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Augmented nodal matrices and normal trees ☆

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ABSTRACT

Augmented nodal matrices play an important role in the analysis of different features of electrical circuit models. Their study can be addressed in an abstract setting involving two- and three-colour weighted digraphs. By means of a detailed characterization of the structure of proper and normal trees, we provide a unifying framework for the rank analysis of augmented matrices. This covers in particular Maxwell's tree-based determinantal expansions of (non-augmented) nodal matrices, which can be considered as a one-colour version of our results. Via different colour assignments to circuit devices, we tackle the DC-solvability problem and the index characterization of certain differential–algebraic models which arise in the nodal analysis of electrical circuits, extending several known results of passive circuits to the non-passive context.

1. Introduction

In literature on electrical circuits, it is well known that the determinant of a nodal matrix can be written in terms of a sum of tree weights. This result, which can be traced back to J. C. Maxwell (cf. [3, p. 192] or [12, p. 112]), can be stated for connected, weighted digraphs and says that the determinant of the nodal matrix AWA^{T} of a connected circuit equals the sum of weight products over the set of network trees. Here A is a reduced incidence matrix and W is a (diagonal) matrix of weights, which may correspond e.g. to admittances in linear circuits in the transformed domain or to conductances in a purely resistive circuit. As it will be detailed later, this is a consequence of the Cauchy–Binet formula.

However, in many real situations, circuit models involve so-called *augmented* nodal matrices of the form

$$M = \begin{pmatrix} A_{\mathcal{B}} W_{\mathcal{B}} A_{\mathcal{B}}^{\mathrm{T}} & A_{\mathcal{G}} \\ A_{\mathcal{G}}^{\mathrm{T}} & 0 \end{pmatrix}, \tag{1}$$

corresponding to a weighted digraph in which the branches are grouped in three categories, say blue, green and red. The incidence matrix $A_{\mathcal{B}}$ and the weights $W_{\mathcal{B}}$ of blue branches enter explicitly the augmented nodal matrix; the incidence matrix $A_{\mathcal{G}}$ of green branches also enters the matrix, in the form depicted in (1); finally, no information about red branches is explicitly displayed in M even though they certainly play a role in the problem. This type of augmented matrices will be shown later to be a key element in nodal modeling techniques such as Modified Nodal Analysis (MNA) (used in circuit simulation programs, e.g. in SPICE or TITAN [17,22,23,47]) or Augmented Nodal Analysis (ANA) [30,44]; they are also involved in other issues arising in circuit analysis, for instance in the DC-solvability problem.

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From certain applications in circuit theory (which can be understood to result from specific assignments of blue, green and red branches to different types of circuit devices), it is already known that M is nonsingular if $W_{\mathcal{B}}$ is a positive definite (not necessarily diagonal) matrix and the digraph has neither red cutsets nor green loops; see e.g. Theorem 4 in [44]. The positive definiteness property characterizes *strictly passive* devices in circuit theory. Moreover, under a definiteness assumption on $W_{\mathcal{B}}$, in the presence of red cutsets and/or green loops it is true that (cf. the proof of Theorem 3 in [41])

$$\ker M = \ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix} \times \ker A_{\mathcal{G}} \tag{2}$$

or, equivalently,

$$\operatorname{cork} M = \dim \ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix} + \dim \ker A_{\mathcal{G}}, \tag{3}$$

where $\operatorname{cork} M = p - \operatorname{rk} M$ for a $p \times p$ matrix. The identities displayed in (2) and (3) can be understood as follows: the circuit topology or, more precisely, the number of independent red cutsets and green loops defines the maximal rank (or minimal corank) attainable by the augmented nodal matrix (1); then a positive definiteness assumption on $W_{\mathcal{B}}$ guarantees that this maximal rank is actually reached.

Our first goal is to assert these identities without positive definiteness assumptions, thereby driving different known properties of nodal models of passive circuits to the non-passive setting. Specifically, by removing positive definiteness assumptions we will connect the nonsingularity (in the absence of red cutsets and green loops) or, in general, the maximal rank of M with the *proper* and *normal* trees introduced by Bashkow [5] and Bryant [9–11]. This applies to problems with diagonal matrices $W_{\mathcal{B}}$ and extends previous research in this direction [16]. Moreover, problems with non-diagonal matrices $W_{\mathcal{B}}$, which model the existence of controlled sources and coupling effects in electrical circuits, will be included in the analysis by means of the novel concepts of a *balanced* tree and a *regular pair* of trees.

These issues might seem to be rather technical. However, as it will be shown in Section 5, they play a key role in DC-solvability analyses and also in the differential–algebraic index characterization of MNA/ANA models of non-passive circuits (which is the ultimate goal of our study). The characterization of the structure of proper and normal trees, necessary for subsequent studies, defines a result of independent interest which is addressed in Section 2. Notice that the discussion in Section 2 does not involve the weights or the branch reference directions but only the nature of the different branches; for this reason, in the spirit of Minty, Chua and others [14,35,49] we state the main results of Section 2 in terms of three-coloured graphs. Certain two-colour variants of our results will also arise in the analysis, whereas the one-colour version amounts to the above-mentioned Maxwell's node-admittance property. These properties will pave the way for the characterization of augmented matrices carried out in Sections 3 and 4 for problems with diagonal and non-diagonal matrices $W_{\mathcal{B}}$, respectively. Finally, the circuit applications referred to above are discussed in Section 5.

2. Normal trees of two- and three-colour graphs

In this Section we present a characterization of the set of normal (and proper) trees of three-colour graphs. Background on graph and digraph theory is given in Section 2.1, whereas Section 2.2 addresses the structure of normal trees. We state the results in terms of three-colour graphs just for simplicity in subsequent applications, where different colour assignments to circuit devices are needed in different problems. Many of the ideas discussed in Section 2.2 directly stem from the original work of Bryant, focused on the state formulation problem for linear circuits; see particularly [10].

2.1. Elementary notions from digraph theory

We refer the reader to [1,2,4,6,12,15,19] for extensive introductions to graph theory. In this subsection we compile several results which will be useful later. The proof of these properties can be found in the above-mentioned references.

Loops, cutsets. Recall that the reduced incidence matrix $A \in \mathbb{R}^{(n-1)\times m}$ of a connected digraph with n nodes and m branches is defined as (a_{ij}) with

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i. \end{cases}$$

If $\mathcal{K}$ is a subset of the set of branches of a connected digraph $\mathbf{G}$, we will denote by $A_{\mathcal{K}}$ (resp. $A_{\mathbf{G}-\mathcal{K}}$) the submatrix of A defined by the columns which correspond to branches in $\mathcal{K}$ (resp. not in $\mathcal{K}$). By a $\mathcal{K}$-loop we mean a loop formed only by branches belonging to $\mathcal{K}$.

Lemma 1. *Let $\mathcal{K}$ be a subset of branches of a connected digraph* $\mathbf{G}$. *Then* $\dim \ker A_{\mathcal{K}}$ *equals the number of independent $\mathcal{K}$-loops. In particular, $A_{\mathcal{K}}$ has full column rank if and only if $\mathcal{K}$ does not contain loops.*

Here, the notion of independence relies on the representation of a given oriented loop as the vector u of $\mathbb{R}^m$ defined componentwise by

$$u_j = \begin{cases} 1 & \text{if branch } j \text{ is in the loop with the same orientation} \\ -1 & \text{if branch } j \text{ is in the loop with the opposite orientation} \\ 0 & \text{if branch } j \text{ is not in the loop.} \end{cases}$$

The subset $\mathcal{K}$ of branches of a digraph is said to be a *cutset* if the deletion of $\mathcal{K}$ increases the number of connected components, and additionally it is minimal with respect to this property (that is, the removal of any proper subset of $\mathcal{K}$ does not increase the number of connected components). For a connected digraph, a cutset is a minimal disconnecting set. Again, by a $\mathcal{K}$-cutset we mean a cutset defined exclusively by branches belonging to $\mathcal{K}$. An oriented cutset can be identified with a vector of $\mathbb{R}^m$ in the manner explained above for loops.

Lemma 2. *Let $\mathcal{K}$ be a subset of branches of a connected digraph* **G**. *Then* $\dim\ker A^{\mathrm{T}}_{\mathbf{G}-\mathcal{K}}$ *equals the number of independent $\mathcal{K}$-cutsets. In particular, $\mathcal{K}$ does not contain cutsets if and only if $A_{\mathbf{G}-\mathcal{K}}$ has a full row rank.*

Via the identity $\dim\ker A^{\mathrm{T}}_{\mathbf{G}-\mathcal{K}} = (n-1) - \mathrm{rk}\, A^{\mathrm{T}}_{\mathbf{G}-\mathcal{K}}$, the property stated in Lemma 2 can be understood to express the fact that the removal of k independent cutsets in a connected digraph defines a subgraph with $k+1$ connected components.

Trees. In a connected digraph, a *tree* is an acyclic spanning subgraph. That is, a subset T of branches of a connected digraph, together with their terminal nodes, defines a tree if it does not include any loops and every node in the digraph is incident with at least one branch of T. As usual in circuit theory, we use 'tree' to mean *spanning tree*. Branches in a tree are called *twigs*. The branches which are not in T (termed *links* or *chords*) are said to define a *cotree*. The choice of a tree in every connected component of a non-connected digraph defines a *forest*, the remaining branches yielding a *coforest*.

Lemma 3. *Let T be a subset of branches of a connected digraph. T defines a tree if and only if A_T is a non-singular matrix, being* $\det A_T = \pm 1$ *in this case.*

Contractions and minors. The *contraction* of a given branch is a digraph operation defined by the identification of the terminal nodes of that branch into a single one, which inherits the adjacencies of the original pair of nodes. In the electrical circuit literature such a branch is said to be *short-circuited*. A sequence of branch removals (or open-circuits) and contractions defines a *minor* of a given digraph: find details in [15].

A subgraph of a minor can be also considered as a subgraph of the original digraph. It is also true that an acyclic subgraph of a given minor is acyclic in the original digraph; however, typically a spanning subgraph of a minor will not span the original digraph. Therefore, a tree of a given minor defines an acyclic subgraph (but not necessarily a tree) of the original graph.

Coloured (di)graphs. Proper and normal trees. Topological properties of electrical circuits are those which can be assessed just in terms of the electrical nature (resistive, capacitive, etc.) of every branch, regardless of the actual current-voltage characteristic in that branch. In the analysis of different topological features of circuits, it will therefore be useful to consider digraphs in which every branch is assigned a given colour; different assignments will be used for the analysis of different properties (cf. Section 5).

In the sequel we will not make specific use of the branch directions and thus the discussion in the remainder of this Section is stated for graphs, in the understanding that it applies also to digraphs via the underlying graph (that is, the graph without reference directions).

The main role will be played by *three-colour* graphs, in which branches are painted green, blue or red. *Two-colour* graphs (either green/blue or blue/red) will also arise in the analysis. Paint also the nodes of three-colour graphs as follows: a node will be painted green if it is incident with at least one green branch, blue if it is not incident with green branches but it is incident with at least one blue branch, and red if it is just incident with red branches. This applies also to two-colour graphs. The green and green/blue subgraphs will be hence defined by the green and the green and blue branches, respectively, together with their incident nodes.

In this setting it is possible to define *proper* and *normal* trees, which were introduced by Bashkow and Bryant [5,10,11] (cf. also [28]) in order to tackle the state formulation problem for RLC circuits.

Definition 1. In a three-colour connected graph, a tree will be called *proper* if it contains all green branches and no red branches, together with (possibly) some blue branches.

The existence of a proper tree imposes that the graph has neither green loops nor red cutsets. In their presence we will have to deal with *normal* trees instead.

Definition 2. In a three-colour connected graph, a tree will be called *normal* if it contains the maximum possible number of green branches, the minimum possible number of red branches, and (possibly) some blue branches.

This notion is well-defined; the number of green branches in a normal cotree can be actually shown to be given by the number x_g of independent green loops, and similarly the number of red branches in a normal tree equals the number x_r of independent red cutsets [8,10]. The notions of a proper and a normal forest in a non-connected graph are defined componentwise. Note that the term 'normal tree' is sometimes used in another sense (specifically, to mean depth-first search trees [15]).

In two-colour graphs the notions above apply with just one requirement, being defined by all or the maximum possible number of green branches in green/blue graphs, and by none or the minimum possible number of red branches in blue/red graphs. Note, incidentally, that from the point of view of proper and normal trees there is no difference between both types of two-colour graphs; a difference will be made by the augmented nodal matrices of both types, though. One-colour graphs can be modeled as blue graphs, and the augmented nodal matrix will amount in this setting to the nodal matrix $A_{\mathcal{B}}W_{\mathcal{B}}A_{\mathcal{B}}^{\mathrm{T}}$. In this one-colour context the notions of a proper and a normal tree would simply amount to that of a tree.

We will also make use of the coloured branch theorem as stated in Lemma 4 below [14,49]. This is actually a corollary of a more general result proved by Minty (cf. Theorem 3.1 in [35]).

Lemma 4. *In a three-colour graph with just one blue branch, this branch either forms a loop exclusively with green branches or a cutset exclusively with red branches, but not both.*

Certainly, the loop or cutset arising in this result need not be unique.

Weights. As indicated in the Introduction, the analysis of augmented nodal matrices performed in Sections 3 and 4 relies on the assignment of *weights* to blue branches in coloured digraphs; these real-valued parameters will define the entries in the diagonal of the weight matrix $W_{\mathcal{B}}$. Given a tree in a coloured digraph with a diagonal weight matrix $W_{\mathcal{B}}$, we will define its *b-weight* as the product of weights in the blue twigs; this weight would be set to 1 in the absence of blue branches in the tree. Note that in Section 4 we will deal, additionally, with problems in which the existence of non-vanishing entries away from the diagonal of $W_{\mathcal{B}}$ will generalize the notion of the b-weight of a tree.

2.2. The structure of normal trees

The analysis of augmented nodal matrices carried out in Section 3 makes fundamental use of the characterization of the set of normal trees of two- and three-colour graphs presented here. As indicated above, the main ideas supporting Theorems 1 and 2 below are already present in the construction of a normal tree within the seminal work of Bryant [10,11]. The results in this subsection apply in particular to proper trees since they are a particular case of normal trees, being displayed in graphs without green loops and red cutsets. Other related types of trees (namely, balanced trees and regular tree pairs) will arise in the analysis of problems with non-diagonal weight matrices $W_{\mathcal{B}}$, and are discussed in Section 4.

2.2.1. Two-colour graphs

Given a green/blue connected graph, let us construct its *blue-cut minor* as detailed in the sequel. Note first that, by the coloured branch theorem, every blue branch either defines a loop exclusively with green branches or a cutset exclusively with other blue branches; indeed, for every single blue branch, paint the remaining blue ones in red and apply Lemma 4. Remove every blue branch that defines a loop exclusively with green branches, so that all the remaining blue branches define blue cutsets of the original graph; the blue-cut minor is then obtained after contracting all green branches. By construction, the blue-cut minor has $k_g + n_b$ nodes, where k_g is the number of components of the green subgraph and n_b the number of blue nodes, which in a green/blue graph are the nodes incident with blue branches only.

Theorem 1. *The normal trees of a green/blue connected graph are defined by all possible combinations of a forest of the green subgraph and a tree of the blue-cut minor.*

Proof. Let us first show that any such combination defines a normal tree, namely, that it defines an acyclic, spanning subgraph with the maximum possible number of green branches.

1. To see that such a combination must be acyclic, note that green loops are precluded in a green forest, so that any loop should include at least one blue branch. However, the contraction of all green branches in such a loop would yield a loop in the blue-cut minor, against the hypothesis that blue branches come from a tree of this minor.

2. Any such combination spans the whole graph; indeed, every green node (i.e. a node incident with at least one green branch) must belong to the green forest, whereas blue nodes (incident with blue branches only) are necessarily in the blue-cut tree.

3. Finally, to check that these combinations attain the maximum possible number of green branches in a tree, keep in mind that a green forest has n_g–k_g green branches, n_g and k_g being the number of green nodes and connected components of the green subgraph, respectively. A set of more than $n_g - k_g$ green branches must include a subset of branches coming from the same green component (say the j-th one) and exceeding the number $(n_g)_j - 1$ of twigs in that component, and should therefore include a green loop.

Conversely, in order to see that every normal tree has this structure, proceed as follows.

4. As indicated above, by the coloured branch theorem every blue branch either defines a loop with green branches only, or a blue cutset, but not both. Now, a blue branch defining a loop with green branches cannot take part in a normal tree.

Assume it does. Then the fundamental cutset defined by the blue twig must include at least another branch from the loop; this follows from the fact that any cutset includes an even number of branches from any loop (see item 3.28 in [2]). This extra branch must be green. Replacing in the tree the blue branch by this green one we get another tree (since all branches in the cutset connect the two separated components), with more green branches than the original one. This contradicts the hypothesis that the tree is normal. Hence, the normal trees belong to the subgraph defined by the green branches together with the blue branches forming blue cutsets.

5. From items 1–3 above, a normal tree has $n_g - k_g$ green branches. In order to distribute them among the components of the green subgraph without forming loops, for $j = 1 \dots k_g$ the j-th green component must include $(n_g)_j - 1$ branches, which by the absence of loops define a tree in that component.

6. Since the graph has $n_g + n_b$ nodes, any tree must have $n_g + n_b - 1$ branches. We know that in a normal one there are $n_g - k_g$ green branches, so that there must be $n_g + n_b - 1 - (n_g - k_g) = k_g + n_b - 1$ blue branches. By item 4 they all belong to the blue-cut minor. Additionally, the set of blue branches in the tree must be incident with all green trees in the forest and with the n_b blue nodes. This means that the $k_g + n_b - 1$ blue branches in the normal tree span the blue-cut minor, which has $k_g + n_b$ nodes; since a spanning subgraph with $n - 1$ branches in a connected graph with n nodes must necessarily be acyclic (if it were not, remove a branch from a loop to spuriously obtain a spanning subgraph with $n-2$ branches), it follows that the blue branches in the normal tree define a tree of the blue-cut minor, thus completing the proof. □

2.2.2. Three-colour graphs

The *red-cut minor* of a green/blue/red connected graph is defined by the removal of all red branches which define loops just with green and/or blue branches, and by the subsequent contraction of green and blue branches. It has $k_{gb} + n_r$ nodes, k_{gb} being the number of components of the green/blue subgraph and n_r the number of red nodes (that is, nodes incident with red branches only).

Theorem 2. *The normal trees of a three-colour connected graph are defined by all possible combinations of a normal forest of the green/blue subgraph and a tree of the red-cut minor.*

The normal forests of the green/blue subgraph are defined by the choice of a normal tree (as given by Theorem 1) in each component of the green/blue subgraph.

Proof. The proof of the first assertion parallelizes the one of Theorem 1 and can be obtained in a straightforward manner by replacing "green loop" by "green/blue loop", "green tree" by "green/blue normal tree", "blue-cut" by "red-cut", etc. The only thing needing to be proved is that such a combination has the minimum possible number of red branches and hence defines indeed a normal tree. In this regard, note that the red branches must connect k_{gb} green/blue components and n_r red nodes, so that there must be no less than $k_{gb} + n_r - 1$ red branches in the tree. Since this is the number of branches in a tree of the red-cut contraction, then normal trees actually have $k_{gb} + n_r - 1$ red branches and the combinations arising in the statement of the theorem actually define normal trees. □

The second assertion in Theorem 2 is self-evident: it is included only to provide a full account of the complete structure of the set of normal trees.

3. Augmented nodal matrices of three-colour weighted digraphs

3.1. Cauchy–Binet determinantal expansions

The analysis of augmented nodal matrices presented below will crucially rely on the Cauchy–Binet formula (see e.g. [27]) stated in Lemma 5 for products of three matrices $D \in \mathbb{R}^{p\times m}$, $E \in \mathbb{R}^{m\times m}$ and $F \in \mathbb{R}^{m\times p}$, with $p \leq m$. We will use α, β to denote subsets of $\{1, \dots, m\}$ with $\operatorname{card}\alpha = \operatorname{card}\beta = p$, whereas ω will stand for $\{1, \dots, p\}$. This way, $E^{\alpha,\beta}$ is the $p \times p$ submatrix of E defined by the rows indexed by α and the columns indexed by β, whereas $D^{\omega,\alpha}$ (resp. $F^{\beta,\omega}$) is the submatrix of D (resp. of F) including entries from all rows in D (resp. all columns in F) and the columns indexed by α (resp. the rows indexed by β).

Lemma 5. *If $D \in \mathbb{R}^{p\times m}$, $E \in \mathbb{R}^{m\times m}$, and $F \in \mathbb{R}^{m\times p}$, with $p \leq m$, then*

$$\det DEF = \sum_{\alpha,\beta} \det D^{\omega,\alpha} \det E^{\alpha,\beta} \det F^{\beta,\omega}, \tag{4}$$

the sum being taken over all index sets $\alpha, \beta \subseteq \{1, \dots, m\}$ with $\operatorname{card}\alpha = \operatorname{card}\beta = p$.

3.2. Maximal rank characterization of augmented nodal matrices

The results of Sections 2.2 and 3.1 make it possible to undertake the rank analysis of the augmented nodal matrix M (cf. (1)) of a three-colour digraph. The matrix $W_{\mathcal{B}}$ of weights in blue branches will be assumed throughout this Section to be diagonal. Problems with non-diagonal matrices $W_{\mathcal{B}}$ will be addressed in Section 4.

Lemma 6. *In a three-colour connected digraph, the corank of M is greater than or equal to the number of independent red cutsets plus the number of independent green loops.*

Proof. Indeed, it is straightforward to check that

$$u \in \ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix}, \qquad v \in \ker A_{\mathcal{G}} \Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} \in \ker \begin{pmatrix} A_{\mathcal{B}} W_{\mathcal{B}} A_{\mathcal{B}}^{\mathrm{T}} & A_{\mathcal{G}} \\ A_{\mathcal{G}}^{\mathrm{T}} & 0 \end{pmatrix}, \tag{5}$$

and therefore

$$\ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix} \times \ker A_{\mathcal{G}} \subseteq \ker \begin{pmatrix} A_{\mathcal{B}} W_{\mathcal{B}} A_{\mathcal{B}}^{\mathrm{T}} & A_{\mathcal{G}} \\ A_{\mathcal{G}}^{\mathrm{T}} & 0 \end{pmatrix} = \ker M. \tag{6}$$

The result is then due to the fact that the number of independent red cutsets and independent green loops equal the dimensions of $\ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix}$ and $\ker A_{\mathcal{G}}$, respectively (cf. Lemmas 1 and 2). □

The main results in this Section are Theorem 3 and, specially, Theorem 4. The former, which can be considered as a particular case of the latter, is stated separately because of its independent interest and also for the sake of clarity in the proof of Theorem 4. The proof of Theorem 3 follows the ideas introduced in [16] for the analysis of circuits without VC-loops and IL-cutsets. Recall from Section 2.1 that the b-weight of a tree in a digraph with a diagonal matrix $W_{\mathcal{B}}$ is the product of the weights of all blue twigs.

Theorem 3. *For a connected digraph, the matrix M in (1) is nonsingular if and only if*

(i) *there are neither red cutsets nor green loops, and*
(ii) *the sum of b-weights in proper trees does not vanish.*

Proof. From Lemma 6 it is clear that the nonsingularity of M precludes the existence of green loops and red cutsets. We then need to show that, in the absence of these configurations, the nonsingularity of M is equivalent to the non-vanishing of the sum of products of the weights of blue branches in proper trees. Notice that the absence of green loops and red cutsets yields the existence of at least one proper tree.

In order to achieve this, we will make use of the factorization

$$M = \begin{pmatrix} A_{\mathcal{B}} W_{\mathcal{B}} A_{\mathcal{B}}^{\mathrm{T}} & A_{\mathcal{G}} \\ A_{\mathcal{G}}^{\mathrm{T}} & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} A_{\mathcal{B}} & A_{\mathcal{G}} & 0 \\ 0 & 0 & I_{\mathcal{G}} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} W_{\mathcal{B}} & 0 & 0 \\ 0 & 0 & I_{\mathcal{G}} \\ 0 & I_{\mathcal{G}} & 0 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} & 0 \\ A_{\mathcal{G}}^{\mathrm{T}} & 0 \\ 0 & I_{\mathcal{G}} \end{pmatrix}}_{F}, \tag{7}$$

following the idea introduced in [16]. The size of the matrix $I_{\mathcal{G}}$ above equals the number of green branches. Applying the Cauchy–Binet formula stated in Lemma 5 we can write

$$\det M = \det DEF = \sum_{\alpha,\beta} \det D^{\omega,\alpha} \det E^{\alpha,\beta} \det F^{\beta,\omega}, \tag{8}$$

for certain submatrices (with the same size as M) $D^{\omega,\alpha}$, $E^{\alpha,\beta}$ and $F^{\beta,\omega}$ of D, E and F, respectively. As explained in Section 3.1, the index sets α, β, ω within the superscripts specify the rows and columns defining the different submatrices; in particular, ω indicates that entries from all rows of D and all columns of F are present in $D^{\omega,\alpha}$ and $F^{\beta,\omega}$, respectively.

The structure of the matrices D, E and F makes it possible to restrict considerably the set of submatrices $D^{\omega,\alpha}$, $E^{\alpha,\beta}$ and $F^{\beta,\omega}$ for which the corresponding determinants in (8) do not vanish. Specifically, the only nonzero determinants come from submatrices of the form

$$D^{\omega,\alpha} = \begin{pmatrix} A_{\tilde{\mathcal{B}}} & A_{\mathcal{G}} & 0 \\ 0 & 0 & I_{\mathcal{G}} \end{pmatrix}, \qquad E^{\alpha,\beta} = \begin{pmatrix} \tilde{W} & 0 & 0 \\ 0 & 0 & I_{\mathcal{G}} \\ 0 & I_{\mathcal{G}} & 0 \end{pmatrix}, \qquad F^{\beta,\omega} = \begin{pmatrix} A_{\tilde{\mathcal{B}}}^{\mathrm{T}} & 0 A_{\mathcal{G}}^{\mathrm{T}} & 0 \\ 0 & I_{\mathcal{G}} \end{pmatrix},$$

the tilde ˜ indicating that only the entries corresponding to some blue branches must be present in each non-vanishing factor. This structure is due to the facts explained in the sequel. First, the full row rank and full column rank requirements in $D^{\omega,\alpha}$ and $F^{\beta,\omega}$, respectively, imply that the $I_{\mathcal{G}}$ blocks must be fully present in these submatrices and, subsequently, in $E^{\alpha,\beta}$. In turn, this means that the blocks $A_{\mathcal{G}}$ from D and $A_{\mathcal{G}}^{\mathrm{T}}$ from F must be entirely present in $D^{\omega,\alpha}$ and $F^{\beta,\omega}$, respectively. Additionally, the diagonal structure of the weight matrix $W_{\mathcal{B}} = \mathrm{diag}\,(w_1, w_2, \ldots, w_{m_b})$ (where w_i is the weight of the i-th blue branch and m_b the number of blue branches) implies that the same set of blue branches (denoted by $\tilde{\mathcal{B}}$) enters $A_{\tilde{\mathcal{B}}}$ and $A_{\tilde{\mathcal{B}}}^{\mathrm{T}}$ within $D^{\omega,\alpha}$ and $F^{\beta,\omega}$. By $\tilde{W}$ we denote the (diagonal) matrix of weights of the blue branches within $\tilde{\mathcal{B}}$.

The block diagonal structure of $\tilde{W}$ yields

$$\det E^{\alpha,\beta} = (-1)^{m_g} \prod_{m_i \in \tilde{\mathcal{B}}} w_i, \tag{9}$$

where m_g is the number of green branches (note that m_g column exchanges transforms $E^{\alpha,\beta}$ into a diagonal matrix); with the expression $m_i \in \tilde{\mathcal{B}}$ we specify the branches that belong to $\tilde{\mathcal{B}}$, whereas w_i denotes their corresponding weights.

In turn, the set $\tilde{\mathcal{B}}$ must make the square matrix $(A_{\tilde{\mathcal{B}}}\ A_{\mathcal{G}})$ nonsingular, meaning that this set of blue branches together with *all* green branches (mind the full presence of $A_{\mathcal{G}}$ and $A_{\mathcal{G}}^{\mathrm{T}}$ in $D^{\omega,\alpha}$ and $F^{\beta,\omega}$) must define a tree. Since it includes all green branches and no red ones, it will actually be a proper tree. We then have

$$\det D^{\omega,\alpha} = \det F^{\beta,\omega} = \pm 1. \tag{10}$$

The expressions depicted in (9) and (10) transform (8) into

$$\det M = (-1)^{m_g} \sum_{T\in\mathcal{T}_p} \prod_{m_i \in T\cap\mathcal{B}} w_i, \tag{11}$$

$\mathcal{T}_p$ being the set of proper trees. This expression completes the proof. □

Remark that in special cases there may exist a proper green tree, that is, a tree comprising all green branches and no blue or red ones. In this situation $A_{\mathcal{G}}$ is a square nonsingular matrix, and this makes M in (1) nonsingular because of its block structure. It is easy to check that in this situation there cannot exist other proper trees and, since in a tree without blue branches the b-weight is assumed to be 1, this setting is implicitly comprised in the statement above. It is also worth remarking that, if there are no proper green trees, then all proper trees must have at least one blue branch.

In the presence of red cutsets or green loops, the matrix M will be a singular one. Because of their applications in circuit theory, it is important to characterize the situations in which the corank of M actually equals the number x_r of independent red cutsets plus the number x_g of independent green loops or, equivalently, the situations in which

$$\ker M = \ker \begin{pmatrix} A_{\mathcal{B}}^{\mathrm{T}} \\ A_{\mathcal{G}}^{\mathrm{T}} \end{pmatrix} \times \ker A_{\mathcal{G}}, \tag{12}$$

as done in Theorem 4 below. Note that Theorem 3 can be considered a particular case of Theorem 4 with $x_r = x_g = 0$, corresponding to $\ker M = \{0\}$.

Theorem 4. *The corank of the augmented nodal matrix M of a three-colour connected digraph equals the number of independent red cutsets plus the number of independent green loops (or, equivalently, the identity (12) holds) if and only if the sum of b-weights in normal trees does not vanish.*

Proof. We will first perform a sequence of transformations which result in the identity $\operatorname{cork} M = \operatorname{cork} M''' + x_g + x_r$ for a certain matrix M'''. After that we will prove that the nonsingularity of M''' (that is, the vanishing of its corank) is equivalent to the non-vanishing of the sum of products of the weights of the blue branches in normal trees.

1. Let us first delete from the digraph a coforest of the green subgraph; this is equivalent to removing x_g green branches from the digraph in a way such that no green loop remains in the resulting subgraph. Note that the deletion of these branches does not disconnect the graph. The incidence matrix of this subgraph reads

$$M' = \begin{pmatrix} A_{\mathcal{B}} W_{\mathcal{B}} A_{\mathcal{B}}^{\mathrm{T}} & A_{\overline{\mathcal{G}}} \\ A_{\overline{\mathcal{G}}}^{\mathrm{T}} & 0 \end{pmatrix}, \tag{13}$$

where $A_{\overline{\mathcal{G}}}$ is the submatrix of $A_{\mathcal{G}}$ corresponding to the green branches in the forest, that is, to the green branches which were not deleted. The columns removed from $A_{\mathcal{G}}$ belong to the subspace spanned by $A_{\overline{\mathcal{G}}}$, since all the deleted branches form a loop with some of the green branches which were not removed. This means that $\operatorname{rk} M' = \operatorname{rk} M$, and then

$$\operatorname{cork} M = \operatorname{cork} M' + x_g. \tag{14}$$

2. The green/blue subgraph of the digraph defined this way has the same number of components (k_{gb}) as the original one. Hence, after a reordering of columns and rows (which does not affect the rank) in (13), we obtain a block-diagonal matrix of the form

$$M'' = \begin{pmatrix}
\tilde{A}_{\mathcal{B}_1} W_{\mathcal{B}_1} \tilde{A}_{\mathcal{B}_1}^{\mathrm{T}} & \tilde{A}_{\overline{\mathcal{G}}_1} & 0 & 0 & \dots & 0 & 0 & 0 \\
\tilde{A}_{\overline{\mathcal{G}}_1}^{\mathrm{T}} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \tilde{A}_{\mathcal{B}_2} W_{\mathcal{B}_2} \tilde{A}_{\mathcal{B}_2}^{\mathrm{T}} & \tilde{A}_{\overline{\mathcal{G}}_2} & \dots & 0 & 0 & 0 \\
0 & 0 & \tilde{A}_{\overline{\mathcal{G}}_2}^{\mathrm{T}} & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \tilde{A}_{\mathcal{B}_{k_{gb}}} W_{\mathcal{B}_{k_{gb}}} \tilde{A}_{\mathcal{B}_{k_{gb}}}^{\mathrm{T}} & \tilde{A}_{\overline{\mathcal{G}}_{k_{gb}}} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{A}_{\overline{\mathcal{G}}_{k_{gb}}}^{\mathrm{T}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

where $W_{\mathcal{B}_j}$ stands for the (diagonal) matrix of weights in blue branches of the j-th green/blue component, and $\tilde{A}_{\mathcal{B}_j}, \tilde{A}_{\overline{\mathcal{G}}_j}$ are incidence submatrices of that component. The last vanishing rows and columns correspond to red nodes, namely, nodes which are not incident with the green/blue subgraph.

Remark that, if the j-th component does not include the circuit reference node, we can choose a reference node in that component and therefore remove a row and the corresponding column of the j-th block without affecting the rank. Denote by $\hat{A}_{\mathcal{B}_j}, \hat{A}_{\overline{\mathcal{G}}_j}$ the corresponding reduced incidence matrices; for the block including the reference node (if it is not red) set $\hat{A}_{\mathcal{B}_j} = \tilde{A}_{\mathcal{B}_j}, \hat{A}_{\overline{\mathcal{G}}_j} = \tilde{A}_{\overline{\mathcal{G}}_j}$. This means that every block (except the one including the reference node, if it is not red) contributes in one to the corank of M''. Together with the contribution to the corank coming from the last vanishing rows and columns, and using the fact that the number of components of the green/blue subgraph plus the number of red nodes exceeds by one the number of independent red cutsets, we derive the relation $\operatorname{cork} M'' = \operatorname{cork} M''' + x_r$, with

$$M''' = \begin{pmatrix} \hat{A}_{\mathcal{B}_1} W_{\mathcal{B}_1} \hat{A}^{\mathrm{T}}_{\mathcal{B}_1} & \hat{A}_{\overline{\mathcal{G}}_1} & 0 & 0 & \dots & 0 & 0 \\ \hat{A}^{\mathrm{T}}_{\overline{\mathcal{G}}_1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \hat{A}_{\mathcal{B}_2} W_{\mathcal{B}_2} \hat{A}^{\mathrm{T}}_{\mathcal{B}_2} & \hat{A}_{\overline{\mathcal{G}}_2} & \dots & 0 & 0 \\ 0 & 0 & \hat{A}^{\mathrm{T}}_{\overline{\mathcal{G}}_2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \hat{A}_{\mathcal{B}_{k_{gb}}} W_{\mathcal{B}_{k_{gb}}} \hat{A}^{\mathrm{T}}_{\mathcal{B}_{k_{gb}}} & \hat{A}_{\overline{\mathcal{G}}_{k_{gb}}} \\ 0 & 0 & 0 & 0 & 0 & \hat{A}^{\mathrm{T}}_{\overline{\mathcal{G}}_{k_{gb}}} & 0 \end{pmatrix}.$$

Together with $\operatorname{cork} M'' = \operatorname{cork} M'$ and (14), this yields

$$\operatorname{cork} M = \operatorname{cork} M''' + x_g + x_r. \tag{15}$$

3. The matrix M''' is nonsingular (i.e. has a corank of zero) if and only if the blocks

$$\begin{pmatrix} \hat{A}_{\mathcal{B}_j} W_{\mathcal{B}_j} \hat{A}^{\mathrm{T}}_{\mathcal{B}_j} & \hat{A}_{\overline{\mathcal{G}}_j} \\ \hat{A}^{\mathrm{T}}_{\overline{\mathcal{G}}_j} & 0 \end{pmatrix} \tag{16}$$

are nonsingular for $j = 1, \dots, k_{gb}$. Note that the digraph in item 1 above has been constructed in a way such that each one of these blocks defines the augmented nodal matrix of a connected component which does not include green loops or red cutsets. According to Theorem 3, the nonsingularity of each block of the form (16) is then equivalent to the non-vanishing of the sum of b-weights in proper trees of that component.

4. It then remains to be shown that the condition arising in item 3 is equivalent to the non-vanishing of the sum of b-weights in normal trees of the original digraph.

4.1. Due to the structure of normal trees presented in Theorem 2, the sum of products of the weights of blue branches in normal trees can be written as

$$t_r \sum_{F \in \mathcal{F}_{gb}} \prod_{m_i \in F \cap \mathcal{B}} w_i. \tag{17}$$

Here t_r is the number of trees in the red-cut contraction (if there are no red cutsets set $t_r = 1$, green/blue forests amounting in this case to trees), whereas $\mathcal{F}_{gb}$ is the set of normal forests of the green/blue subgraph, and $F \cap \mathcal{B}$ is the set of blue branches in the forest F.

4.2. These normal forests are defined by all possible combinations of normal trees coming from different components of the green/blue subgraph. This means that (17) can be recast as

$$t_r \prod_{C_j \in \mathcal{C}} \sum_{T \in \mathcal{T}_j} \prod_{m_i \in T \cap \mathcal{B}} w_i, \tag{18}$$

$\mathcal{C}$ being the set of connected components of the green/blue subgraph, and $\mathcal{T}_j$ the set of normal trees in its j-th component.

4.3. Additionally, according to Theorem 1, the normal trees of every green/blue component are defined by all possible combinations of a forest of the corresponding green subgraph and a tree of the blue-cut minor coming from that component. This means that

$$\sum_{T \in \mathcal{T}_j} \prod_{m_i \in T \cap \mathcal{B}} w_i = (f_g)_j \sum_{T \in \mathcal{T}_j^b} \prod_{m_i \in T} w_i, \tag{19}$$

where $(f_g)_j$ is the number of green forests in the j-th component (if there are no green branches in that component set $(f_g)_j = 1$) and $\mathcal{T}_j^b$ is the set of trees in the blue-cut minor of the j-th component.

From (19) we can rewrite (18) as

$$t_r \prod_{C_j \in \mathcal{C}} \left((f_g)_j \sum_{T \in \mathcal{T}_j^b} \prod_{m_i \in T} w_i \right) = t_r f_g \prod_{C_j \in \mathcal{C}} \sum_{T \in \mathcal{T}_j^b} \prod_{m_i \in T} w_i, \tag{20}$$

f_g standing for the total number of forests in the green subgraph (in the absence of green branches in the digraph set $f_g = 1$).

4.4. Since t_r and f_g are positive integers, the non-vanishing of (20) is equivalent to the non-vanishing of the sum of tree weights in the blue-cut minor of each component of the green/blue subgraph. However, by construction, the trees in this blue-cut minor correspond to the blue branches in the proper trees referred to in item 3 above. This means that, indeed, the nonsingularity of M''' is equivalent to the non-vanishing of the sum of b-weights in normal trees. In the light of (15), this completes the proof. □

4. Controlled branches and coupling effects

The results so far are focused on problems in which the matrix $W_{\mathcal{B}}$ within M (cf. (1)) is diagonal. However, many important applications in circuit theory lead to problems which can be modeled by augmented nodal matrices of the form (1) with a non-diagonal structure in $W_{\mathcal{B}}$. This is the case in circuits including certain types of controlled sources, in particular those arising in most transistor models, and also in the presence of coupling effects.

In this Section we extend some of the previous results to digraphs with a non-diagonal matrix $W_{\mathcal{B}}$. We do not attempt to accommodate in our analysis all possible non-diagonal structures for $W_{\mathcal{B}}$; instead, the attention will be focused on certain structures which allow for a reasonably simple rank characterization in terms of trees and, at the same time, lead to relevant applications in circuit theory: see, in this regard, Sections 5.4 and 5.5.

4.1. Controlled branches

Non-diagonal matrices $W_{\mathcal{B}}$ often come in practice from electrical circuits including controlled sources for which the controlling device is modeled by a blue branch. In digraph terms, we will now classify blue branches in two classes, as follows. The blue branches of Sections 2 and 3 will now be *dark blue* branches, having non-zero weights w_{ii} ($i = 1, \ldots, m_d$). Additionally, there will be a second class of so-called *light blue* branches, with $w_{ii} = 0$ ($i = m_d + 1, \ldots, m_b$), each one of which will be controlled by a dark blue branch; this means that for every $i \in \{m_d + 1, \ldots, m_b\}$ there exists a unique $j \in \{1, \ldots, m_d\}$ such that $w_{ij} \neq 0$. The parameter w_{ii} of dark blue branches will be termed the *branch weight*; in turn, $w_{ij} \neq 0$ is the *control parameter* of the i-th branch, the j-th one being its controlling branch. This framework may model for instance electrical circuits in which dark blue branches are resistors and light blue branches are voltage-controlled current sources in which the controlling device is a resistor.

The setting described above gives $W_{\mathcal{B}}$ the structure

$$W_{\mathcal{B}} = \begin{pmatrix} W_1 & 0 \\ W_2 & 0 \end{pmatrix}, \tag{21}$$

where $W_1 \in \mathbb{R}^{m_d \times m_d}$ is diagonal and $W_2 \in \mathbb{R}^{m_l \times m_d}$ has exactly one non-vanishing entry per row; here $m_l = m_b - m_d$ stands for the number of light blue branches.

The results of previous Sections can be extended to this context by means of the concept of a *balanced tree*. In order to introduce this notion, let us classify the blue branches into mutually-disjoint groups, each one defined by a unique dark blue branch together with the set of light blue branches that it controls, if any; this dark blue branch is called the *representative* of all the branches in its group, including itself. Let us also recall that that a proper tree is a tree which contains all green branches, (possibly) some blue ones –which can now be dark or light ones–, but no red branches.

Definition 3. A *balanced tree* is a proper tree which satisfies the following:

(1) it contains no more than one blue twig from each one of the mutually-disjoint groups defined above; and
(2) it is such that the removal of all light blue twigs and the inclusion of their representatives still result in a (proper) tree.

We will call the tree without light blue branches arising in item (2) the *associated tree* of the original balanced one.

The *b-weight of a balanced tree* is defined by the product of

(a) the branch weights of the dark blue twigs;
(b) the control parameters of the light blue twigs; and
(c) the signature of the tree, defined as $+1$ or -1 if $\det A_{T_1} = \det A_{T_2}$ or $\det A_{T_1} \neq \det A_{T_2}$, respectively; here A_{T_1} and A_{T_2} are the incidence matrices of the balanced tree and its associated one, and the columns in A_{T_1} are assumed to be ordered according to the order of the representatives in A_{T_2}.

By definition, any proper tree in which all blue branches are dark is balanced and coincides with its associated tree. Furthermore, its signature is $+1$ and its b-weight is simply defined by the product of its branch weights, as in Sections 2 and 3. These trees play the role of the proper trees in previous Sections, but in the rank characterization of Theorem 5 below there is an additional contribution coming from balanced trees which include light blue branches.

Theorem 5. *For a connected digraph with controlled branches in which $W_{\mathcal{B}}$ has the structure depicted in (21), the matrix M in (1) is nonsingular if and only if*

(i) *there are neither cutsets defined by red and/or light blue branches, nor green loops, and*
(ii) *the sum of b-weights in balanced trees does not vanish.*

Proof. Regarding item (i), it is only worth clarifying that the presence of at least one balanced tree precludes the existence of cutsets defined by red and/or light blue branches (besides green loops), since the associated tree of that balanced one just includes dark blue branches (apart from all the green ones).

For item (ii) we proceed as in the proof of Theorem 3. The non-trivial determinants in the Cauchy–Binet expansion come from submatrices of the form

$$D^{\omega,\alpha} = \begin{pmatrix} A_{\hat{\mathcal{B}}} & A_{\mathcal{G}} & 0 \\ 0 & 0 & I_{\mathcal{G}} \end{pmatrix}, \qquad E^{\alpha,\beta} = \begin{pmatrix} \tilde{W} & 0 & 0 \\ 0 & 0 & I_{\mathcal{G}} \\ 0 & I_{\mathcal{G}} & 0 \end{pmatrix}, \qquad F^{\beta,\omega} = \begin{pmatrix} A^{\mathrm{T}}_{\check{\mathcal{B}}} & 0A^{\mathrm{T}}_{\mathcal{G}} & 0 \\ 0 & I_{\mathcal{G}} & \end{pmatrix}. \tag{22}$$

Note that now the sets $\hat{\mathcal{B}}$ and $\check{\mathcal{B}}$ of blue branches entering $D^{\omega,\alpha}$ and $F^{\beta,\omega}$ need not be the same, and that $\tilde{W}$ will be non-diagonal in many cases.

For $\det \tilde{W}$ not to vanish, it is easy to see that no more than one blue branch from each group may enter the tree T_1 defined by $\hat{\mathcal{B}}$ and the green branches, and that the corresponding representatives must enter T_2 (defined by $\check{\mathcal{B}}$ and the green branches). This means that the only tree pairs leading to non-vanishing terms in the expansion are defined by balanced trees and their associated ones; these include, of course, the cases in which $\hat{\mathcal{B}} = \check{\mathcal{B}}$, that is, the pairs of coincident trees in which all blue twigs are dark.

In cases in which T_1 includes at least one light blue twig (ie when $\hat{\mathcal{B}} \neq \check{\mathcal{B}}$), reordering the rows in $\tilde{W}$ and, correspondingly, the columns in $(A_{\hat{B}}\ A_{\mathcal{G}})$ we do not alter the sign of the determinantal products; once the branches in the balanced tree are ordered according to their representatives, $\tilde{W}$ gets a diagonal form and its determinant equals the product of branch weights and control parameters. The result then follows from the fact that the determinantal product of the incidence matrices coming from $D^{\omega,\alpha}$ and $F^{\beta,\omega}$ yields the signature of the balanced tree as defined above. □

For brevity we do not address the analogue of Theorem 4 in this setting, although the reader can derive it without difficulty just redefining balanced trees in terms of normal ones.

4.2. Coupled problems

Another family of problems with non-diagonal matrices $W_{\mathcal{B}}$ come from electrical circuits displaying coupling effects among the devices modeled by blue branches, which lead to non-trivial entries away from the main diagonal in $W_{\mathcal{B}}$. In digraph terms, a non-vanishing value w_{ij} in the (i, j)-th entry of $W_{\mathcal{B}}$, with $i \neq j$, may be seen as a weight which describes the influence of the j-th branch on the i-th one. It may certainly happen that also the i-th branch has a non-trivial influence on the j-th one, namely, that $w_{ji} \neq 0$, possibly with $w_{ij} \neq w_{ji}$. In any of these situations, the i-th and j-th branches are said to be coupled; w_{ij} and w_{ji} are called the *coupling coefficients* between the i-th and the j-th branches. This setting does not imply anything about the values of w_{ii} and w_{jj}, which will still be referred to as the *branch weights* and may vanish or not.

We will assume in the sequel that every blue branch is coupled to at most another one. Without loss of generality coupled branches will be supposed to be the first c ones and to be numbered consecutively, so that $W_{\mathcal{B}}$ has the form

$$W_{\mathcal{B}} = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ w_{21} & w_{22} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & w_{33} & w_{34} & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & w_{43} & w_{44} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & w_{c+1\,c+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & w_{c+2\,c+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & w_{m_b m_b} \end{pmatrix}. \tag{23}$$

By the weight of a pair of coupled branches (say the i-th and the $(i+1)$-th ones) we will mean the determinant $w_{i\,i}w_{i+1\,i+1} - w_{i\,i+1}w_{i+1\,i}$.

Theorem 3 can be extended to augmented nodal matrices M in which $W_{\mathcal{B}}$ has the form depicted in (23), as detailed in Theorem 6 below. Now, in the sum arising in item (ii), there is a new term involving certain (so-called *regular*) tree pairs which are intrinsic to the existence of coupled branches. Note also that, in the contribution of proper trees to that sum, the b-weight is defined as the determinant of the submatrix of $W_{\mathcal{B}}$ defined by the blue twigs, and now it equals the product of the weights of pairs of coupled twigs times the product of the branch weights of the twigs which are not coupled to other twigs (regardless of the fact that some of them may be coupled to links).

Definition 4. A pair $\{T_1, T_2\}$ of proper trees is said to be *regular* if their sets of blue twigs can be described as $\mathcal{B}_0 \cup \mathcal{B}_1$ and $\mathcal{B}_0 \cup \mathcal{B}_2$, respectively, and satisfy the following:

(1) $\mathcal{B}_1$ and $\mathcal{B}_2$ are not empty;
(2) every branch in $\mathcal{B}_1$ is coupled to one branch of $\mathcal{B}_2$; and
(3) no branch of $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is coupled to any other blue branch of T_1 (resp. T_2).

This way, the regular pairs $\{T_1, T_2\}$ are defined by all possible distributions of (one or more pairs of) coupled branches between T_1 and T_2 in a way such that there exists a common set of blue branches which (together with the set of green branches) yield a pair of proper trees.

Now, the *b-weight of a regular pair of trees* $\{T_1, T_2\}$ is defined as the product of

(a) the weight of the coupled pairs in $\mathcal{B}_0$;
(b) the branch weights of the twigs in $\mathcal{B}_0$ which are not coupled to other branches in $\mathcal{B}_0$;
(c) the coupling coefficients w_{ij} between the coupled branches from $\mathcal{B}_1$ and $\mathcal{B}_2$, in the understanding that the i-th branch belongs to $\mathcal{B}_1$ and the j-th one belongs to $\mathcal{B}_2$; and
(d) the signature of $\{T_1, T_2\}$, defined as $+1$ or -1 if $\det A_{T_1} = \det A_{T_2}$ or $\det A_{T_1} \neq \det A_{T_2}$, respectively.

Theorem 6. *For a connected digraph in which every blue branch is coupled to at most another one, the matrix M in (1) is nonsingular if and only if*

(i) *there are neither red cutsets nor green loops, and*
(ii) *the sum of b-weights in proper trees and regular tree pairs does not vanish.*

Proof. The result again relies on the Cauchy–Binet expansion, which leads to factors of the form shown in (22). When $\hat{\mathcal{B}} = \check{\mathcal{B}}$ we are led to the proper trees arising in item (ii), where it is worth emphasizing that now their b-weights $\det \tilde{W}$ may include some factors of the form $w_{i\,i}w_{i+1\,i+1} - w_{i\,i+1}w_{i+1\,i}$ coming from pairs of coupled twigs, as indicated above.

The cases $\hat{\mathcal{B}} \neq \check{\mathcal{B}}$ lead to the contribution coming from the regular tree pairs defined above. This is a consequence of the form of $W_{\mathcal{B}}$ in (23) and can be checked by means of the following remarks.

First, for a given pair of coupled branches, if both of them belong to $\hat{\mathcal{B}}$ then they must also enter $\check{\mathcal{B}}$ (if none of them do, there would be two vanishing rows in $\tilde{W}$, and if only of them enters $\hat{\mathcal{B}}$, $\tilde{W}$ would have two linearly dependent rows). These branches belong to $\mathcal{B}_0$ in the notion of a regular pair and their weights contribute through item (a) above.

If exactly one branch of a coupled pair belongs to $\hat{\mathcal{B}}$, then it must happen that either this branch or the other one in the pair (but not both) must belong to $\check{\mathcal{B}}$; the absence of both would result in a vanishing row in $\tilde{W}$, whereas if both of them enter $\check{\mathcal{B}}$ then $\tilde{W}$ would have two linearly dependent columns. In the first case the branch belongs to $\mathcal{B}_0$ and contributes to the b-weight through item (b), whereas in the second one the coupled branches enter $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, contributing to the b-weight through the coefficient w_{ij} in item (c). Note that at least one coupled pair must meet the second case in order to make $\hat{\mathcal{B}} \neq \check{\mathcal{B}}$; otherwise we would be led to a term in the sum coming from a proper tree.

If no branch of a given coupled pair enters $\hat{\mathcal{B}}$, none of them can appear in $\check{\mathcal{B}}$ either since they would yield vanishing columns in $\tilde{W}$.

Additionally, if an uncoupled branch enters $\hat{\mathcal{B}}$ then it must also belong to $\check{\mathcal{B}}$ since, otherwise, $\tilde{W}$ would have a vanishing row. These branches also belong to $\mathcal{B}_0$ and contribute to the b-weight via item (b). Similarly, if an uncoupled branch does not belong to $\hat{\mathcal{B}}$ then it cannot be displayed in $\check{\mathcal{B}}$ either, in order to prevent vanishing columns.

Finally, the product $\det D^{\omega,\alpha} \det F^{\beta,\omega}$ yields the contribution coming from the signature of the pair in item (d). □

Again, for the sake of brevity we do not tackle the analogue of Theorem 4 in this context, although it essentially parallelizes Theorem 6; we just need to replace proper trees by normal ones (also in the notion of a regular tree pair) and the result follows along the same lines.

5. Applications in electrical circuit theory

The results presented in Sections 2–4 make it possible to tackle different analytical properties of non-passive lumped circuits. Specifically, after introducing some background on nodal analysis of electrical circuits in Section 5.1, we will address the DC-solvability problem in Section 5.2 and the index characterization of different nodal circuit models in Section 5.3, working in both cases without passivity restrictions and in an uncoupled setting without controlled sources. In Sections 5.4 and 5.5 we analyze the corresponding properties for circuits with controlled sources and coupling effects, respectively.

The fact that different applications involving electrical circuits can be discussed in the terms of Sections 2–4 relies on different assignments of green/blue/red branches to circuit devices. Conductances and, occasionally, capacitances will define the weights of blue branches. In uncoupled problems, the passivity conditions amounts to the requirement that the individual conductances, inductances and capacitances are positive. The goal is to assess properties which are known to hold for passive problems in a more general setting, by allowing these weights to take on negative values.

We will assume throughout this Section that the circuits under study are *well-posed*, in the sense that they do not include V-loops (i.e. loops formed exclusively by voltage sources) or I-cutsets (cutsets defined only by current sources). This is (generically) a necessary requirement for solutions to exist.

5.1. Nodal analysis of lumped circuits

Analysis methods currently used to set up circuit equations use a *semistate* formalism, which allows some variable redundancy in order to automatically generate a network model. In the time-domain, this approach leads to models based on *differential–algebraic equations* (DAEs). This is the case of Augmented Nodal Analysis (ANA) [30,44] or Modified Nodal Analysis (MNA), used in actual circuit simulation programs such as SPICE or TITAN [17,22,23,47].

Augmented Nodal Analysis. ANA models are defined by a differential–algebraic system of the form

$$C(v_c)v_c' = i_c \tag{24a}$$

$$L(i_l)i_l' = A_{\mathcal{L}}^{\mathrm{T}} e \tag{24b}$$

$$0 = A_{\mathcal{R}}\gamma(A_{\mathcal{R}}^{\mathrm{T}} e) + A_{\mathcal{C}} i_c + A_{\mathcal{L}} i_l + A_{\mathcal{V}} i_v + A_{\mathcal{I}} i_s(t) \tag{24c}$$

$$0 = A_{\mathcal{C}}^{\mathrm{T}} e - v_c \tag{24d}$$

$$0 = A_{\mathcal{V}}^{\mathrm{T}} e - v_s(t). \tag{24e}$$

We are assuming the circuit to be connected. The matrices $A_{\mathcal{R}}, A_{\mathcal{C}}, A_{\mathcal{L}}, A_{\mathcal{V}}$ and $A_{\mathcal{I}}$ describe the incidence relations in resistive, capacitive, inductive, voltage source and current source branches, respectively. Eq. (24c) describes the Kirchhoff Current Law (KCL) in the form $Ai = 0$, having expressed resistor currents in terms of branch voltages and these in turn in terms of node voltages. In Eqs. (24b)–(24e), we have made use of the formulation of Kirchhoff Voltage Law (KVL) as $v = A^{\mathrm{T}} e$. The variables arising in the model are then defined by the node voltages e, the capacitor voltages v_c and currents i_c, the inductor currents i_l, and the voltage source currents i_v. The functions $i_s(t)$ and $v_s(t)$ describe the currents and voltages in the (assumed independent) corresponding sources. Controlled sources can be included in this model as explained in Section 5.4.

The resistors are voltage-controlled by a C^1 relation of the form $i_r = \gamma(v_r) = \gamma(A_{\mathcal{R}}^{\mathrm{T}} e)$. The incremental conductance matrix will be denoted by $G(v_r) = \gamma'(v_r)$, whereas capacitors and inductors are characterized by their incremental capacitance and inductance matrices $C(v_c)$ and $L(i_l)$, respectively. In the sequel we will often omit the label 'incremental' to refer to the conductances, capacitances and inductances. When the conductance, capacitance and inductance matrices are diagonal the corresponding devices are said to be *uncoupled*. Additionally, the circuit is said to be *strictly locally passive* (or, with terminological abuse, sometimes simply 'passive') if $G(v_r)$, $C(v_c)$ and $L(i_l)$ are positive definite; a square matrix $M \in \mathbb{R}^{p\times p}$ is said to be positive definite if $u^{\mathrm{T}} M u > 0$ for every $u \in \mathbb{R}^p - \{0\}$.

Modified Nodal Analysis. MNA models can be derived from (24) by inserting (24d) into (24a), and this one in turn into (24c), leading to the DAE

$$A_{\mathcal{C}} C(A_{\mathcal{C}}^{\mathrm{T}} e) A_{\mathcal{C}}^{\mathrm{T}} e' + A_{\mathcal{R}}\gamma(A_{\mathcal{R}}^{\mathrm{T}} e) + A_{\mathcal{L}} i_l + A_{\mathcal{V}} i_v = -A_{\mathcal{I}} i_s(t) \tag{25a}$$

$$L(i_l)i_l' - A_{\mathcal{L}}^{\mathrm{T}} e = 0 \tag{25b}$$

$$A_{\mathcal{V}}^{\mathrm{T}} e = v_s(t). \tag{25c}$$

The elimination of capacitive voltages and currents results in a more compact model, better-suited than ANA for computational purposes. The presence of the nodal capacitance matrix $A_{\mathcal{C}} C(A_{\mathcal{C}}^{\mathrm{T}} e) A_{\mathcal{C}}^{\mathrm{T}}$ in the leading term of the system leads, however, to several technical difficulties not displayed by ANA (cf. [17,41,44,47]).

5.2. DC-solvability and L-proper trees

Assume that a given circuit is driven by DC sources, namely, that the excitations in the sources have the form $i_s(t) = I_s$ and $v_s(t) = V_s$ for certain constant vectors I_s, V_s. Let us denote by $f(x)$ the map displayed in the right-hand side of (24), namely

$$f(x) = \begin{pmatrix} i_c \\ A_{\mathcal{L}}^{\mathrm{T}} e \\ A_{\mathcal{R}}\gamma(A_{\mathcal{R}}^{\mathrm{T}} e) + A_{\mathcal{C}} i_c + A_{\mathcal{L}} i_l + A_{\mathcal{V}} i_v + A_{\mathcal{I}} I_s \\ A_{\mathcal{C}}^{\mathrm{T}} e - v_c \\ A_{\mathcal{V}}^{\mathrm{T}} e - V_s \end{pmatrix}. \tag{26}$$

A set of values for $x = (v_c, i_l, e, i_c, i_v)$ which annihilates $f(x)$ is then said to define an *equilibrium point* of the circuit. Note that the equilibrium condition requires $i_c = 0$ and $A_{\mathcal{L}}^{\mathrm{T}} e = v_l = 0$, and therefore these conditions define a *DC operating point* of the network, that is, a solution of the resistive network obtained after open-circuiting capacitors and short-circuiting inductors. Except for the conditions $i_c = 0$ and $v_c = A_{\mathcal{C}}^{\mathrm{T}} e$, which involve capacitive currents and voltages not used in Modified Nodal Analysis, the same set of requirements defines an equilibrium point in the MNA model (25).

Fix in the sequel an equilibrium point x^*, that is, a point for which $f(x^*) = 0$, and let $f'(x^*)$ stand for the Jacobian matrix (i.e. the matrix of partial derivatives) of f evaluated at x^*. The nonsingularity of $f'(x^*)$ implies that x^* is an isolated equilibrium point (in linear cases, the equilibrium would actually be unique) and, from a computational perspective, is a key requirement for its determination in nonlinear problems via Newton-based methods; for this reason the nonsingularity of $f'(x^*)$ (or, equivalently, that of the Jacobian matrix arising in other circuit models) is often referred to as a *DC-solvability condition* [18]. See also [13,20,21,43,45,46] and the bibliography therein.

It is already known that the DC-solvability requirement implies, as a necessary condition, the absence of VL-loops (that is, loops defined by inductors and (maybe) voltage sources) and IC-cutsets (cutsets formed by capacitors and (maybe) current sources). This topological check is performed by current circuit simulators [18], the role of these configurations having been already examined in [24,25,34]. In strictly locally passive circuits, the absence of VL-loops and IC-cutsets is actually enough to guarantee DC-solvability (cf. Theorem 4.1 in [43]).

We are now in a position to improve on this result, extending the characterization of DC-solvability to uncoupled, non-passive circuits. Indeed, the Jacobian matrix at an equilibrium point $x^* = (v_c^*, i_l^*, e^*, i_c^*, i_v^*)$ reads

$$f'(x^*) = \begin{pmatrix} 0 & 0 & 0 & I_c & 0 \\ 0 & 0 & A_{\mathcal{L}}^{\mathrm{T}} & 0 & 0 \\ 0 & A_{\mathcal{L}} & A_{\mathcal{R}} G A_{\mathcal{R}}^{\mathrm{T}} & A_{\mathcal{C}} & A_{\mathcal{V}} \\ -I_c & 0 & A_{\mathcal{C}}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & A_{\mathcal{V}}^{\mathrm{T}} & 0 & 0 \end{pmatrix}, \tag{27}$$

where I_c is an identity matrix whose size is given by the number of capacitors and G stands for $\gamma'(A_{\mathcal{R}}^{\mathrm{T}} e^*)$. The nonsingularity of $f'(x^*)$ is easily seen to be equivalent to that of the augmented nodal matrix

$$\begin{pmatrix} A_{\mathcal{R}} G A_{\mathcal{R}}^{\mathrm{T}} & A_{\mathcal{L}} & A_{\mathcal{V}} \\ A_{\mathcal{L}}^{\mathrm{T}} & 0 & 0 \\ A_{\mathcal{V}}^{\mathrm{T}} & 0 & 0 \end{pmatrix}. \tag{28}$$

Inspired in the notion of an *L-normal tree* [37], let us define a tree as *L-proper* if it contains all voltage sources and all inductors, possibly some resistors, but neither current sources nor capacitors. Assign blue branches to resistors, their conductances defining the weights of these branches, green branches to voltage sources and inductors, and red branches to current sources and capacitors. The following result, which extends the DC-solvability characterization to non-passive circuits without coupling among resistors, then follows in a straightforward manner from Theorem 3.

Proposition 1. *The DC-solvability condition holds at an equilibrium point of a well-posed, connected circuit without resistive coupling if and only if*

(i) *the circuit has neither IC-cutsets nor VL-loops, and*
(ii) *the sum of conductance products in L-proper trees does not vanish.*

Note that in uncoupled, strictly locally passive circuits the incremental conductances are positive and thus the sum of conductance products in item (ii) above is always positive. Therefore, in the uncoupled setting the characterization of DC-solvability for passive circuits in terms of the absence of VL-loops and IC-cutsets can be seen as a corollary of Proposition 1 above. The advantage of the present result relies on the fact that it accommodates, for instance, negative differential resistances (NDR's) arising e.g. in tunnel diodes or in Chua's circuit family. In Proposition 6 (Section 5.4) we will address the analogue of this result for problems with certain types of controlled sources which arise in other circuit applications, involving e.g. transistor models.

5.3. Index analysis of DAE circuit models

The systems (24) and (25) arising in nodal analysis define *differential–algebraic* models of lumped electrical circuits. Stemming from the seminal work of Newcomb and others (cf. [36]), the DAE or semistate approach is nowadays pervasive in time-domain circuit modeling and simulation [17,22,23,39,42,44,47,48,50].

A major issue in the characterization of semistate circuit models involve their *index*; cf. [7,26,29,31,32,38,42] for extensive discussions of this notion for general DAEs and [17,40,41,44,47,48] for index analyses of passive circuits. Roughly speaking, the index measures how much a given DAE differs from an explicit ODE, and reflects the sensitivity of the system to the derivatives of the excitation terms. It also provides a measure of the difficulties expected in the numerical treatment of the DAE.

Much research on the index characterization of lumped circuits is based on the so-called *tractability* index, supported on projector methods (see [31,32,42,48]). Focusing on quasilinear DAEs of the form $A(x)x' = f(x,t)$, the index is said to be zero if $A(x)$ is a nonsingular matrix; if it is not, provided that it has a constant kernel and letting Q be a constant projector onto $\ker A(x)$, the DAE is said to have a tractability index of one if $A_1(x) = A(x) - f_x(x,t)Q$ is non-singular; we are using the fact that for the circuit models (24) and (25) the derivative $f_x(x,t)$ is actually independent of t, so that A_1 only depends on x. If $A_1(x)$ is singular with constant rank and $Q_1(x)$ is a continuous projector onto its kernel, the DAE is said to have a tractability index of two if $A_2(x) = A_1(x) - f_x(x,t)(I-Q)Q_1(x)$ is nonsingular. These notions support not only analytical characterizations of the system solutions but also numerical techniques (cf. [17,32,33,42,47,48]).

For strictly locally passive circuits, the index behavior of the MNA system (25) is known to depend crucially on the presence or absence of VC-loops (loops defined by capacitors and (maybe) voltage sources) and IL-cutsets (cutsets formed by inductors and (maybe) current sources) [47]. This is also the case for the ANA model (24) [41,42,44]. MNA systems are index zero when modeling a passive circuit with a capacitive tree and without voltage sources [48]. However, the index of non-passive circuits is largely unexplored in the literature. In [16], low (≤ 1) index characterizations are presented for uncoupled circuits without passivity requirements. In the forthcoming subsections we show how different properties discussed in [16] can be seen as a consequence of the general results of previous Sections; we also present some new results aimed at the analysis of index-two configurations in non-passive circuits.

5.3.1. ANA models and C-normal trees

Assume that in the Augmented Nodal Analysis model (24) the capacitance and inductance matrices C, L are nonsingular. Since the leading matrix $A(x)$ of this system has the block-diagonal structure $\{C, L, 0, 0, 0\}$, a constant projector Q onto $\ker A(x)$ is defined by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n-1} & 0 & 0 \\ 0 & 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & 0 & I_v \end{pmatrix}, \tag{29}$$

the sizes of the identity submatrices being given by the number of nodes different from the reference one, of capacitors and of voltage sources, respectively. The matrix $A_1 = A - f_x Q$ reads

$$A_1 = \begin{pmatrix} C & 0 & 0 & -I_c & 0 \\ 0 & L & -A_{\mathcal{L}}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & -A_{\mathcal{R}} G A_{\mathcal{R}}^{\mathrm{T}} & -A_{\mathcal{C}} & -A_{\mathcal{V}} \\ 0 & 0 & -A_{\mathcal{C}}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & -A_{\mathcal{V}}^{\mathrm{T}} & 0 & 0 \end{pmatrix}. \tag{30}$$

Due to the nonsingularity of C and L, the matrix A_1 is nonsingular if and only if so it is

$$J_1 = \begin{pmatrix} A_{\mathcal{R}} G A_{\mathcal{R}}^{\mathrm{T}} & A_{\mathcal{C}} & A_{\mathcal{V}} \\ A_{\mathcal{C}}^{\mathrm{T}} & 0 & 0 \\ A_{\mathcal{V}}^{\mathrm{T}} & 0 & 0 \end{pmatrix}. \tag{31}$$

Define a tree as *C-proper* if it contains all voltage sources and all capacitors, possibly some resistors, but neither current sources nor inductors. This concept corresponds to the original use of the term 'proper tree' in [5]. Assigning now blue branches to resistors, green branches to voltage sources and capacitors, and red branches to current sources and inductors, we may consider Proposition 2 below (cf. [16]) as a corollary of Theorem 3.

Proposition 2. *Consider a well-posed, connected circuit with nonsingular capacitance and inductance matrices C, L, and no coupling among resistors. The ANA model (24) has tractability index one if and only if*

(i) *the circuit has neither IL-cutsets nor VC-loops, and*
(ii) *the sum of conductance products in C-proper trees does not vanish.*

As in Section 5.2, Proposition 2 allows one to include devices with negative incremental resistance in the index analysis. Note also that the parallelism between Propositions 1 and 2 extends to non-passive settings the reactive duality explored for passive circuits in [41]. Again, Proposition 7 in Section 5.4 extends this characterization to problems with controlled sources.

The existence of IL-cutsets and/or VC-loops may lead to an index two configuration in the ANA model (24). Even though a complete index two characterization of ANA models for non-passive circuits is beyond the scope of this paper, a key requirement in this direction is that the identity

$$\ker J_1 = \ker \begin{pmatrix} A_{\mathcal{R}}^{\mathrm{T}} \\ A_{\mathcal{C}}^{\mathrm{T}} \\ A_{\mathcal{V}}^{\mathrm{T}} \end{pmatrix} \times \ker\, (A_{\mathcal{C}}\; A_{\mathcal{V}}) \tag{32}$$

holds. Equivalently, the dimension of the kernel of the matrix J_1 in (31) (or, equivalently, that of the kernel of A_1 in (30)) must not exceed the number of independent IL-cutsets plus the number of independent VC-loops. In this regard, the following result can be derived from Theorem 4 if we define a tree as *C-normal* when it contains all voltage sources, no current source, the maximum possible number of capacitors and the minimum possible number of inductors, together with (possibly) some resistors. This is the type of tree used by Bryant [10,11] and corresponds to the common use of the term 'normal tree' in electrical circuit theory (cf. [28]).

Proposition 3. *Consider a well-posed, connected circuit with nonsingular capacitance and inductance matrices C, L, and no coupling among resistors. In the presence of IL-cutsets and/or VC-loops, the corank of the matrix A_1 in (30) equals the number of independent IL-cutsets plus the number of independent VC-loops (or, equivalently, the identity (32) holds) if and only if the sum of conductance products in C-normal trees does not vanish.*

Proof. The nonsingularity of C and L guarantees that the corank of A_1 equals that of J_1. In turn, the well-posedness assumption precludes V-loops and I-cutsets in the circuit. Let us short-circuit (contract) voltage sources and open-circuit (remove) current sources, and then assign blue branches to resistors, green branches to capacitors, and red branches to inductors. It is easy to check that the C-normal trees of the original circuit correspond to the normal trees (in the sense of Definition 2) of the modified circuit. The result then follows in a straightforward manner from Theorem 4. □

5.3.2. Two-colour problems, MNA models and capacitive-normal trees

In index characterizations of Modified Nodal Analysis, an important role is played by the nodal capacitance matrix $A_{\mathcal{C}}C(A_{\mathcal{C}}^{\mathrm{T}}e)A_{\mathcal{C}}^{\mathrm{T}}$ because of its appearance in the leading term of (25). Under a positive definiteness assumption on C, the nonsingularity of this matrix is known to rely on the existence of a capacitive tree [48]; this makes the MNA model (25) index zero for strictly locally passive circuits without voltage sources. In the absence of capacitive trees, the positive definiteness requirement on C guarantees that the identity

$$\ker A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}} = \ker A_{\mathcal{C}}^{\mathrm{T}} \tag{33}$$

holds, a property which is needed in index one analyses of MNA [16,17,42,47,48].

The extension of these properties to non-passive circuits (more precisely, to circuits without the restriction of definiteness on C) has been examined in [16]. We show below that such analysis can be also derived from the results of Sections 2 and 3, by an appropriate assignment of colours to circuit devices. In our present context we will be led to a *two-colour* setting, in which capacitors are painted in blue, and all the remaining devices in red.

Proposition 4. *Consider a well-posed, connected circuit without capacitive coupling. Then the nodal capacitance matrix $A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}}$ is nonsingular if and only if*

(i) *the circuit has no VILR-cutsets (i.e. there exists at least one capacitive tree), and*
(ii) *the sum of capacitance products in capacitive trees does not vanish.*

The result above is an immediate consequence of Theorem 3. Indeed, since capacitors correspond to blue branches and all the other devices define red branches, item (i) is just a restatement of item (i) in Theorem 3 because of the fact that there are no green branches; for item (ii), notice that proper trees amount in this setting to capacitive trees. Remark also that the absence of VILR-cutsets makes the capacitive subgraph connected, which is equivalent to the existence of at least one capacitive tree. Proposition 8 in Section 5.5 extends this result to circuits with coupled capacitors.

The presence of VILR-cutsets rules out the existence of capacitive trees and makes $A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}}$ a singular matrix. In this context, the assessment of the identity depicted in (33) can be performed in terms of Theorem 4; to achieve this, let us define a *capacitive-normal* tree (not to be confused with the C-normal trees of 5.3.1) as a tree which contains as many capacitors as possible.

Proposition 5. *Consider a well-posed, connected circuit without capacitive coupling. In the absence of capacitive trees, the corank of the nodal capacitance matrix $A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}}$ equals the number of independent VILR-cutsets (or, equivalently, the identity (33) holds) if and only if the sum of capacitance products in capacitive-normal trees does not vanish.*

Proceeding as in Section 2, it is easy to check that capacitive-normal trees are defined by all possible combinations of a capacitive forest coming from the capacitive subgraph and a tree of the VILR-cut minor. Hence, the sum of capacitance products in capacitive-normal trees can be rewritten as a product (ranging over all connected components of the capacitive subgraph) of sums of capacitance products in capacitive components, times a non-vanishing factor given by the number of trees in the VILR-cut minor. This proves the equivalence of the statement in Proposition 5 with the non-degeneracy of capacitive blocks referred to in [16], showing that the results involving nodal capacitance matrices in [16] can be also derived as a two-colour version of the results discussed in Sections 2 and 3 above.

5.4. Controlled sources

The results discussed above can be extended to circuits including certain types of controlled sources by means of Theorem 5 (cf. Section 4.1). We illustrate this by addressing the analogues of Propositions 1 and 2 for problems with voltage-controlled current sources (VCCS's), which are pervasive in active circuits because of its appearance in most transistor models (see e.g. [17,46]).

The ANA model for circuits including VCCS's can be written as

$$C(v_c)v_c' = i_c \tag{34a}$$

$$L(i_l)i_l' = A_{\mathcal{L}}^{\mathrm{T}} e \tag{34b}$$

$$0 = A_{\mathcal{R}}\gamma(A_{\mathcal{R}}^{\mathrm{T}} e) + A_{\mathcal{C}} i_c + A_{\mathcal{L}} i_l + A_{\mathcal{V}} i_v + A_{\mathcal{I}} i_s(t) + A_{\mathcal{J}}\gamma_j(A_{\mathcal{R}}^{\mathrm{T}} e, v_c, v_s(t)) \tag{34c}$$

$$0 = A_{\mathcal{C}}^{\mathrm{T}} e - v_c \tag{34d}$$

$$0 = A_{\mathcal{V}}^{\mathrm{T}} e - v_s(t). \tag{34e}$$

Notice the presence in (34c) of the new term $A_{\mathcal{J}}\gamma_j(A_{\mathcal{R}}^{\mathrm{T}} e, v_c, v_s(t))$ which models the contribution of the VCCS's. The matrix $A_{\mathcal{J}}$ describes the incidence of these devices, whereas $i_j = \gamma_j(v_r, v_c, v_s(t)) = \gamma_j(A_{\mathcal{R}}^{\mathrm{T}} e, v_c, v_s(t))$ stands for the controlling relation of the sources; for later-detailed reasons, only resistors, capacitors and independent voltage sources will be allowed to act as controlling devices. As discussed below, this is not a too restrictive condition regarding real applications in circuit theory.

5.4.1. DC-solvability

The DC-solvability condition can be now characterized for circuits with VCCS's in which the controlling device is a resistor using Theorem 5. Assign dark blue branches to resistors, light blue branches to VCCS's and, as in Section 5.2, green branches to voltage sources and inductors, and red branches to current sources and capacitors. Define an *L-balanced tree* as a tree which contains all voltage sources and all inductors, neither independent current sources nor capacitors, and which satisfies the following requirements involving resistors and controlled sources: first, it does not simultaneously include as twigs a resistor and any of its controlled VCCS's, nor two (or more) VCCS's controlled by the same resistor; second, the removal of all VCCS's from the tree and the inclusion of their controlling resistors still define a tree. The *b-weight* of an L-balanced tree is defined as the product of the conductances of its resistors, the control parameters of its VCCS's, given by the non-zero entries of $K \equiv \partial\gamma_j/\partial v_r$, and its signature as defined in item (c) of Section 4.1.

Theorem 5 then yields the following DC characterization, which can be derived in a straightforward manner once we note that the block $A_{\mathcal{R}}GA_{\mathcal{R}}^{\mathrm{T}}$ in $f'(x^*)$ (cf. (27)) is now replaced by

$$A_{\mathcal{R}}GA_{\mathcal{R}}^{\mathrm{T}} + A_{\mathcal{J}}KA_{\mathcal{R}}^{\mathrm{T}} = (A_{\mathcal{R}}\ A_{\mathcal{J}})\begin{pmatrix} G & 0 \\ K & 0 \end{pmatrix}\begin{pmatrix} A_{\mathcal{R}}^{\mathrm{T}} \\ A_{\mathcal{J}}^{\mathrm{T}} \end{pmatrix} \tag{35}$$

Proposition 6. *Assume that a given well-posed, connected circuit without resistive coupling includes VCCS's, each one being controlled by exactly one resistor. The DC-solvability condition then holds at an equilibrium point if and only if*

(i) *the circuit has neither IJC-cutsets nor VL-loops, and*
(ii) *the sum of b-weights in L-balanced trees does not vanish.*

Notice that the absence of cutsets defined by red and/or light blue branches precludes the presence of *all* types of current sources (namely, independent and controlled ones) in the cutsets of item (i).

5.4.2. Index characterization of ANA models of circuits with VCCS's

Similarly, Theorem 5 makes it possible to include certain voltage-controlled current sources in the index analysis of ANA models. A *C-balanced* tree is a tree which contains all voltage sources and all capacitors, neither independent current sources nor inductors, and which satisfies the requirements involving resistors and VCCS's detailed in Section 5.4.1 above. The b-weight of these trees is defined exactly as in 5.4.1.

Proposition 7. *Assume that a well-posed, connected circuit with nonsingular capacitance and inductance matrices C, L, and no coupling among resistors, includes VCCS's, each one being controlled by the voltage of one resistor and/or some capacitors and/or some voltage sources. The ANA model (24) of these circuits has a tractability index of one if and only if*

(i) *the circuit has neither IJL-cutsets nor VC-loops, and*
(ii) *the sum of b-weights in C-balanced trees does not vanish.*

This result follows again from Theorem 5 noting that the product $A_{\mathcal{R}}GA_{\mathcal{R}}^{\mathrm{T}}$ in A_1 (cf. (30)) now takes on the expression depicted in (35).

It is worth emphasizing that the voltages of capacitors (and also those of voltage sources) may act as controlling variables for the sources; this is due to the fact that the derivatives with respect to v_c are not involved in the matrix A_1. This is important in applications since in many transistor models the controlling variables are capacitor voltages (see for instance [17,46]). Finally, remark that also the voltage of devices which define loops with capacitors and voltage sources may act as controlling variables without affecting the characterization above: this follows from the fact that Kirchhoff's voltage law allows one to write γ_j still in terms of $A_{\mathcal{R}}^{\mathrm{T}}e$, v_c and $v_s(t)$, and shows that the requirements on the controlling devices are not unduly restrictive (compare with Table III of [17]).

5.5. Coupling effects

Propositions 1 and 2 can also be extended to problems with resistive coupling along the lines defined by Theorem 6 in Section 4.2. This allows one, for instance, to include *gyrators* [13] in the above-discussed DC-solvability analysis and index characterization. For the sake of brevity, however, we just illustrate the analysis of coupled problems by means of the analogue of Proposition 4 in problems with coupled capacitors, arising often in real applications.

Assume that each capacitor may be coupled to at most another one; this means that the capacitance matrix has a block-diagonal structure in which the blocks have the form

$$\begin{pmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj} \end{pmatrix}.$$

Here C_{ii} and C_{jj} are the capacitances of the i-th and the j-th capacitors, whereas C_{ij} and C_{ji} are the coupling coefficients between both capacitors. These parameters stand for the branch weights w_{ii}, w_{jj} and the coefficients w_{ij}, w_{ji} of Section 4.2. Recall from 5.3.2 that, in the characterization of nonsingular nodal capacitance matrices $A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}}$, capacitors were painted in blue and the remaining circuit devices in red, so that proper trees amount in this setting to capacitive trees. Define the b-weight of a capacitive tree as the product of the determinants $C_{ii}C_{jj} - C_{ij}C_{ji}$ of pairs of coupled capacitors in the tree times the product of the capacitances of the twigs which are not coupled to other twig capacitors. Additionally, let us then define a pair of capacitive trees as *regular* if their twigs satisfy the requirements displayed in Definition 4; the b-weight of a regular capacitive tree pair is then defined exactly as in Section 4.2. The following result then follows from Theorem 6.

Proposition 8. *Consider a well-posed, connected circuit in which every capacitor is coupled to at most another one. Then the nodal capacitance matrix $A_{\mathcal{C}}CA_{\mathcal{C}}^{\mathrm{T}}$ is nonsingular if and only if*

(i) *the circuit has no VILR-cutsets (i.e. there exists at least one capacitive tree), and*
(ii) *the sum of the b-weights in capacitive trees and regular capacitive tree pairs does not vanish.*

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