. This completes the proof of the Claim.

For any $t \geqslant t_{k_0}$, we find an integer $l \geqslant 1$ such that $t_{k_0} + l - 1 \leqslant t \leqslant t_{k_0} + l$. From (5.19) and (5.20), we get

$$\begin{aligned} u(t, x) &\geqslant m(t_{k_0} + l) \\ &\geqslant \min\{m_\partial(t_{k_0} + l), m_\partial(t_{k_0} + l - 1), ..., m_\partial(t_{k_0}), 1\} \\ &\geqslant 1 - \varepsilon. \end{aligned}$$

Therefore,

$$\liminf_{t \to \infty} u(t, x) \geqslant 1 - \varepsilon,$$

uniformly for $x \in \overline{\tilde{\Omega}^1_\infty}$. Since ε is arbitrary, (5.18) follows. This completes the proof.

With the help of Lemma 5.6, we are now ready to prove the following lemma.

LEMMA 5.7. *Suppose* $e < \beta \leqslant e^2$ *and there exist* $T_0 \geqslant 1$, $c_0 > 0$ *such that*

$$M(t) = \max_{(\xi, x) \in [t-1, t] \times \overline{\tilde{\Omega}^1_\infty}} |u(\xi, x) - \phi(x)| = |u(t_0, x_0) - \phi(x_0)| \geqslant c_0,$$

where $(t_0, x_0) \in [t-1, t] \times \tilde{\Omega}^1_\infty$ *and* $t \geqslant T_0$. *Then, there exists* $T_1 \geqslant T_0$ *such that* $M(t)$ *is monotone decreasing for* $t \geqslant T_1$.

Proof. According to Lemma 5.6, for any given $0 < \varepsilon < c_0/2e^2$, we can find $T_1 \geqslant T_0$, such that $u(t-1, x) + \varepsilon \geqslant 1$ for all $t \geqslant T_1$ and $x \in \overline{\tilde{\Omega}^1_\infty}$. In the rest of the proof, we assume $t \geqslant T_1$. We consider $[t-1, t] \times \overline{\tilde{\Omega}^1_\infty}$. Clearly, if the maximum of $|u(\xi, x) - \phi(x)|$ is obtained at $t-1$, then we have

$$M(t) \leqslant M(t-1). \tag{5.21}$$

On the other hand, suppose $(t_0, x_0) \in (t-1, t] \times \tilde{\Omega}^1_\infty$. We will show that (5.21) still holds. Suppose

$$M(t) = \max_{(\xi, x) \in [t-1, t] \times \overline{\tilde{\Omega}^1_\infty}} |u(\xi, x) - \phi(x)| = u(t_0, x_0) - \phi(x_0)$$

is the positive maximum in $(t-1, t] \times \tilde{\Omega}^1_\infty$, where $t_0 \in (t-1, t]$ and $x_0 \in \tilde{\Omega}^1_\infty$. Then, we have $\partial(u-\phi)/\partial t \geqslant 0$ and $d\Delta(u-\phi) \leqslant 0$ at (t_0, x_0). By (5.17), this leads to

$$u(t_0, x_0) - \phi(x_0) \leqslant \beta[u(t_0-1, x_0)\, e^{-u(t_0-1, x_0)} - \phi(x_0)\, e^{-\phi(x_0)}]. \tag{5.22}$$

Clearly, $u(t_0-1, x_0) < \phi(x_0)$. Otherwise, since $x \mapsto xe^{-x}$ is decreasing for $x \geqslant 1$ we have

$$u(t_0, x_0) - \phi(x_0) \leqslant 0,$$

which is a contradiction. Moreover, there exists $\xi \in [u(t_0-1, x_0), \phi(x_0)]$ such that

$$\begin{aligned} &u(t_0-1, x_0)\, e^{-u(t_0-1, x_0)} - \phi(x_0)\, e^{-\phi(x_0)} \\ &\quad = (1-\xi)\, e^{-\xi}(u(t_0-1, x_0) - \phi(x_0)) \\ &\quad \leqslant e^{-2}(\phi(x_0) - u(t_0-1, x_0)), \end{aligned} \tag{5.23}$$

Substituting (5.23) into (5.22) gives rise to

$$\begin{aligned} M(t) &= u(t_0, x_0) - \phi(x_0) \\ &\leqslant \beta e^{-2}(\phi(x_0) - u(t_0 - 1, x_0)) \\ &\leqslant \beta e^{-2} M(t-1) \leqslant M(t-1). \end{aligned} \tag{5.24}$$

On the other hand, suppose

$$M(t) = \max_{(\xi, x) \in [t-1, t] \times \overline{\tilde{\Omega}^1_\infty}} |u(\xi, x) - \phi(x)| = -[u(t_0, x_0) - \phi(x_0)],$$

i.e., $u(t_0, x_0) - \phi(x_0)$ is the negative minimum in $(t-1, t] \times \tilde{\Omega}^1_\infty$, where $t_0 \in (t-1, t]$ and $x_0 \in \tilde{\Omega}^1_\infty$. We divide our discussion into two cases.

Case 1. $u(t_0 - 1, x_0) > \phi(x_0)$. We use the same arguments as above to get

$$u(t_0, x_0) - \phi(x_0) \geqslant -\beta e^{-2}(u(t_0 - 1, x_0) - \phi(x_0)). \tag{5.25}$$

Case 2. $u(t_0 - 1, x_0) < \phi(x_0)$. Note that $u(t_0 - 1, x_0) + \varepsilon \geqslant 1$. Therefore we obtain

$$\begin{aligned} &u(t_0, x_0) - \phi(x_0) \\ &\quad \geqslant \beta[(u(t_0 - 1, x_0) + \varepsilon)\, e^{-(u(t_0 - 1, x_0) + \varepsilon)} - (\phi(x_0) + \varepsilon)\, e^{-(\phi(x_0) + \varepsilon)}] \\ &\qquad + \beta[u(t_0 - 1, x_0)\, e^{-u(t_0 - 1, x_0)} - (u(t_0 - 1, x_0) + \varepsilon)\, e^{-(u(t_0 - 1, x_0) + \varepsilon)}] \\ &\qquad - \beta[\phi(x_0)\, e^{-\phi(x_0)} - (\phi(x_0) + \varepsilon)\, e^{-(\phi(x_0) + \varepsilon)}] \\ &\quad \geqslant -2\beta\varepsilon \\ &\quad \geqslant -\beta e^{-2} M(t-1). \end{aligned} \tag{5.26}$$

Combining (5.25) with (5.26) one obtains

$$M(t) \leqslant \beta e^{-2} M(t-1) \leqslant M(t-1). \tag{5.27}$$

Now for any $s \geqslant T_1$ and $t - 1 \leqslant s \leqslant t$, if $t_0 \in (s-1, s]$ and $t_1 \in (s-1, s]$, then

$$|u(t_0, x_0) - \phi(x_0)| \leqslant M(s),$$

and

$$|u(t_1, x_1) - \phi(x_1)| \leqslant M(s),$$

which imply $M(t) \leqslant M(s)$. If $t_0 \notin (s-1, s]$ or $t_1 \notin (s-1, s]$, then $(t_0 - 1) \in (s-1, s]$ or $(t_1 - 1) \in (s-1, s]$, we still get the same conclusion. Now if $T_1 \leqslant s < (t-1)$, then there exists a positive integer $l \geqslant 1$ such that $s \in [t-l-1, t-l]$. According to (5.19), (5.24), and (5.27), we have

$$M(s) \geqslant M(t-k) \geqslant M(t).$$

This completes the proof.

LEMMA 5.8. *Suppose $e < \beta < e^2$ and let $u(t, x)$ be a nontrivial nonnegative solution of (2.1)–(2.2). Then, $u(t, x) \to \phi(x)$ in $C(\overline{\tilde{\Omega}^1_\infty})$ as $t \to \infty$.*

Proof. Let

$$M(t) := |u(\xi(t), \eta(t)) - \phi(\eta(t))| = \max_{(\xi, x) \in [t-1, t] \times \overline{\tilde{\Omega}^1_\infty}} |u(\xi, x) - \phi(x)|,$$

and

$$M_\partial(t) := \max_{(\xi, x) \in [t-1, t] \times \partial\tilde{\Omega}^1_\infty} |u(\xi, x) - \phi(x)|.$$

CLAIM. *For any sufficiently small $\varepsilon > 0$, there exists t_ε such that*

$$M(t_\varepsilon) < \varepsilon.$$

Proof of the Claim. Suppose the Claim fails, i.e. there exist $\varepsilon_0 > 0$ and $T_0 > 1$ such that

$$M(t) \geqslant \varepsilon_0, \qquad \text{for all} \quad t \geqslant T_0. \tag{5.28}$$

We will show that there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ satisfying

$$\begin{aligned} &T_0 \leqslant t_k \leqslant t_{k+1}, && \text{for all} \quad k \geqslant 1; \\ &t_k \to \infty, && \text{as } k \to \infty; \quad \text{and} \\ &M(t_k) = M_\partial(t_k), && \text{for all} \quad k \geqslant 1. \end{aligned}$$

Indeed, if this is not the case, i.e. for all $t \geqslant T_0$, the maximum of $|u(\xi, x) - \phi(x)|$ on $[t-1, t] \times \overline{\tilde{\Omega}^1_\infty}$ is obtained in $[t-1, t] \times \tilde{\Omega}^1_\infty$, then, according to Lemma 5.7 $M(t)$ is monotone decreasing for $t \geqslant T_1$. We denote $M_0 := \lim_{t \to \infty} M(t)$. We will discuss two cases to show $M_0 = 0$.

Case 1. Suppose that $\xi(t)=t-1$ and $\eta(t)\in\tilde{\Omega}^1_\infty$, for all $t\geqslant T_1$. For any $0<\varepsilon<\min\{\tau(1-\beta e^{-2})\,\varepsilon_0,\, \varepsilon_0/2e^2\}$, we use the arguments similar to Part B of the proof of Lemma 5.4 to find a sequence $\{\tilde{t}_k\}_{k=1}^\infty$ such that

$$\begin{aligned} &\tilde{t}_k\geqslant T_1, && 0<\tilde{t}_{k+1}-\tilde{t}_k<1, && \text{for all}\quad k\geqslant 1;\\ &\tilde{t}_k\to\infty && \text{as}\quad k\to\infty; && \text{and}\\ &-\varepsilon<\frac{dM(\tilde{t}_k+1)}{dt}\leqslant 0, && && \text{for all}\quad k\geqslant 1. \end{aligned} \tag{5.29}$$

For any $k\geqslant 1$, we consider

$$M(\tilde{t}_k+1)=u(\tilde{t}_k,\eta(\tilde{t}_k+1))-\phi(\eta(\tilde{t}_k+1)),$$

i.e., $u(\tilde{t}_k,\eta(\tilde{t}_k+1))-\phi(\eta(\tilde{t}_k+1))$ is the positive maximum. Without loss of generality, we may assume, by using the continuity of $u(t,x)$ and $\phi(x)$, that there exists a sequence $\{h_j\}_{j=1}^\infty$, satisfying $0<|h_j|<1$ and $h_j\to 0$ as $j\to\infty$, such that

$$M(\tilde{t}_k+1+h_j)=u(\tilde{t}_k+h_j,\eta(\tilde{t}_k+1+h_j))-\phi(\eta(\tilde{t}_k+1+h_j)).$$

Now we follow Part B of the proof of Lemma 5.4 to obtain

$$\frac{dM(\tilde{t}_k+1)}{dt}=\frac{\partial u(\tilde{t}_k,\eta(\tilde{t}_k+1))}{\partial t}. \tag{5.30}$$

On the other hand, suppose

$$M(\tilde{t}_k+1)=-[u(\tilde{t}_k,\eta(\tilde{t}_k+1))-\phi(\eta(\tilde{t}_k+1))],$$

i.e., $u(\tilde{t}_k,\eta(\tilde{t}_k+1))-\phi(\eta(\tilde{t}_k+1))$ is the negative minimum. Proceeding as before, we get

$$\frac{dM(\tilde{t}_k+1)}{dt}=-\frac{\partial u(\tilde{t}_k,\eta(\tilde{t}_k+1))}{\partial t}. \tag{5.31}$$

Using (5.29)–(5.31) and following a similar argument as in the proof of Lemma 5.7, we obtain

$$\tau M(\tilde{t}_k+1)\leqslant\varepsilon+\tau\beta e^{-2}M(\tilde{t}_k), \tag{5.32}$$

Therefore, we take the limit as $k\to\infty$ to get

$$\tau M_0\leqslant\varepsilon+\tau\beta e^{-2}M_0,$$

that is

$$M_0 \leqslant \frac{\varepsilon}{\tau(1-\beta e^{-2})}.$$

This implies $M_0 = 0$ since ε can be arbitrarily small.

Case 2. Next we assume that there exists an increasing sequence, still denoted by $\{\tilde{t}_k\}_{k=1}^{\infty}$, such that on $[\tilde{t}_k, \tilde{t}_k+1] \times \overline{\tilde{\Omega}^1_\infty}$, we have $\xi(\tilde{t}_k+1) > \tilde{t}_k$ and $\eta(\tilde{t}_k+1) \in \tilde{\Omega}^1_\infty$.

In this case, we follow the proof of Lemma 5.7 to get

$$M(\tilde{t}_k+1) \leqslant \beta e^{-2} M(\tilde{t}_k).$$

Then we take the limit as $k \to \infty$ to obtain

$$M_0 \leqslant \beta e^{-2} M_0,$$

which implies $M_0 = 0$ since $\beta e^{-2} < 1$.

Therefore, the aforementioned sequence $\{t_k\}$exists. By Lemma 5.5,

$$\lim_{t \to \infty} u(t, x) = \phi(x), \qquad \text{for} \quad x \in \partial \tilde{\Omega}^1_\infty.$$

Hence, for any $0 < \varepsilon < \varepsilon_0$, there exists $k_0 \geqslant 1$ such that

$$M_\partial(t) < \varepsilon, \qquad \text{for all} \quad t \geqslant t_{k_0}. \tag{5.33}$$

Next, we show

$$M(t_{k_0}+1) \leqslant \max\{M_\partial(t_{k_0}+1), M_\partial(t_{k_0})\}. \tag{5.34}$$

Clearly, (5.34) is true if $M(t_{k_0}+1) = M_\partial(t_{k_0}+1)$. Now suppose $M(t_{k_0}+1) > M_\partial(t_{k_0}+1)$. On $[t_{k_0}, t_{k_0}+1] \times \tilde{\Omega}^1_\infty$, if $\xi(t_{k_0}+1) = t_{k_0}$, then, $M(t_{k_0}+1) \leqslant M_\partial(t_{k_0})$. On the other hand, suppose $\xi(t_{k_0}+1) > t_{k_0}$. We follow the proof of Lemma 5.7 to obtain $M(t_{k_0}+1) \leqslant M(t_{k_0}) = M_\partial(t_{k_0})$. Therefore (5.34) is true. We combine (5.33) and (5.34) to get $M(t_{k_0}+1) \leqslant \varepsilon < \varepsilon_0$. This contradicts (5.28) as well. The proof of the Claim is complete.

Now that the claim holds, we conclude $\lim_{t \to \infty} u(t, x) = \phi(x)$ according to the local asymptotic stability of the positive steady state (Theorem 4.2). This completes the proof.

Proof of Theorem 5.3. Using Lemma 5.5, Proposition 2.2, the continuity of $\phi(x)$, and the compactness of $\overline{\Omega^1_\infty}$, we can show that for any $\varepsilon > 0$, there exists $T_1 > 0$, such that

$$|u(t, x) - \phi(x)| \leqslant \varepsilon \qquad \text{for all} \quad x \in \overline{\Omega^1_\infty} \quad \text{and} \quad t \geqslant T_1. \tag{5.35}$$

The approach is similar to what we have shown in Lemma 5.5, and hence the details are omitted here. According to Lemma 5.8, we can also find $T_2 > 0$, such that

$$|u(t, x) - \phi(x)| \leqslant \varepsilon \qquad \text{for all} \quad x \in \overline{\tilde{\Omega}^1_\infty} \quad \text{and} \quad t \geqslant T_2. \tag{5.36}$$

Therefore, for $t \geqslant T_* := \max\{T_1, T_2\}$, we obtain

$$\begin{aligned} \|u(t, \cdot) - \phi(\cdot)\|_{C(\bar{\Omega})} &= \max_{x \in \bar{\Omega}} |u(t, x) - \phi(x)| \\ &\leqslant \max\{ \max_{x \in \overline{\Omega^1_\infty}} |u(t, x) - \phi(x)|, \max_{x \in \overline{\tilde{\Omega}^1_\infty}} |u(t, x) - \phi(x)| \} \\ &\leqslant \varepsilon. \end{aligned}$$

This implies $\lim_{t \to \infty} \|u(t, \cdot) - \phi(\cdot)\|_{C(\bar{\Omega})} = 0$.

The L^2-convergence is an immediate consequence of (5.35), (5.36) and the boundedness of Ω. Thus the proof of Theorem 5.3 is complete.

So far, we have shown the global attractivity of the zero solution or of the positive steady state in the sense of $L^2(\Omega)$. We will now show that the convergence results (Theorem 3.2 and Theorem 5.3) can be enhanced to $C^1(\Omega)$ by using an a priori estimate and an interpolation inequality.

THEOREM 5.9. *Let $u(t, x)$ be a solution of* (2.1)–(2.2) *and let $U(x)$ be the corresponding steady state, i.e. the zero solution or the positive steady state $\phi(x)$. Then, there exists a constant K, independent of time t, such that*

$$\|u(t, \cdot) - U(\cdot)\|_{C^1(\Omega)} \leqslant K \|u(t, \cdot) - U(\cdot)\|^{1-\alpha}_{L^2(\Omega)}, \qquad \text{for all} \quad t > 1,$$

where $0 < \alpha < 1$ is a constant decided by (5.42).

Proof. Throughout the proof, we will use K to denote the various constants independent of t. For any $n < p < \infty$, define the operator A: $D(A) \to L^p(\Omega)$ as in Section 2. Clearly, A^{-1} is bounded in $L^p(\Omega)$. Therefore we have

$$\|u(t, \cdot) - U(\cdot)\|_{L^p(\Omega)} \leqslant K \|A[u(t, \cdot) - U(\cdot)]\|_{L^p(\Omega)}, \tag{5.37}$$

for some positive constant K. Now since $u(t, x)$ is a solution of (2.1)–(2.2) with $u(0, \cdot) \in L^p(\Omega)$, we have $u(t, \cdot) \in W^{2, p}(\Omega) \cap W^{1, p}_0(\Omega)$ for $t > 1$. Using (5.37) and an a priori estimate (cf. Pazy [20, p. 242]), we get, for $t > 1$,

$$\begin{aligned} &\|u(t, \cdot) - U(\cdot)\|_{W^{2, p}(\Omega)} \\ &\quad \leqslant K(\|A[u(t, \cdot) - U(\cdot)]\|_{L^p(\Omega)} + \|u(t, \cdot) - U(\cdot)\|_{L^p(\Omega)}) \\ &\quad \leqslant K \|A[u(t, \cdot) - U(\cdot)]\|_{L^p(\Omega)}. \end{aligned} \tag{5.38}$$

Following an argument similar to that in Section 2, we have

$$\|A[u(t,\cdot)-U(\cdot)]\|_{L^p(\Omega)} \leqslant K, \qquad \text{for} \quad t>1. \tag{5.39}$$

By combining (5.38) and (5.39), one obtains

$$\|u(t,\cdot)-U(\cdot)\|_{W^{2,p}(\Omega)} \leqslant K. \tag{5.40}$$

Now using [6, Theorem 10.1], we have

$$\|u(t,\cdot)-U(\cdot)\|_{C^1(\Omega)} \leqslant K\, \|u(t,\cdot)-U(\cdot)\|^{\alpha}_{W^{2,p}(\Omega)}\, \|u(t,\cdot)-U(\cdot)\|^{1-\alpha}_{L^2(\Omega)}, \tag{5.41}$$

where $p>n$, $0<\alpha<1$, and,

$$-1=\left(\frac{n}{p}-2\right)\alpha+(1-\alpha)\frac{n}{2}. \tag{5.42}$$

Substituting (5.40) into (5.41) gives rise to our conclusion. This completes the proof.

APPENDIX: LIST OF SYMBOLS

$\partial\Omega$	the boundary of Ω
Δ	the Laplace operator
$L^p(\Omega)$	the set of functions which are L^p integrable on Ω
$C^m(\Omega)$	the set of functions which are continuous in Ω together with all their derivatives up to order m
$C^{m+\mu}(\Omega)$	the set of functions in $C^m(\Omega)$ whose$m$-derivatives are Hölder continuous with exponent μ
$C_0^m(\Omega)$	the subset of $C^m(\Omega)$ consisting of those functions which have compact support in Ω
$W^{2,p}(\Omega)$	the set of functions in $L^p(\Omega)$ whose weak derivatives of order $\leqslant 2$ exist and belong to $L^p(\Omega)$
$W_0^{1,p}(\Omega)$	the completion of the set of functions in $C_0^1(\Omega)$ with respect to the norm of $W^{1,p}(\Omega)$
$H^2(\Omega)$	the Sobolev space $W^{2,2}(\Omega)$
$H_0^1(\Omega)$	the Sobolev space $W_0^{1,2}(\Omega)$
$\Gamma(\alpha)$	the Γ function
$\|\cdot\|_X$	the norm over Banach space X

ACKNOWLEDGMENTS

We thank Yin-Xi Huang of the University of Memphis for making many invaluable suggestions on an earlier version of this paper. We also appreciate the generous comments of an anonymous referee which greatly improved the readability of the paper.

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