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Barriers in metric spaces

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ABSTRACT

Defining a subset $\mathcal{B}$ of a connected topological space T to be a *barrier* (in T) if $\mathcal{B}$ is connected and its complement $T-\mathcal{B}$ is disconnected, we will investigate barriers $\mathcal{B}$ in the tight span

$$T(D)=\left\{f\in\mathbb{R}^X:\forall_{x\in X} f(x)=\sup_{y\in X}\Big(D(x,y)-f(y)\Big)\right\}$$

of a metric D defined on a finite set X (endowed, as a subspace of $\mathbb{R}^X$, with the metric and the topology induced by the ℓ_∞-norm) that are of the form

$$\mathcal{B}=\mathcal{B}_\varepsilon(f):=\{g\in T(D):\|f-g\|_\infty\le\varepsilon\}$$

for some $f\in T(D)$ and some $\varepsilon\ge 0$. In particular, we will present some conditions on f and ε which ensure that such a subset of $T(D)$ is a barrier in $T(D)$. More specifically, we will show that $\mathcal{B}_\varepsilon(f)$ is a barrier in $T(D)$ if there exists a bipartition (or *split*) of the ε-support $\mathrm{supp}_\varepsilon(f):=\{x\in X:f(x)>\varepsilon\}$ of f into two non-empty sets A and B such that $f(a)+f(b)\le ab+\varepsilon$ holds for all elements $a\in A$ and $b\in B$ while, conversely, whenever $\mathcal{B}_\varepsilon(f)$ is a barrier in $T(D)$, there exists a bipartition of $\mathrm{supp}_\varepsilon(f)$ into two non-empty sets A and B such that, at least, $f(a)+f(b)\le ab+2\varepsilon$ holds for all elements $a\in A$ and $b\in B$.

1. Introduction

Given a set X and a metric $D:X\times X\to\mathbb{R}:(x,y)\mapsto xy$ defined on X, consider the tight span

$$T(D):=\{f\in\mathbb{R}^X:f(x)=\sup(xy-f(y):y\in X)\text{ for all }x\in X\}$$

of D endowed, as a subspace of $\mathbb{R}^X$, with the metric and the topology induced by the ℓ_∞-norm; recall (cf. [1–3]) that

$$\|f-g\|_\infty=\sup(f(x)-g(x):x\in X)=\sup(g(x)-f(x):x\in X)\tag{1}$$

holds for all $f,g\in T(D)$, and recall that, denoting the so-called *Kuratowski map* $X\to\mathbb{R}:y\mapsto xy$ associated with an element $x\in X$ by $k_x=k_x^D$, one has $k_x\in T(D)$ and $\|f-k_x\|_\infty=f(x)$.

Next, given a map $f\in T(D)$ and a non-negative number $\varepsilon\in\mathbb{R}_{\ge0}$, let $\mathcal{B}_\varepsilon(f):=\{g\in T(D):\|f-g\|_\infty\le\varepsilon\}$ denote the (closed) ε-ball centered at f, put $T_{(f,\varepsilon)}(D):=T(D)-\mathcal{B}_\varepsilon(f)$, and denote the ε-*support* $\{x\in X:f(x)>\varepsilon\}$ of f by $\mathrm{supp}_\varepsilon(f)$. Note that $x\in\mathrm{supp}_\varepsilon(f)\iff k_x\in T_{(f,\varepsilon)}(D)$ holds for every $x\in X$ and every f in $T(D)$, and define f to be

– a *topological* ε-cutpoint of D if $\mathcal{B}_\varepsilon(f)$ is a *barrier* in $T(D)$, i.e., if $T_{(f,\varepsilon)}(D)$ is disconnected,

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– a *virtual* ε-cutpoint of D if there exists a bipartition of $\mathrm{supp}_\varepsilon(f)$ into two non-empty disjoint subsets A and B such that

$$f(a)+f(b) \le ab+\varepsilon$$

holds for all elements $a \in A$ and $b \in B$, and

– a *weak* virtual ε-cutpoint of D if there exists a bipartition of $\mathrm{supp}_\varepsilon(f)$ into two non-empty disjoint subsets A and B such that

$$f(a)+f(b) \le ab+2\varepsilon$$

holds for all elements $a \in A$ and $b \in B$.

Building on results obtained in [4–6] and motivated by related work (see e.g. [7]), we will show in this note that these notions are, in fact, closely related to one another: A map $f \in T(D)$ is a topological ε-cutpoint of D whenever it is a virtual ε-cutpoint of D while, conversely, if it is a topological ε-cutpoint of D, then it is, at least, a weak virtual ε-cutpoint.

2. Virtual ε-cutpoints are topological ε-cutpoints

With X, D, f, and ε as above, let $\Gamma = \Gamma_{(f,\varepsilon)} = (\mathrm{supp}_\varepsilon(f), E_{(f,\varepsilon)})$ denote the graph with vertex set $\mathrm{supp}_\varepsilon(f)$ and edge set

$$E = E_{(f,\varepsilon)} := \left\{ \{a,b\} \in \binom{\mathrm{supp}_\varepsilon(f)}{2} : f(a)+f(b) > ab+\varepsilon \right\}$$

so that f is a virtual ε-cutpoint if and only if Γ is disconnected. Further, given a subset A of $\mathrm{supp}_\varepsilon(f)$, let $O(A) = O_f^\varepsilon(A)$ denote the (necessarily open) subset $O(A) = O_f^\varepsilon(A) := \{g \in T_{(f,\varepsilon)}(D) : \forall_{x \in \mathrm{supp}_\varepsilon(f)-A} f(x) < g(x)\}$ of the (also open) subset $T_{(f,\varepsilon)}(D)$ of $T(D)$. Note that

– $k_a \in O(A)$ holds for every connected component $A \in \pi_0(\Gamma)$, the set of connected components of Γ, and every $a \in A$ as

$$f(x) < f(x)+f(a)-\varepsilon \le xa = k_a(x)$$

holds for all $a, x \in \mathrm{supp}_\varepsilon(f)$ with $\{a,x\} \notin E_{(f,\varepsilon)}$ and, hence, for all $a \in A$ and $x \in \mathrm{supp}_\varepsilon(f) - A$ if A is a connected component of Γ;

– $O(A) \cap O(B) = \emptyset$ holds for any two subsets A, B of $\mathrm{supp}_\varepsilon(f)$ with $A \cap B = \emptyset$ as $g \in O(A) \cap O(B)$ for some $g \in T_{(f,\varepsilon)}(D)$ would imply that $g(x)$ exceeds $f(x)$ for all x in $\mathrm{supp}_\varepsilon(f) - A$ as well as in $\mathrm{supp}_\varepsilon(f) - B$ and, hence, for all $x \in \mathrm{supp}_\varepsilon(f)$ implying (cf. (1)) the contradiction

$$\varepsilon < \|f-g\|_\infty \le \sup\bigl(f(x) : g(x) \le f(x)\bigr) \le \sup\bigl(f(x) : x \notin \mathrm{supp}_\varepsilon(f)\bigr) \le \varepsilon;$$

– if $g \in T_{(f,\varepsilon)}(D)$ holds for some map $g \in T(D)$, there must exist some $a \in X$ with $g(a)+\varepsilon < f(a)$ and, therefore, with $a \in \mathrm{supp}_\varepsilon(f)$ as well as $g \in O\bigl(\Gamma(a)\bigr)$ (with $\Gamma(a)$ denoting the connected component of Γ containing a) as $f(x) < f(x)+f(a)-\varepsilon-g(a) \le xa-g(a) \le g(x)$ must hold for every $x \in \mathrm{supp}_\varepsilon(f) - \Gamma(a)$, i.e., $T_{(f,\varepsilon)}(D) = \bigcup_{A \in \pi_0(\Gamma)} O(A)$ holds for every f and ε as above.

Together, these imply most of

Theorem 1. *With X, D, f, and ε as above, the collection*

$$\mathcal{O} = \mathcal{O}_{(f,\varepsilon)} := \bigl\{O(A) : A \in \pi_0(\Gamma)\bigr\}$$

of open subsets of $T_{(f,\varepsilon)}(D)$ forms a partition of $T_{(f,\varepsilon)}(D)$ into a family of pairwise disjoint and non-empty subsets of $T_{(f,\varepsilon)}(D)$, each such subset $O(A)\bigl(A \in \pi_0(\Gamma)\bigr)$ containing all Kuratowski maps k_a with $a \in A$.

More generally, given any partition $\mathcal{A}$ of $\mathrm{supp}_\varepsilon(f)$ into non-empty subsets for which $f(a)+f(a') \le aa'+\varepsilon$ or, equivalently, $\{a,a'\} \notin \Gamma_{(f,\varepsilon)}$ holds, for all $a \in A$ and $a' \in A'$, for any two distinct subsets $A, A' \in \mathcal{A}$, the corresponding collection $\mathcal{O}(\mathcal{A}) := \{O(A) : A \in \mathcal{A}\}$ of open subsets of $T_{(f,\varepsilon)}(D)$ forms a partition of $T_{(f,\varepsilon)}(D)$ such that $k_a \in O(A)$ holds for all $a \in A \in \mathcal{A}$.

In particular, there exists a canonical surjective mapping $\Pi_f = \Pi_{(f,\varepsilon)}$ from $\pi_0\bigl(T_{(f,\varepsilon)}(D)\bigr)$, the set of connected components of $T_{(f,\varepsilon)}(D)$, into $\pi_0(\Gamma_{(f,\varepsilon)})$ defined by associating, with each connected component C of $T_{(f,\varepsilon)}(D)$, the unique connected component $A = A_f(C)$ of Γ for which $C \subseteq O(A)$ holds.

Proof. Clearly, the assertions not yet established above follow from the fact that $\bigcup_{a \in A} O\bigl(\Gamma(a)\bigr) = O(A)$ holds for any subset $A \in \mathcal{A}$ which follows immediately from the fact that, as established already above, $T_{(f,\varepsilon)}(D)$ is the disjoint union of its subsets of the form $O(A')$ with $A' \in \pi_0(\Gamma)$ and that, by definition, $O(U') \subseteq O(U)$ holds for all $U, U' \subseteq \mathrm{supp}_\varepsilon(f)$ with $U' \subseteq U$: Indeed, this implies that $\bigcup_{a \in A} O\bigl(\Gamma(a)\bigr) \subseteq O(A)$ as well as $\bigcup_{b \in B} O\bigl(\Gamma(b)\bigr) \subseteq O(B)$ holds for any subset $A \in \mathcal{A}$ and its complement $B := \mathrm{supp}_\varepsilon(f) - A$ relative to $\mathrm{supp}_\varepsilon(f)$ (as $a \in A$ and $b \in B$ implies $\Gamma(a) \subseteq A$ and $\Gamma(b) \subseteq B$) and hence, in view of $O(A) \cap O(B) = \emptyset$, also $O(A) \subseteq T_{(f,\varepsilon)}(D) - O(B) \subseteq \bigcup_{x \in \mathrm{supp}_\varepsilon(f)} O\bigl(\Gamma(x)\bigr) - \bigcup_{b \in B} O\bigl(\Gamma(b)\bigr) = \bigcup_{a \in A} O\bigl(\Gamma(a)\bigr)$. ■

Note that the converse of the second part of Theorem 1 does not hold in general. More precisely, Example 1 below presents a metric space (X, D) together with a map $f \in T(D)$ and a number $\varepsilon > 0$ such that $T_{(f,\varepsilon)}(D)$ is disconnected while the corresponding graph $\Gamma_{(f,\varepsilon)}$ is connected (see Fig. 1).

Example 1. Put $X := \{a, b, a', b'\}$, define D by $ab = a'b' := 1$, $aa' = bb' := 10$ and $ab' = a'b := 11$, put $\varepsilon := 0.5$, and consider the map f on X with $f(a) = f(a') = f(b) = f(b') := 5.5$. Then $f(x)+f(y) > xy+\varepsilon$ holds for all $x, y \in X$

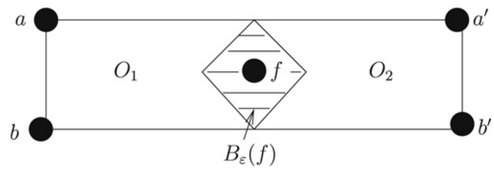


Fig. 1. The tight span $T(D) = O_1 \dot\cup \mathcal{B}_\varepsilon(f) \dot\cup O_2$ for the space (X, D) considered in the example in the text.

except in the case $\{x, y\} = \{a, b'\}$ and $\{x, y\} = \{a', b\}$ implying that $\Gamma_{(f,\varepsilon)}$ is connected while $T_{(f,\varepsilon)}(D)$ is the disjoint union of the two open subsets $O_1 := \{g \in T_{(f,\varepsilon)}(D) : g(a) < g(a')\}$ and $O_2 : \{g \in T_{(f,\varepsilon)}(D) : g(a') < g(a)\}$: Indeed, according to [1, p. 335], $g(a) + g(b') = g(a') + g(b) = 11$ must hold for every $g \in T(D)$ while, by definition of $T(D)$, we must have $g(a) + g(a'), g(b) + g(b') \geq 10$. So, $g(a) = g(a')$ can hold only in the case $g(a) = g(a') \in [5, 6]$ and $g(b) = g(b') \in [5, 6]$ and, therefore, $\|g - f\| \leq \varepsilon$. So, $T_{(f,\varepsilon)}(D) = O_1 \dot\cup O_2$ must hold.

3. Topological ε-cutpoints are weak virtual ε-cutpoints

We now establish a partial converse of Theorem 1. To this end, we introduce the following notation.

With X, D, and f as above, we denote by $\Gamma^* = \Gamma^*_{(f,\varepsilon)}$ the graph with vertex set $\mathrm{supp}_\varepsilon(f)$ (just as for Γ) and edge set the subset

$$E^* = E^*_{(f,\varepsilon)} := \left\{ \{a, b\} \in \binom{\mathrm{supp}_\varepsilon(f)}{2} : f(a) + f(b) > ab + 2\varepsilon \right\}$$

of $E = E_{(f,\varepsilon)}$ (implying that f is a weak virtual ε-cutpoint if and only if Γ^* is disconnected), we denote, for every $a \in \mathrm{supp}_\varepsilon(f)$, by $\Gamma^*(a)$ the unique connected component of Γ^* that contains the vertex a, and we denote, for every $g \in T_{(f,\varepsilon)}$, by $T_{(f,\varepsilon)}(D|g)$ the unique connected component of $T_{(f,\varepsilon)}(D)$ that contains the map g. Then, the following holds:

Theorem 2. *There exists a canonical surjective mapping*

$$\Pi^*_f = \Pi^*_{(f,\varepsilon)} : \pi_0(\Gamma^*) \to \pi_0\big(T_{(f,\varepsilon)}(D)\big)$$

from the set $\pi_0(\Gamma^*)$ of connected components of Γ^* onto $\pi_0\big(T_{(f,\varepsilon)}(D)\big)$ induced by associating, with every connected component $\Gamma^*(a) \in \mathrm{supp}_\varepsilon(f)$, the connected component $T_{(f,\varepsilon)}(D|k_a)$ of $T_{(f,\varepsilon)}(D)$, that is, there exists, for every $g \in T_{(f,\varepsilon)}$, some $a = a_g \in \mathrm{supp}_\varepsilon(f)$ with $T_{(f,\varepsilon)}(D|g) = T_{(f,\varepsilon)}(D|k_a)$, and $T_{(f,\varepsilon)}(D|k_a) = T_{(f,\varepsilon)}(D|k_b)$ holds for any two elements a, b in $\mathrm{supp}_\varepsilon(f)$ for which the connected components $\Gamma^*(a)$ and $\Gamma^*(b)$ of Γ^* coincide.

In particular, given a bipartition of $\mathrm{supp}_\varepsilon(f)$ into two non-empty subsets A and B such that the corresponding two open subsets $O_{(f,\varepsilon)}(A)$ and $O_{(f,\varepsilon)}(B)$ of $T_{(f,\varepsilon)}(D)$ form a bipartition of $T_{(f,\varepsilon)}(D)$, one has $f(a) + f(b) \leq ab + 2\varepsilon$ for all $a \in A$ and $b \in B$.

Proof. To establish this theorem, we use the following well-known fact (cf. [1, Section 1.10]):

(**Geod**) $T(D)$ is a *geodesic* space relative to the metric induced by the ℓ_∞-norm, i.e., there exists, for any two maps $f_1, f_2 \in T(D)$, an isometry $\varphi = \varphi_{(f_1,f_2)}$ from the real interval $[0, \|f_1 - f_2\|_\infty] \subset \mathbb{R}$ into $T(D)$ with $\varphi(0) = f_1$ and $\varphi(\|f_1 - f_2\|_\infty) = f_2$.

Clearly, this implies that the following holds:

(i) The *metric interval*

$$[f_1, f_2]_D := \{h \in T(D) : \|f_1 - h\|_\infty + \|h - f_2\|_\infty = \|f_1 - f_2\|_\infty\}$$

and the sets $\mathcal{B}_\varepsilon(f)$ are connected subsets of $T(D)$ for all f_1, f_2, f in $T(D)$ and all $\varepsilon \geq 0$.

(ii) Restricting Kuratowski's mapping $k : X \to T(D) : a \mapsto k_a$ to the subset $\mathrm{supp}_\varepsilon(f)$ of X induces a surjective mapping

$$k_{(f,\varepsilon)} : \mathrm{supp}_\varepsilon(f) \to \pi_0\big(T_{(f,\varepsilon)}(D)\big)$$

because (cf. (1)) there exists, for every $g \in T_{(f,\varepsilon)}(D)$, some $a = a_g \in X$ with

$$\|f - g\|_\infty = f(a) - g(a)$$

and, hence, $f(a) \geq \|f - g\|_\infty > \varepsilon$ (i.e., $a \in \mathrm{supp}_\varepsilon(f)$) as well as $g \in [k_a, f]_D$ in view of

$$\|k_a - f\|_\infty = f(a) = g(a) + \|f - g\|_\infty = \|k_a - g\|_\infty + \|f - g\|_\infty$$

which, in turn, implies $[k_a, g]_D \subseteq T_{(f,\varepsilon)}(D)$ as $h \in [k_a, g]_D \subseteq [k_a, f]_D$ implies $g \in [h, f]_D$ and, therefore $\|h - f\|_\infty = \|h - g\|_\infty + \|g - f\|_\infty > \varepsilon$ as well as $T_{(f,\varepsilon)}(D|g) = T_{(f,\varepsilon)}(D|k_a)$.

(iii) And finally, given any two maps $f_1, f_2 \in T_{(f,\varepsilon)}(D)$, the connected components $T_{(f,\varepsilon)}(D|f_1)$ and $T_{(f,\varepsilon)}(D|f_2)$ of $T_{(f,\varepsilon)}(D)$ containing the two maps f_1, f_2, respectively, must coincide whenever

$$\|f_1 - f_2\|_\infty + 2\varepsilon < \|f_1 - f\|_\infty + \|f - f_2\|_\infty$$

holds: Indeed, $[f_1, f_2]_D \subseteq T_{(f,\varepsilon)}(D)$ must hold in this case because $h \in [f_1, f_2]_D$ implies

$$\begin{aligned} 2\varepsilon &< \|f_1 - f\|_\infty + \|f - f_2\|_\infty - \|f_1 - f_2\|_\infty \\ &= \|f_1 - f\|_\infty + \|f - f_2\|_\infty - \|f_1 - h\|_\infty - \|h - f_2\|_\infty \\ &= (\|f_1 - f\|_\infty - \|f_1 - h\|_\infty) + (\|f - f_2\|_\infty - \|h - f_2\|_\infty) \\ &\le \|h - f\|_\infty + \|h - f\|_\infty = 2\|h - f\|_\infty, \end{aligned}$$

i.e., it implies $h \in T_{(f,\varepsilon)}(D)$ for all $h \in [f_1, f_2]_D$ as claimed. In particular, $T_{(f,\varepsilon)}(D|k_a) = T_{(f,\varepsilon)}(D|k_b)$ holds for any two elements a, b in $\text{supp}_\varepsilon(f)$ with $\{a, b\} \in E^*_{(f,\varepsilon)}$ (as this implies $\|k_a - f\|_\infty + \|f - k_b\|_\infty = f(a) + f(b) > ab + 2\varepsilon = \|k_a - k_b\|_\infty + 2\varepsilon$) and, hence, for any two elements a, b in $\text{supp}_\varepsilon(f)$ for which the connected components $\Gamma^*(a)$ and $\Gamma^*(b)$ of Γ^* containing a and b, respectively, coincide.

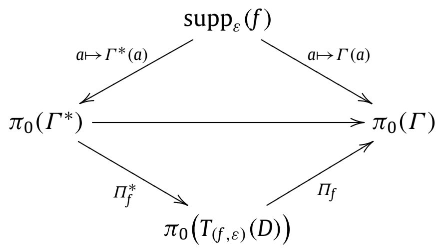
Clearly this establishes Theorem 2. ■

Remark. Note that the factor 2 in the definition of Γ^* is optimal in the sense that there are topological ε-cutpoints f such that the graph

$$\Gamma^k := (\text{supp}_\varepsilon(f), E^k) := \left\{ \{a, b\} \in \binom{\text{supp}_\varepsilon(f)}{2} : f(a) + f(b) > ab + k\varepsilon \right\}$$

is connected for any $k \in [1, 2)$: Indeed, for the space (X, D) considered in Example 1, f is a topological ε-cutpoint while the graph Γ^k is connected.

Our results suggest considering the following commutative diagram of canonical surjective maps:



Clearly, our results imply:

Corollary 3.1. *Continuing with the notation introduced above, the maps Π_f and Π_f^* are mutually inverse bijections if and only if the canonical surjective map from $\pi_0(\Gamma^*)$ onto $\pi_0(\Gamma)$ that associates with any connected component C of Γ^* the unique connected component of Γ that contains C is a bijection.*

Note finally that in the particular case $\varepsilon := 0$, we clearly have $\Gamma = \Gamma^*$ and, hence, recover a result from [6]: Π_f is a bijection from $\pi_0(T(D) - \{f\})$ onto the set of connected components of the graph

$$\Gamma_f := \left(\text{supp}(f), \left\{ \{a, b\} \in \binom{\text{supp}(f)}{2} : f(a) + f(b) > ab \right\} \right).$$

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