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Cohomological Invariants of Algebras with Involution

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With a central simple algebra, A, we can associate a quadratic form, namely the trace form $\mathrm{Trd}_A(x^2)$. This form has been studied by Rowen [19], Formanek [5], Lewis [11], and Kersten [7]. Its Hasse invariant has been computed by Saltman (unpublished), Serre [23, annexe], Lewis and Morales [13], and Tignol [25].

In this paper, we consider a central simple algebra A endowed with an involution σ. With (A, σ) we associate the quadratic form T_σ defined by $T_\sigma(x) = \mathrm{Trd}_A(\sigma(x)x)$, which has values in the subfield of the center of A fixed by the involution. This form has been introduced by Weil [26]. It has been used in [14, 16] to define the signature of the involution σ. We also consider the restriction T_σ^+ of T_σ to the subspace of σ-invariant elements of A. The invariants of T_σ and T_σ^+ are invariants of the algebra with involution. The main purpose of this paper is to study the determinant and the Hasse invariant of these quadratic forms.

In Section 2, we assume that the involution σ is of the first kind. Then, we have a notion of determinant, due to Jacobson [6] and Knus, Parimala, and Sridharan [10]. We give here a new definition of the determinant of σ, in terms of the trace form T_σ^+. We then compute the Hasse invariant of the trace form T_σ, in terms of the degree of the algebra, its class in the Brauer group, and the determinant of the involution. This last result has also been obtained by Lewis [12].

If the involution is of the second kind, we first prove that the determinant and the Hasse invariant of the trace form T_σ are both trivial. Then, we see that the Hasse invariant of the trace form T_σ^+ defines a non-trivial invariant of (A, σ), which we call determinant class mod 2. It is related to the class in the Brauer group of the discriminant algebra of (A, σ) defined

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in [8] by Knus, Merkurjev, Rost, and Tignol. Finally, we define a cohomological invariant of (A, σ), with values in the quotient of $H^2(k, {}_\alpha\mu_n)$ by the action of μ_2, where ${}_\alpha\mu_n$ is a twisted form of the group μ_n, which we call the determinant class of (A, σ), and we prove that the determinant class mod 2 of (A, σ) is the reduction modulo 2 of this invariant. The formula that gives the Hasse invariant of the form $\mathrm{Trd}_A(x^2)$ can be viewed as a particular case of this result.

In the last section, as an application of the previous results, we construct an indecomposable involution of the second kind on a biquaternion division algebra.

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1. NOTATIONS

Let k be a field of characteristic different from 2 and let n be an integer. If A is a k-algebra, we denote by $Z(A)$ its center. We recall that a quadratic étale extension of k is either $k \times k$ or a separable field extension of degree 2 of k.

We consider a k-algebra A, such that either A is a central simple algebra of degree n over k (case (i)), or A is semi-simple, its center $Z(A)$ is a quadratic étale extension of k and $\dim_k(A) = 2n^2$ (case (ii)).

An involution on A is an anti-automorphism of order 2 of the k-algebra A. We suppose here that A is endowed with an involution σ. We assume moreover that if $Z(A)$ is not k, then σ acts on $Z(A)$ as the only non-trivial k-automorphism of $Z(A)$, which we will denote by $\bar{\ }$. In case (i), the involution acts trivially on $Z(A)$ and is said to be of the first kind, while in case (ii) it is said to be of the second kind.

In case (ii), if $Z(A) = k \times k$, then there exists a central simple algebra B of degree n over k such that $A = B \times B^0$, where B^0 is the opposite algebra of B, and the involution is given by $\sigma(x, y) = (y, x)$. Otherwise, $Z(A)$ is a quadratic field extension K of k, and A a central simple algebra of degree n over K.

We let $A^+ = \{a \in A, \sigma(a) = a\}$ and $A^- = \{a \in A, \sigma(a) = -a\}$. They are supplementary k-subvector spaces of A.

Trace Forms

If C is a central simple algebra, we denote by Trd_C the reduced trace on C. Let us now define a trace T_A on A. If A is a central simple algebra, then T_A is the reduced trace Trd_A. Otherwise, there exists a central simple algebra B over k such that $A = B \times B^0$, and we define $T_A(x, y) = \frac{1}{2}\mathrm{Trd}_B(x + y)$.

In this paper, we are interested in studying the quadratic form T_σ, defined by

$$T_\sigma(x) = T_A(\sigma(x)x) \in k, \qquad \text{for any } x \in A.$$

We also consider the restrictions of T_σ to A^+ and A^-, which are respectively denoted by T_σ^+ and T_σ^-. It is easy to check that A^+ and A^- are orthogonal for the trace form. Hence, we have

$$T_\sigma = T_\sigma^+ \oplus T_\sigma^-.$$

Remarks. (i) If the center of A is a quadratic field extension K of k, then T_A has values in K. But since $\sigma(\mathrm{Trd}_A(\sigma(x)x)) = \mathrm{Trd}_A(\sigma(x)x)$, T_σ actually has values in k.

(ii) If $A = B \times B^0$, then T_σ^+ is isometric to the quadratic form $\mathrm{Trd}_B(x^2)$, which has been studied by Rowen [19], Formanek [5], Serre [23, annexe], Lewis and Morales [13], and Tignol [25].

Involutions on Central Simple Algebras

We recall here a few well-known facts about central simple algebras and involutions (see for instance [20] or [24]).

First of all, by Wedderburn's theorem, a central simple algebra A is isomorphic to a matrix algebra $A \simeq M_r(D)$, with coefficients in a division algebra D, with $Z(D) = Z(A)$. Moreover, it is known that if A is endowed with an involution, then there exists an involution τ over D, of the same kind. Once τ is fixed, the classification of involutions on A corresponds to the classification of hermitian forms up to similarity.

More precisely, if the involutions are of the first kind, then for any σ on A, there exists a δ-hermitian form $H_\sigma\colon D^r \times D^r \to (D, \tau)$, where $\delta = \pm 1$, such that σ is the adjoint involution with respect to H_σ, i.e., for any $x, y \in D^r$ and for any $f \in A$, we have $H_\sigma(f(x), y) = H_\sigma(x, \sigma(f)(y))$. This form H_σ is uniquely determined, up to a scalar factor $\lambda \in k^*$.

Let us assume now that the involutions are of the second kind. Then for any σ on A, there exists a δ-hermitian form $H_\sigma\colon D^r \times D^r \to (D, \tau)$, such that σ is the adjoint involution with respect to H_σ, and where δ is an element of $Z(A)$ such that $\sigma(\delta)\delta = 1$. By Hilbert's theorem 90, this form H_σ may actually be chosen to be a 1-hermitian form, in which case it is uniquely determined up to a scalar factor $\lambda \in k^*$.

If A is the split algebra $A = \mathrm{End}_k(V)$, where V is an n-dimensional k-vector space, this means that any involution of the first kind on A is the adjoint involution σ_b with respect to some bilinear form $b\colon V \times V \to k$, which is either symmetric or skew-symmetric. If B is the matrix of b in a fixed basis of V, then σ_b acts on $M \in M_n(k) \simeq \mathrm{End}_k(V)$ by $\sigma_b(M) = B^{-1}M^tB$, where M^t denotes the transposed matrix of M. By using a well chosen basis of V, one may always assume that B is a diagonal matrix if b is symmetric, and $B = J$, where J is the matrix consisting of m diagonal blocs all equal to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ if b is skew-symmetric.

Now if $A = \mathrm{End}_K(V)$, where V is an n-dimensional K-vector space, any involution of the second kind on A is the adjoint involution with respect to a 1-hermitian form h: $V \times V \to K$. If B is a matrix of h, σ_h acts on $M \in M_n(K)$ by $\sigma_h(M) = B^{-1}\overline{M}^tB$, where if M is the matrix (m_{ij}), $\overline{M}^t$ denotes the matrix $(\overline{m}_{ji})$. We will usually choose B in diagonal form, in which case its coefficients lie in k.

Algebraic Groups

The groups $O(q)$, $Spin(q)$, and $GO(q)$ are respectively the orthogonal group, the spinor group, and the group of similarities of the quadratic form q. We denote by Sp_n the symplectic group of order n, and by GSp_n the corresponding group of similarities. For any algebraic group G, PG is the corresponding adjoint group, and G^+ the connected component of the identity in G.

Galois Cohomology

Let k_s be a separable closure of k. We denote by Γ_k the Galois group $\mathrm{Gal}(k_s/k)$. If G is an algebraic group defined over k, and if $i = 0$ or 1, $H^i(k, G)$ is the cohomology set $H^i(\Gamma_k, G(k_s))$. For any Γ_k-module C, and for any integer i, we denote by $H^i(k, C)$ the cohomology group $H^i(\Gamma_k, C)$. In particular, if $C = \mu_2$, we recall that $H^1(k, \mu_2)$ is isomorphic to k^*/k^{*2} and $H^2(k, \mu_2)$ is isomorphic to the 2-part of the Brauer group of k, $\mathrm{Br}_2(k)$.

Quadratic and Hermitian Forms

Let q be an r-dimensional quadratic form over k, and let $\langle a_1, \ldots, a_r \rangle$ be any diagonalization of q. The determinant of q is $d(q) = a_1 \cdots a_r \in k^*/k^{*2}$. The Hasse invariant of q is defined by $w_2(q) = \sum_{1 \leq i < j \leq r}(a_i, a_j) \in H^2(k, \mu_2)$, where (a_i, a_j) denotes the cup-product of the square classes of a_i and a_j. Both definitions are independent of the choice of a diagonalization of q [20]. Moreover, using the formulas given in [20, Chap. II, 11.13], one can easily check the following:

LEMMA 1. *The Hasse invariant of the direct sum of two quadratic forms* q_1 *and* q_2 *is given by*:

(i) $w_2(q_1 \oplus q_2) = w_2(q_1) + w_2(q_2) + (d(q_1), d(q_2))$. *Moreover if we denote by* r *the dimension of* q, *then for any integer* k *and any* $\lambda \in k^*$ *we have*:

(ii) $w_2(kq) = kw_2(q) + (k(k-1)/2)(-1, d(q))$, *where* kq *is the direct sum of* k *copies of* q;

(iii) $w_2(\langle \lambda \rangle \otimes q) = w_2(q) + (r(r-1)/2)(-1, \lambda) + (r-1)(\lambda, d(q))$;

(iv) $w_2(\langle 1, \lambda \rangle \otimes q) = (r(r-1)/2)(-1, \lambda) + (-\lambda, d(q))$.

If q is defined on the k-vector space V, and if L/k is any field extension, we denote by q_L the extension of q to the field L, that is, the form defined on $V \otimes_k L$ by $q_L(v \otimes \lambda) = \lambda^2 q(v)$.

For any elements $a, b \in k^*/k^{*2}$, we denote by $\langle\langle a, b \rangle\rangle$ the Pfister form $\langle\langle a, b \rangle\rangle = \langle 1, -a \rangle \otimes \langle 1, -b \rangle$.

Let now h be a hermitian form with values in $(K, {}^-)$, and let us consider any diagonalization $\langle a_1, \dots, a_n \rangle$ of h. Then the coefficients a_i lie in k, and the determinant of h is $d(h) = a_1 \cdots a_n \in k^*/N_{K/k}(K^*)$.

2. INVOLUTIONS OF THE FIRST KIND

We assume in this section that we are in case (i), which means that A is a central simple algebra over k of degree n, endowed with an involution of the first kind σ. We also assume that n is even, say $n = 2m$.

Let F be a splitting field of A, and let b be a bilinear form associated with the involution $\sigma \otimes \mathrm{Id}$ defined on the split algebra $A \otimes_k F$. We recall that the involution σ is said to be of orthogonal type if b is symmetric and of symplectic type if b is skew-symmetric [20, 24]. This definition does not depend upon the choice of a splitting field F.

In the first part of this section, we are going to give some examples in which the trace forms and their invariants can be explicitly computed, and which will be used later.

2.1. *Examples*

2.1.1. *Split Case*

Let us consider first the case when $A = M_n(k)$. Let σ_b be the adjoint involution with respect to the bilinear form b. It is easy to check that its trace form is $T_{\sigma_b} = b \otimes b$ [14, Corollary 1]. Hence we have $d(T_{\sigma_b}) = 1$. Moreover, if b is symmetric, we have $w_2(T_{\sigma_b}) = (-1, d(b))$. In particular, if t denotes the transposition, its trace form is given by $T_t = n^2\langle 1 \rangle$.

If b is symmetric, we can also easily compute the trace form $T^+_{\sigma_b}$. Let B be a diagonal matrix of b, and let us call $b_1, \dots, b_n$ its coefficients. The involution is given by $\sigma_b(M) = B^{-1}M^tB$, for any $M \in M_n(k)$. Hence $M_n(k)^+$ consists of the matrices $B^{-1}S$, where S is any symmetric matrix with coefficients in k. If $(e_{ij})_{1 \le i,j \le n}$ denotes the canonical basis of $M_n(k)$, then $(B^{-1}e_{ii})_{1 \le i \le n} \cup (B^{-1}(e_{ij} + e_{ji}))_{1 \le i < j \le n}$ is a basis of $M_n(k)^+$. Moreover, one can check that this basis is orthogonal for the trace form, and we get that

$$T^+_{\sigma_b} = \left\langle b_1^{-2}, \dots, b_n^{-2}, 2b_1^{-1}b_2^{-1}, 2b_1^{-1}b_3^{-1} \dots, 2b_{n-1}^{-1}b_n^{-1} \right\rangle.$$

Hence the determinant of $T^+_{\sigma_b}$ is $2^{n(n-1)/2}(b_1^{-1} \cdots b_n^{-1})^{n-1} = 2^m d(b) \in k^*/k^{*2}$. In particular, for the transposition, we have that

$$T_t^+ = n\langle 1 \rangle \oplus (n(n-1)/2)\langle 2 \rangle.$$

Moreover, using the same method, we get

$$T_t^- = (n(n-1)/2)\langle 2 \rangle.$$

2.1.2. *Quaternion Algebras*

Let Q be a quaternion algebra over k. We denote by $\tau\colon Q \to Q$ its canonical involution. It is known that τ is the only involution of symplectic type on Q. Now, if σ is an involution of orthogonal type on Q, there exists an element $i \in Q$ such that $\tau(i) = -i$ and such that σ is given by $\sigma(x) = i^{-1}\tau(x)i$. Let j be an element in Q such that $ij = -ji$, and set $i^2 = a$ and $j^2 = b$. Then $(1, i, j, ij)$ is a basis of Q over k, which is orthogonal for the trace forms, and one can easily check

$$T_\tau = \langle 2 \rangle \otimes \langle\langle a, b \rangle\rangle, \qquad T_\tau^+ = \langle 2 \rangle, \qquad T_\tau^- = \langle -2a, -2b, 2ab \rangle;$$

$$T_\sigma = \langle 2 \rangle \otimes \langle\langle a, -b \rangle\rangle, \qquad T_\sigma^+ = \langle 2, 2b, -2ab \rangle, \qquad T_\sigma^- = \langle -2a \rangle.$$

It is now easy to check that the trace forms T_τ and T_σ both have trivial determinant, and that their Hasse invariants are given by

$$\begin{cases} w_2(T_\tau) = (-1,-1) + (a,b), \\ w_2(T_\sigma) = (-1,-a) + (a,b). \end{cases}$$

2.1.3. *Case of the Algebra $M_m(Q)$*

Let us consider now the involutions $\tau' = t \otimes \tau$ and $\sigma' = t \otimes \sigma$ on the algebra $M_m(k) \otimes_k Q$. The trace forms are given by

$$\begin{cases} T_{\tau'} = T_t \otimes T_\tau = m^2\langle 2, -2a, -2b, 2ab \rangle, \\ T_{\sigma'} = T_t \otimes T_\sigma = m^2\langle 2, -2a, 2b, -2ab \rangle. \end{cases}$$

Hence again they both have trivial determinant, and we have $w_2(T_{\tau'}) = mw_2(T_\tau) = m(-1,-1) + m(a,b)$, and $w_2(T_{\sigma'}) = mw_2(T_\sigma) = m(-1,-a) + m(a,b)$.

Remark. These formulas also hold if Q is split. Hence, let us assume now $a = b = 1$. Since τ' is of symplectic type, and since all involutions of symplectic type on $M_n(k)$ are isomorphic, if b is any skew-symmetric bilinear form, the Hasse invariant of T_{σ_b} is given by

$$w_2(T_{\sigma_b}) = m(-1,-1) + m(1,1) = m(-1,-1).$$

Moreover, the form $T^+_{\sigma_b}$ can also be computed this way. Indeed,

$$(M_m(k) \otimes_k Q)^+ = M_m(k)^+ \otimes_k Q^+ \oplus M_m(k)^- \otimes_k Q^-.$$

Hence we have

$$\begin{aligned} T^+_{\sigma_b} &= T^+_t \otimes T^+_\tau \oplus T^-_t \otimes T^-_\tau \\ &= \left(m\langle 1 \rangle + \frac{m(m-1)}{2}\langle 2 \rangle \right) \otimes \langle 2 \rangle \oplus \frac{m(m-1)}{2}\langle 2 \rangle \otimes \langle -2,-2,2 \rangle. \end{aligned}$$

In particular, its determinant is equal to 2^m.

2.2. *Determinant of an Involution*

The determinant of involutions of the first kind was first introduced by Jacobson in [6]. Later, Knus, Parimala, and Sridharan gave a more direct definition [10]. We can also define it in terms of the trace forms. Precisely, we give the following definition:

DEFINITION 1. The determinant of σ is $d(\sigma) = 2^m d(T^+_\sigma)$.

Let us assume that $A = M_n(k)$, and that σ is the adjoint involution σ_b with respect to the bilinear form b. We then have the following result:

PROPOSITION 1. *The determinant of the involution σ_b is $d(\sigma_b) = 1$ if b is skew-symmetric, and $d(\sigma_b) = d(b)$ if b is symmetric.*

Proof. We have computed the determinant of $T^+_{\sigma_b}$ in the previous section. It is given by $d(T^+_{\sigma_b}) = 2^m$ if b is skew-symmetric (cf. Remark 2.1.3), and $d(T^+_{\sigma_b}) = 2^m d(b)$ if b is symmetric (cf. Subsection 2.1.1). This proves the proposition.

Remarks. (i) The form b is only defined up to a scalar factor $\lambda \in k^*$. But as n is even, its determinant $d(b)$ does not depend upon the choice of b.

(ii) (See also [7, 11].) Let F be a generic splitting field of the algebra A. One may take for F the function field of the Severi-Brauer variety of A. It is easy to check that the trace form of the involution $\sigma \otimes \mathrm{Id}$ defined on the split algebra $A \otimes_k F \simeq M_n(F)$ is simply the extension to the field F of the trace form T_σ. We also have $T^+_{\sigma\otimes \mathrm{Id}} = (T^+_\sigma)_F$. Hence the monomorphism $\psi: k^*/k^{*2} \hookrightarrow F^*/F^{*2}$ maps $d(T^+_\sigma)$ to $d(T^+_{\sigma\otimes\mathrm{Id}})$. This implies

$$d(\sigma \otimes \mathrm{Id}) = \psi(d(\sigma)).$$

Since ψ is injective, this means we can extend scalars to F to compute the determinant of σ.

(iii) The definition we gave for the determinant of an involution coincides with the usual one. Indeed, because of the previous remark, it is enough to prove it in the split case, and this was done in Proposition 1. One though has to be aware that what we call the determinant here, following [8], is called the discriminant in [9, 10].

(iv) We have also seen in Subsection 2.1.1 that in the split case, $d(T_{\sigma_b}) = 1$. Again by extending scalars to a generic splitting field of the algebra A, it is easy to check that for any (A, σ), we have $d(T_\sigma) = 1$. Hence, since $T_\sigma = T^+_\sigma \oplus T^-_\sigma$, we actually have

$$d(T^-_\sigma) = d(T^+_\sigma) = 2^m d(\sigma).$$

2.3. *Hasse Invariant of the Trace Form*

In this section, we compute the Hasse invariant of the trace form T_σ.

THEOREM 1 (see also [12]). *The Hasse invariant of T_σ is given by*

$$w_2(T_\sigma) = \begin{cases} m(-1,-1) + m[A] & \textit{if } \sigma \textit{ is of symplectic type,} \\ (-1, d(\sigma)) + m[A] & \textit{if } \sigma \textit{ is of orthogonal type.} \end{cases}$$

Remarks. (i) Again let F be a generic splitting field of the algebra A. It is known that the kernel of the morphism $\mathrm{Br}(k) \to \mathrm{Br}(F)$ is generated by $[A]$ [2, 18]. Since A is endowed with an involution of the first kind, its class in $\mathrm{Br}(k)$ is of order 2. Hence, if we denote by Ψ the morphism $\mathrm{Br}_2(k) \to \mathrm{Br}_2(F)$, we have $\mathrm{Ker}\,\Psi = \{0, [A]\}$.

We have already seen that $T_{\sigma\otimes\mathrm{Id}} = (T_\sigma)_F$. Hence $\Psi(w_2(T_\sigma))$ is equal to $w_2(T_{\sigma\otimes\mathrm{Id}})$. Since the algebra $A \otimes_k F$ is split, the Hasse invariant $w_2(T_{\sigma\otimes\mathrm{Id}})$ has been computed in Subsections 2.1.1 and 2.1.3. We have

$$w_2(T_{\sigma\otimes\mathrm{Id}}) = \begin{cases} m(-1,-1) & \text{if } \sigma \text{ is of symplectic type,} \\ (-1, d(\sigma)) & \textit{if } \sigma \text{ is of orthogonal type.} \end{cases}$$

From all this, we deduce

$$w_2(T_\sigma) = \begin{cases} m(-1,-1) + \varepsilon[A] & \text{if } \sigma \text{ is of symplectic type,} \\ (-1, d(\sigma)) + \varepsilon[A] & \text{if } \sigma \text{ is orthogonal type,} \end{cases}$$

where $\varepsilon \in \{0, 1\}$. Hence we only have to prove that ε is equal to 1 when m is odd and 0 when m is even.

(ii) The Hasse invariant of the restricted trace forms T_σ^+ and T_σ^- can also be computed. We get the following formulas [17]:

$$w_2(T_\sigma^+) = \begin{cases} \dfrac{m(m-1)}{2}(-1,-1) + \dfrac{m(m-1)}{2}[A] \\ \quad \text{if } \sigma \text{ is of symplectic type,} \\ (m-1)(-2, d(\sigma)) + \dfrac{m(m+1)}{2}[A] \\ \quad \text{if } \sigma \textit{ is of orthogonal type;} \end{cases}$$

$$w_2(T_\sigma^-) = \begin{cases} \dfrac{m(m+1)}{2}(-1,-1) + \dfrac{m(m+1)}{2}[A] \\ \quad \textit{if } \sigma \textit{ is of symplectic type,} \\ (m-1)(-2, d(\sigma)) + \dfrac{m(m-1)}{2}[A] \\ \quad \textit{if } \sigma \textit{ is of orthogonal type.} \end{cases}$$

Proof of Theorem 1. The proof we give here is based on Galois cohomology, using a method introduced by Serre in [22]. Very recently, Lewis gave another proof, using the formula that gives the Hasse invariant of the form $\mathrm{Trd}(x^2)$ [12].

Cohomological Definition of the Hasse Invariant. Let us first recall a few basic facts on Galois cohomology and quadratic forms (see for instance [23]).

Let q_0 be an n-dimensional quadratic form over k. Then, $H^1(k, O(q_0))$ classifies isometry classes of n-dimensional quadratic forms q over k. From the exact sequence $1 \to O^+(q_0) \to O(q_0) \stackrel{d}{\to} \mu_2 \to 1$, we deduce a map $d^*\colon H^1(k, O(q_0)) \to k^*/k^{*2}$. It maps the isometry class of a quadratic form q to the class of $d(q)/d(q_0)$ in k^*/k^{*2}. Moreover, the set $H^1(k, O^+(q_0))$ is in bijection with $\mathrm{Ker}\, d^*$. Hence, $H^1(k, O^+(q_0))$ classifies isometry classes of n-dimensional quadratic forms over k with the same determinant as q_0.

Now, from the exact sequence $1 \to \mu_2 \to \mathit{Spin}(q_0) \to O^+(q_0) \to 1$, we deduce a connecting map $\delta_2\colon H^1(k, O^+(q_0)) \to \mathrm{Br}_2(k)$. The image under δ_2 of the isometry class of q is $w_2(q) - w_2(q_0)$.

Classification of Algebras with Involution (A, σ). Let q_0 be an n-dimensional quadratic form over k. The following proposition holds:

PROPOSITION 2. *We have one-to-one correspondences between*

(i) $H^1(k, PSp_n)$ *and isomorphism classes of degree n central simple algebras endowed with an involution of symplectic type*;

(ii) $H^1(k, PGO(q_0))$ *and isomorphism classes of degree n central simple algebras endowed with an involution of orthogonal type*.

Before proving this proposition, we establish a lemma. Let A be a central simple algebra of degree n over k. The algebra A splits over k_s. Let ϕ be an isomorphism ϕ: $M_n(k_s) \to A \otimes_k k_s$. Via this isomorphism, the involution $\sigma \otimes \mathrm{Id}$ defined on the algebra $A \otimes_k k_s$ corresponds to an involution σ_b on the algebra $M_n(k_s)$, which is the adjoint involution with respect to some bilinear form b. The form b is symmetric if σ is of orthogonal type, and skew-symmetric if σ is of symplectic type.

For any matrix P, we denote by $\mathrm{Int}(P)$ the associated inner automorphism, that is, $\mathrm{Int}(P)(M) = PMP^{-1}$. We then have the following:

LEMMA 2. *Let G_σ be the group of similarities of the bilinear form b. Then the group of automorphisms of the algebra with involution $(M_n(k_s), \sigma_b)$ is* $\{\mathrm{Int}(P), P \in G_\sigma\} \simeq PG_\sigma$.

Proof of Lemma 2. Let B be a matrix of b, and let f be an automorphism of $(M_n(k_s), \sigma_b)$. There exists a matrix $P \in M_n(k_s)$ such that $f = \mathrm{Int}(P)$. The fact that $\sigma_b \circ f = f \circ \sigma_b$ implies that the matrices BP^{-1} and $P^t B$ define the same inner automorphism, and hence are proportional. This means that P is the matrix of a similarity for b, and this proves the lemma.

Proof of Proposition 2. The proof of Proposition 2 is now easy, following [21, Chap. X]. First, one can show that the application c: $\gamma \in \Gamma_k \mapsto c(\gamma) \in PG_\sigma$, determined by $\mathrm{Int}(c(\gamma)) = \phi^{-1} \circ {}^\gamma\phi$, is a cocycle. Moreover, if we start with another isomorphism ϕ': $M_n(k_s) \to A \otimes_k k_s$, the cocycle we obtain is cohomologically equivalent to c. Finally, one can prove that the isomorphism classes of algebras with involutions defined on k that become isomorphic to $(M_n(k_s), \sigma_b)$ when we extend scalars to k_s are in bijection with $H^1(k, PG_\sigma)$.

If b is skew-symmetric, then $PG_\sigma = PGSp_n \simeq PSp_n$, and this proves part (i) of Proposition 2. Now if b is symmetric, then $PG_\sigma = PGO(b)$, and since all n-dimensional quadratic forms are isomorphic over k_s, this proves part (ii).

We will also use the following result:

PROPOSITION 3. *The morphism* Int: $PG_\sigma \to O^+(T_{\sigma_b})$ *induces a morphism* Int*: $H^1(k, PG_\sigma) \to H^1(k, O^+(T_{\sigma_b}))$ *which maps the isomorphism class* $[A, \sigma]$ *to the isometry class of* T_σ.

Proof of Proposition 3. Let P be a matrix in G_σ, and let $\lambda(P)$ be its similarity factor. We have $P^tBP = \lambda(P)B$. Hence, for any $M \in M_n(k)$,

$$\begin{aligned} T_{\sigma_b}(\mathrm{Int}(P)(M)) &= \mathrm{Tr}\Big(B^{-1}(P^{-1})^t M^t P^t BPMP^{-1}\Big) \\ &= \mathrm{Tr}\Big(P^{-1}B^{-1}(P^{-1})^t M\,{}^tP^tBPM\Big) \\ &= \mathrm{Tr}\Big(\lambda(P)^{-1}B^{-1}M^t\lambda(P)BM\Big) = T_{\sigma_b}(M). \end{aligned}$$

If A is a central simple algebra over k, endowed with an involution of the first kind σ, then it is known that $A \otimes_k A$ is isomorphic to $\mathrm{End}_k(A)$, the isomorphism being given by $(a \otimes b)(z) = az\sigma(b)$, for any a, b and $z \in A$. In particular, $M_n(k) \otimes M_n(k) \simeq \mathrm{End}_k(M_n(k))$, and modulo this identification, the automorphism $\mathrm{Int}(P)$ corresponds to the tensor product $P \otimes (P^{-1})^t$. Hence, the determinant of $\mathrm{Int}(P)$ is 1, and Int is a morphism $PG_\sigma \to O^+(T_{\sigma_b})$.

Now, if c is the cocycle representing $[A, \sigma]$ introduced in the proof of Proposition 2, then for any $\gamma \in \Gamma_k$, $\mathrm{Int}(c(\gamma)) = \phi^{-1} \circ {}^\gamma\phi$. Since σ is an isomorphism of algebras with involution $(M_n(k_s), \sigma_b) \to (A \otimes_k k_s, \sigma \otimes \mathrm{Id})$, it is also an isometry $(M_n(k_s), T_{\sigma_b}) \to (A \otimes k_s, (T_\sigma)_{k_s})$. Indeed for any $x \in A$, we have

$$\begin{aligned} T_{\sigma_b}(\phi(x \otimes 1)) &= \mathrm{Tr}(\sigma_b(\phi(x \otimes 1))\phi(x \otimes 1)) \\ &= \mathrm{Tr}(\phi((\sigma \otimes \mathrm{Id})(x \otimes 1))\phi(x \otimes 1)) \\ &= \mathrm{Tr}(\phi(\sigma(x)x \otimes 1)) = \mathrm{Trd}_A(\sigma(x)x) = T_\sigma(x). \end{aligned}$$

Hence the cocycle Int*(c): $\gamma \mapsto \mathrm{Int}(c(\gamma))$ represents the isometry class of the form T_σ in $H^1(k, O^+(T_{\sigma_b}))$. This completes the proof of Proposition 3.

Hence, we now have the diagram

$$\begin{array}{ccccccccc} & & & & & & PG_\sigma & & \\ & & & & & & \downarrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(T_{\sigma_b}) & \longrightarrow & O^+(T_{\sigma_b}) & \longrightarrow & 1. \end{array}$$

To complete the proof of Theorem 1, we have to distinguish the symplectic and orthogonal cases. Let us assume first that σ is of symplectic type. In that case, b is skew-symmetric and $PG_\sigma = PSp_n$. Since we have a morphism $PSp_n \to O^+(T_{\sigma_b})$, we also have a morphism between the corresponding simply connected covers $Sp_n \to Spin(T_{\sigma_b})$ [4, Proposition 2.24(i), p. 262]. Hence we have the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Sp_n & \longrightarrow & PSp_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Int}} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(T_{\sigma_b}) & \longrightarrow & O^+(T_{\sigma_b}) & \longrightarrow & 1. \end{array}$$

This diagram induces

$$\begin{array}{ccc} H^1(k, PSp_n) & \xrightarrow{\Delta_2} & H^2(k, \mu_2) \\ \downarrow{\scriptstyle \mathrm{Int}^*} & & \downarrow{\scriptstyle \tau_1^*} \\ H^1(k, O^+(T_{\sigma_b})) & \xrightarrow{\delta_2} & H^2(k, \mu_2). \end{array}$$

Moreover, we can check that the map Δ_2 maps the isomorphism class of the algebra with involution (A, σ) to $[A] \in \mathrm{Br}_2(k)$. Indeed, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Sp_n & \rightarrow & PSp_n & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n & \longrightarrow & SL_n & \longrightarrow & PGL_n & \longrightarrow & 1, \end{array}$$

which induces

$$\begin{array}{ccc} H^1(k, PSp_n) & \xrightarrow{\Delta_2} & H^2(k, \mu_2) \\ \downarrow & & \downarrow \\ H^1(k, PGL_n) & \xrightarrow{\Delta} & H^2(k, \mu_n). \end{array}$$

It is known that Δ maps the isomorphism class of a degree n algebra A to its class $[A] \in \mathrm{Br}_n(k)$ (see for instance [21]). Hence, the isomorphism class $[A, \sigma] \in H^1(k, PSp_n)$ is mapped to $[A] \in \mathrm{Br}_2(k)$ under Δ_2.

Moreover, we proved in Proposition 3 that the image of $[A, \sigma]$ under Int* is $[T_\sigma]$. Hence, we have $\delta_2([T_\sigma]) = \tau_1^*([A])$. From this, and from the cohomological definition of the Hasse invariant, we deduce

$$w_2(T_\sigma) = w_2\big(T_{\sigma_b}\big) + \delta_2\big([T_\sigma]\big) = m(-1, -1) + \tau_1^*\big([A]\big).$$

Replacing k, if necessary, by the rational function field $k(t_1, t_2)$, we may assume that there exists a division quaternion algebra Q over k. Since τ_1 is a morphism $\mu_2 \to \mu_2$, it can only be zero or the identity. But the computation we made in Subsection 2.1.3 for the Hasse invariant of the trace form of the involution $\tau' = t \otimes \tau$ on the algebra $M_m(k) \otimes_k Q$ proves that τ_1 is zero if m is even and the identity if m is odd. Hence $\tau_1([A]) = m[A]$ and this proves the result in the symplectic case.

Since the group $PGO(q_0)$ is not connected, the orthogonal case is slightly more complicated. We actually have to work with a fixed determinant. Precisely, we first prove the following:

PROPOSITION 4. *Isomorphism classes of degree n central simple algebras endowed with an involution of orthogonal type and of determinant* $d(q_0)$ *are in one-to-one correspondence with* $H^1(k, PGO^+(q_0))$.

Proof of Proposition 4. To prove this proposition, we first construct an exact sequence. Over k_s, the groups PO and PGO are isomorphic. Hence, we also have $PO^+ \simeq PGO^+$. Since n is even, the determinant $O(q_0) \to \mu_2$ maps both Id and $-$Id to 1. So, we have the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mu_2 & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & O^+(q_0) & \longrightarrow & O(q_0) & \xrightarrow{\det} & \mu_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & PO^+(q_0) & \longrightarrow & PO(q_0) & \longrightarrow & \mu_2 & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

from which we deduce the exact sequence

$$1 \to PGO^+(q_0) \to PGO(q_0) \overset{p}{\to} \mu_2 \to 1.$$

Over k, this sequence induces

$$H^1(k, PGO^+(q_0)) \to H^1(k, PGO(q_0)) \overset{p^*}{\to} k^*/k^{*2}.$$

We proved in Proposition 2 that $H^1(k, PGO(q_0))$ is in one-to-one correspondence with isomorphism classes of degree n central simple algebras endowed with an involution of orthogonal type. Since the set $H^1(k, PGO^+(q_0))$ is in bijection with the kernel of p^*, the next lemma now clearly implies Proposition 4.

LEMMA 3. *The map p^* maps the isomorphism class $[A, \sigma]$ to the quotient $d(\sigma)/d(q_0) \in k^*/k^{*2}$.*

Let us denote by σ_0 the adjoint involution with respect to q_0. In order to prove Lemma 3, we first prove the following one.

LEMMA 4. *There exists a morphism $\mathrm{Int}_+ \colon PGO(q_0) \to O(T^+_{\sigma_0})$ such that the induced morphism $\mathrm{Int}^*_+ : H^1(k, PGO(q_0)) \to H^1(k, O(T^+_{\sigma_0}))$ maps $[A, \sigma]$ to the isometry class of T^+_σ.*

Proof of Lemma 4. The proof of this lemma is analogous to the proof of Proposition 3. We have already seen that for any $P \in PGO(q_0)$, $\mathrm{Int}(P)$ is an isometry for T_{σ_0}. Let $\mathrm{Int}_+(P)$ be its restriction to M_n^+. It is an isometry for $T^+_{\sigma_0}$.

Let $\phi\colon (M_n(k_s), \sigma_0) \to (A \otimes_k k_s, \sigma \otimes \mathrm{Id})$ be an isomorphism of algebras with involutions. Such a ϕ exists since all orthogonal involutions of $M_n(k_s)$ are isomorphic. The cocycle $c \in Z^1(k, PGO(q_0))$ determined by $\mathrm{Int}(c(\gamma)) = \phi^{-1} \circ {}^\gamma\phi$ represents $[A, \sigma]$.

Moreover, we have proved that ϕ is an isometry between T_{σ_0} and $(T_\sigma)_{k_s}$. Hence, the induced morphism $\phi^+\colon M_n(k_s)^+ \to A^+ \otimes_k k_s$ is an isometry between $T^+_{\sigma_0}$ and $(T^+_\sigma)_{k_s}$. From this, we deduce that the cocycle $\mathrm{Int}^*_+(c)\colon \gamma \mapsto \mathrm{Int}_+(c(\gamma)) = \phi^{+-1} \circ {}^\gamma\phi^+$ represents the isometry class of T^+_σ.

Proof of Lemma 3. Since we have a morphism $PGO(q_0) \to O(T^+_{\sigma_0})$, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & PGO^+(q_0) & \longrightarrow & PGO(q_0) & \xrightarrow{p} & \mu_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{Int}_+} & & \downarrow{\scriptstyle \nu_0} & & \\ 1 & \longrightarrow & O^+(T^+_{\sigma_0}) & \longrightarrow & O(T^+_{\sigma_0}) & \xrightarrow{d} & \mu_2 & \longrightarrow & 1. \end{array}$$

Moreover, it is easy to check that ν_0 is non-trivial. Indeed, let B be a diagonal matrix representing q_0 in a fixed basis, and let A be the diagonal matrix with coefficients $(1, \ldots, 1, -1)$. The matrix of $\mathrm{Int}_+(A)$ in the basis $(B^{-1}e_{ii})_{1 \le i \le n} \cup (B^{-1}(e_{ij} + e_{ji}))_{1 \le i < j \le n}$ of $M_n(k_s)^+$ can be easily computed, and we get $d(\mathrm{Int}_+(A)) = -1$. Since ν_0 is a morphism $\mu_2 \to \mu_2$, this proves that ν_0 is the identity.

We now have the diagram

$$\begin{array}{ccc} H^1(k, PGO(q_0)) & \xrightarrow{p^*} & H^1(k, \mu_2) \\ \downarrow{\scriptstyle \mathrm{Int}_+^*} & & \downarrow{\scriptstyle \mathrm{Id}} \\ H^1(k, O(T_{\sigma_0}^+)) & \xrightarrow{d^*} & H^1(k, \mu_2). \end{array}$$

It is known that d^* maps $[T_\sigma^+]$ to $d(T_\sigma^+)/d(T_{\sigma_0}^+)$. Using the definition of the determinant of an involution given in Subsection 2.2, we get

$$p^*([A, \sigma]) = d(\sigma)/d(\sigma_0).$$

This completes the proof of Lemma 3.

Let us now go back to the proof of Theorem 1. For any $\varepsilon \in k^*/k^{*2}$, we let q_0^ε be the n-dimensional quadratic form $q_0^\varepsilon = \langle 1, -1, \dots, 1, -1, 1, -\varepsilon \rangle$. We call σ_0^ε the associated involution on $M_n(k)$ or $M_n(k_s)$. Its determinant is $(-1)^m \varepsilon$. We will simply denote by q_0 the form q_0^1, and by σ_0 the corresponding involution. From Proposition 4, we deduce that $H^1(k, PGO^+(q_0^\varepsilon))$ classifies isomorphism classes of algebras endowed with an involution of orthogonal type and of determinant $(-1)^m \varepsilon$.

We now use the following lemma:

LEMMA 5. *We have the commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & O^+(q_0^\varepsilon) & \longrightarrow & PGO^+(q_0^\varepsilon) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau_2} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Int}} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(T_{\sigma_0^\varepsilon}) & \longrightarrow & O^+(T_{\sigma_0^\varepsilon}) & \longrightarrow & 1. \end{array}$$

Lemma us assume for the moment that this lemma is proved. We then obtain the diagram

$$\begin{array}{ccc} H^1(k, PGO^+(q_0^\varepsilon)) & \xrightarrow{\Delta_2} & H^2(k, \mu_2) \\ \downarrow{\scriptstyle \mathrm{Int}^*} & & \downarrow{\scriptstyle \tau_2^*} \\ H^1(k, O^+(T_{\sigma_0^\varepsilon})) & \xrightarrow{\delta_2} & H^2(k, \mu_2). \end{array}$$

As in the symplectic case, we get $\delta_2([T_\sigma]) = \tau_2^*([A])$. Hence we have $w_2(T_\sigma) = w_2(T_{\sigma_0^\varepsilon}) + \delta_2([T_\sigma]) = (-1, d(\sigma)) + \tau_2^*([A])$. We now conclude as in the symplectic case.

Replacing k, if necessary, by the rational function field $k(t_1, t_2)$, we may assume that there exists a division quaternion algebra over k. Since τ_2 is a morphism $\mu_2 \to \mu_2$, it can only be zero or the identity. But in subsection

2.1.3, we computed the Hasse invariant of the trace form of the involution $\sigma' = t \otimes \sigma$ on the algebra $A = M_m(k) \otimes_k Q$, and we proved $w_2(T_{\sigma'}) = m(-1, -a) + m[Q] = m(-1, -a) + m[A]$. Moreover, one can easily check that the determinant of the involution σ' is equal to $(-a)^m$. Hence we actually have $w_2(T_{\sigma'}) = (-1, d(\sigma')) + m[A]$. This proves that τ_2 is zero if m is even and the identity if m is odd, and gives the result in the orthogonal case.

Proof of Lemma 5. We have already seen that for any matrix $B \in GO(q_0^\varepsilon)$ the associated inner automorphism is in $O^+(T_{\sigma_0^\varepsilon})$. Hence we have a morphism $O^+(q_0^\varepsilon) \to O^+(T_{\sigma_0^\varepsilon})$, and a morphism between the corresponding simply connected covers $Spin(q_0^\varepsilon) \to Spin(T_{\sigma_0^\varepsilon})$ (see again [4, Proposition 2.24(i), p. 262]), which yields the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(q_0^\varepsilon) & \longrightarrow & O^+(q_0^\varepsilon) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Int}} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(T_{\sigma_0^\varepsilon}) & \longrightarrow & O^+(T_{\sigma_0^\varepsilon}) & \longrightarrow & 1. \end{array}$$

LEMMA 6. *The map ν is equal to* 0.

This lemma implies the previous one. Indeed, the fact that $\nu = 0$ implies that the morphism $Spin(q_0^\varepsilon) \to Spin(T_{0_0^\varepsilon})$ factors through μ_2 and induces a morphism $O^+(q_0^\varepsilon) \to Spin(T_{\sigma_0^\varepsilon})$.

Proof of Lemma 6. Let us first consider the case when $\varepsilon = 1$. The previous diagram induces

$$\begin{array}{ccc} 0^+(q_0)(k) & \xrightarrow{\mathrm{SN}_1} & k^*/k^{*2} \\ \downarrow{\scriptstyle \mathrm{Int}} & & \downarrow{\scriptstyle \nu^*} \\ O^+(T_{\sigma_0})(k) & \xrightarrow{\mathrm{SN}_2} & k^*/k^{*2}, \end{array}$$

where SN_1 and SN_2 are spinor norms, respectively, with respect to q_0 and T_{σ_0}. Since ν can only be 0 or the identity, to prove Lemma 6, it suffices to find an element $f \in O^+(q_0)(k)$ such that $\mathrm{SN}_1(f) \neq 1$ and $\mathrm{SN}_2(\mathrm{Int}(f)) = \nu^*(\mathrm{SN}_1(f)) = 1$.

If v_1 is any non-isotropic vector of $V = k^n$, and if τ_{v_1} is the orthogonal reflexion with respect to v_1, then $\mathrm{SN}_2(\mathrm{Int}(\tau_{v_1})) = 1$. Indeed, let $(v_2, \dots, v_n)$ be such that $(v_1, \dots, v_n)$ is an orthogonal basis of (V, q_0), and let $a_i = q_0(v_i)$. Then one can check that the standard basis $(f_{ij})_{1 \le i, j \le n}$ of $M_n(k) =$

$\mathrm{End}_k(V)$ defined by $f_{ij}(v_l) = \delta_{il} v_j$ is orthogonal for T_{σ_0} and $T_{\sigma_0}(f_{ij}) = a_i^{-1} a_j$. Moreover,

$$\mathrm{Int}(\tau_{v_1})(f_{ij}) = \tau_{v_1} \circ f_{ij} \circ \tau_{v_1} = \begin{cases} f_{ij} & \text{if } \begin{cases} i = 1 \\ j = 1 \end{cases} \text{or } \begin{cases} i \neq 1 \\ j \neq 1 \end{cases} \\ -f_{ij} & \text{if } \begin{cases} i = 1 \\ j \neq 1 \end{cases} \text{or } \begin{cases} i \neq 1 \\ j = 1 \end{cases}. \end{cases}$$

Hence if $\tau_{f_{ij}}$ is the orthogonal reflexion in $(M_n(k),\ T_{\sigma_0})$ with respect to f_{ij}, we get $\mathrm{Int}(\tau_{v_1}) = \tau_{f_{12}} \circ \cdots \circ \tau_{f_{1n}} \circ \tau_{f_{21}} \circ \cdots \circ \tau_{f_{n1}}$. It is now easy to compute $\mathrm{SN}_2(\mathrm{Int}(\tau_{v_1})) = 1$.

The form $\langle 1, -1 \rangle$ represents all elements of k^*, and replacing k if necessary by the rational function field $k(t)$, we may assume that k is not quadratically closed. Hence there exists $v_1 \in V$ such that $q_0(v_1) \notin k^{*2}$. There also exists $w_1 \in V$ such that $q_0(w_1) = 1$. Let $f = \tau_{v_1} \circ \tau_{w_1}$. We have $f \in O^+(q_0)$, $\mathrm{SN}_1(f) = q_0(v_1) \neq 1$ and $\mathrm{SN}_2(\mathrm{Int}(f)) = \mathrm{SN}_2(\mathrm{Int}(\tau_{w_1}) \circ \mathrm{Int}(\tau_{v_1}))$ $= \mathrm{SN}_2(\mathrm{Int}(\tau_{w_1}))\mathrm{SN}_2(\mathrm{Int}(\tau_{v_1})) = 1$. Lemma 6 is now proved in that case.

Let us assume now that $\varepsilon \neq 1 \in k^*/k^{*2}$. Let $L = k(\sqrt{\varepsilon})$. Since ε is a square in L, the extension to L of the form q_0^ε is q_0. Hence we have the two diagrams

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(k) & \longrightarrow & Spin(q_0^\varepsilon)(k) & \longrightarrow & O^+(q_0^\varepsilon)(k) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \nu_1} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2(k) & \longrightarrow & Spin(T_{\sigma_0^\varepsilon})(k) & \longrightarrow & O^+(T_{\sigma_0^\varepsilon})(k) & \longrightarrow & 1, \end{array} \tag{1}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2(L) & \longrightarrow & Spin(q_0)(L) & \longrightarrow & O^+(q_0)(L) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \nu_2} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2(L) & \longrightarrow & Spin(T_{\sigma_0})(L) & \longrightarrow & O^+(T_{\sigma_0})(L) & \longrightarrow & 1, \end{array} \tag{2}$$

with $\nu_2 = 0$. Diagram (2) can be deduced from diagram (1) by scalar extension. Hence, $\nu_2 = 0$ implies $\nu_1 = 0$. This completes the proof of Theorem 1.

3. INVOLUTIONS OF THE SECOND KIND

From now on, we assume that we are in case (ii). Hence we consider algebras with involution (A, σ) that satisfy hypothesis (H): A is a semi-simple algebra over k, $Z(A)$ is a quadratic étale extension of k, $\dim_k(A)$

$= 2n^2$, and σ is of the second kind. This means that either A is a central simple algebra over a quadratic separable field extension K of k, endowed with an involution of the second kind σ, or there exists a central simple algebra B over k such that $A = B \times B^0$, where B^0 denotes the opposite algebra of B, and σ is given by $\sigma(x, y) = (y, x)$.

In both cases, we let $\alpha \in k^*/k^{*2}$ be a generator of the quadratic étale extension $Z(A)/k$, i.e., $Z(A) = k[X]/(X^2 - \alpha)$, and we denote by $\overline{}$ the only non-trivial k-automorphism of $Z(A)$. Moreover, we denote by m the integral part of $n/2$, i.e., $n = 2m$ or $n = 2m + 1$.

3.1. *Classification of the Algebras* (A, σ)

Let A be the group of automorphisms of the algebra with involution $(M_n^2, *)$, where $*$ is defined by $(X, Y)^* = (Y^t, X^t)$. The group A is generated by the morphisms f_0: $(X, Y) \mapsto (Y, X)$, and f_P: $(X, Y) \mapsto (PXP^{-1}, (P^{-1})^t YP^t)$ for all $P \in PGL_n$. Hence, we have the exact sequence

$$1 \to PGL_n \to \mathsf{A} \xrightarrow{p} \mu_2 \to 1,$$

and PGL_n corresponds to the elements in A that act trivially on the center of M_n^2.

PROPOSITION 5. *We have one-to-one correspondences between*

(i) $H^1(k, \mathsf{A})$ *and isomorphism classes of algebras with involution* (A, σ) *satisfying the hypothesis* (H) (*see* [23, III 1.4]).

(ii) *The quotient of* $H^1(k, PGL_n)$ *by the action of* μ_2 *and isomorphism classes of algebras with involution* (A, σ) *satisfying* (H) *and such that* $Z(A)$ *is split*.

Remark. When $Z(A)$ is split, there exists a central simple algebra B over k such that $A = B \times B^0$. It is known that $H^1(k, PGL_n)$ is in one-to-one correspondence with the isomorphism classes of central simple algebras B of degree n over k (see [21, X.5]). But we have to consider here the quotient of $H^1(k, PGL_n)$ by the action of μ_2. Indeed, the algebras with involution $(B \times B^0, (x, y) \mapsto (y, x))$ and $(B^0 \times B, (x, y) \mapsto (y, x))$ are isomorphic even though B and B^0 are not isomorphic in general.

Proof of Proposition 5. The proof of part (i) of Proposition 5 is analogous to the beginning of the proof of Proposition 2. We have to notice that the algebra with involution $(A \otimes_k k_s, \sigma \otimes \mathrm{Id})$ is isomorphic to $(M_n^2(k_s), *)$. As we will need it later, we give here an explicit description of this isomorphism.

Let us assume first that $Z(A)$ is split, and let B be a central simple algebra over k such that $A = B \times B^0$. Let ϕ: $B \otimes_k k_s \to M_n(k_s)$ be an isomorphism. Then ϕ composed with the transposition is an isomorphism

$B^0 \otimes_k k_s \to M_n(k_s)$, and one can easily check the following:

LEMMA 7. *The map*

$$\Phi\colon M_n(k_s)^2 \to A \otimes_k k_s = (B \otimes_k k_s) \times \left(B^0 \otimes_k k_s\right)$$
$$(X,Y) \mapsto \left(\phi^{-1}(X), \phi^{-1}(Y^t)\right)$$

is an isomorphism. Moreover, the inverse of Φ *is given by* $\Phi^{-1}((x,y) \otimes \lambda) = (\phi(x \otimes \lambda), \phi(y \otimes \lambda)^t)$.

Let us assume now that $Z(A)$ is a field extension of k, $Z(A) = K = k(\sqrt{\alpha})$. Let ϕ be an isomorphism $\phi\colon A \otimes_K k_s \to M_n(k_s)$. Again one can easily check the following two lemmas:

LEMMA 8. *Let i and j be the maps*

$$i\colon A \otimes_k k_s \to A \otimes_K k_s$$
$$x \otimes_k \lambda \mapsto x \otimes_K \lambda,$$
$$j\colon A \otimes_K k_s \to A \otimes_k k_s$$
$$x \otimes_K \lambda \mapsto \frac{x}{2} \otimes_k \lambda + \frac{x}{2\sqrt{\alpha}} \otimes_k \lambda\sqrt{\alpha}\,.$$

They are well defined, and satisfy

(i) $i \circ j = \mathrm{Id}$;

(ii) $i \circ (\sigma \otimes \mathrm{Id}) \circ j = 0$;

(iii) $j \circ i + (\sigma \otimes \mathrm{Id}) \circ j \circ i \circ (\sigma \otimes \mathrm{Id}) = \mathrm{Id}$.

LEMMA 9. *The map*

$$\Phi\colon M_n^2(k_s) \to A \otimes_k k_s$$
$$(X,Y) \mapsto j \circ \phi^{-1}(X) + (\sigma \otimes \mathrm{Id}) \circ j \circ \phi^{-1}(Y^t)$$

is an isomorphism. Moreover, the inverse of Φ *is given by* $\Phi^{-1}(x \otimes_k \lambda) = (\phi \circ i(x \otimes_k \lambda), (\phi \circ i \circ (\sigma \otimes \mathrm{Id})(x \otimes_k \lambda))^t)$.

Let us now prove part (ii) of Proposition 5. The exact sequence

$$1 \to PGL_n \to \mathsf{A} \to \mu_2 \to 1$$

induces

$$1 \to PGL_n(k) \to \mathsf{A}(k) \xrightarrow{p} \mu_2 \to H^1(k, PGL_n) \to H^1(k, \mathrm{A}) \xrightarrow{p^*} k^*/k^{*2}.$$

But, in contrast with the previous cases, μ_2 does not act trivially on PGL_n. In particular, if $a\colon \gamma \mapsto a(\gamma)$ is a cocycle in $Z^1(k, PGL_n)$ that does not have all its values in μ_2, and if we denote by $(a^{-1})^t$ the cocycle $\gamma \mapsto (a(\gamma)^{-1})^t$, then a and $(a^{-1})^t$ are not cohomologically equivalent in $Z^1(k, PGL_n)$, but they do correspond to the same class in $H^1(k, \mathsf{A})$. Indeed, one can check that $(a(\gamma)^{-1})^t = f_0^{-1} a(\gamma)^\gamma f_0$. Hence the kernel of p^* is in bijection with the quotient of $H^1(k, PGL_n)$ by the action of μ_2. The next lemma now clearly implies part (ii) of Proposition 5:

Lemma 10. *The image under p^* of the isomorphism class of (A, σ) is a generator α of $Z(A)$ over k.*

Proof. Let $a \in Z^1(k, \mathsf{A})$ be the cocycle defined by

$$a(\gamma) = \Phi^{-1} \circ {}^\gamma\Phi = \Phi^{-1} \circ (\mathrm{Id} \otimes \gamma) \circ \Phi \circ \gamma^{-1}.$$

It represents the class of (A, σ), and we want to compute $p^*(a)$. For any $f \in \mathsf{A}$, $p(f)$ is equal to 1 if f acts trivially on the center $k_s \times k_s$ of $M_n(k_s)^2$, and to -1 if f acts by permuting the two factors.

Let us assume first that $\alpha = 1$. Then it is easy to check that for any $\gamma \in \Gamma_k$, $\Phi^{-1} \circ (\mathrm{Id} \otimes \gamma) \circ \Phi \circ \gamma^{-1}(I_n, 0) = (I_n, 0)$. Hence for any $\gamma \in \Gamma_k$, $p(a(\gamma)) = 1$, and this proves that $p^*(\gamma)$ is the trivial cocycle.

Let us consider now the case when $\alpha \neq 1$. One can easily check the following lemma:

Lemma 11. *We have*

$$i \circ (\mathrm{Id} \otimes \gamma) \circ j = \begin{cases} \mathrm{Id} \otimes \gamma & \textit{if } \gamma(\sqrt{\alpha}) = \sqrt{\alpha} \\ 0 & \textit{if } \gamma(\sqrt{\alpha}) = -\sqrt{\alpha}\,; \end{cases}$$

$$i \circ (\sigma \otimes \gamma) \circ j = \begin{cases} 0 & \textit{if } \gamma(\sqrt{\alpha}) = \sqrt{\alpha} \\ \sigma \otimes \gamma & \textit{if } \gamma(\sqrt{\alpha}) = -\sqrt{\alpha}\,. \end{cases}$$

Then, using the description of Φ given in Lemma 9, one can check that

$$a(\gamma)(I_n, 0) = \begin{cases} (I_n, 0) & \text{if } \gamma(\sqrt{\alpha}) = \sqrt{\alpha} \\ (0, I_n) & \text{if } \gamma(\sqrt{\alpha}) = -\sqrt{\alpha}\,. \end{cases}$$

Hence the cocycle $p^*(a)$ is given by

$$p^*(a)(\gamma) = \begin{cases} 1 & \text{if } \gamma(\sqrt{\alpha}) = \sqrt{\alpha} \\ -1 & \text{if } \gamma(\sqrt{\alpha}) = -\sqrt{\alpha}\,, \end{cases}$$

and this proves the result.

3.2. *Trace Forms*

The trace forms associated with the algebra with involution (A, σ) have been defined in Section 1. Let us denote by T the trace form of the algebra with involution $(M_n^2, *)$, and by T^+ its restriction to $(M_n^2)^+$. The form T is defined by $T(X, Y) = \mathrm{Tr}(Y^t X)$. The following proposition holds:

PROPOSITION 6. *There exists a morphism $\rho\colon \mathsf{A} \to O(T^+)$ such that the induced morphism $\rho^*\colon H^1(k, \mathsf{A}) \to H^1(k, O(T^+))$ maps the isomorphism class of the algebra with involution (A, σ) to the isometry class of T_σ^+.*

Proof. The proof is more or less the same as for Proposition 3 and Lemma 4. For any $f \in \mathsf{A}$, f is an isometry for the trace form T. Hence its restriction $\rho(f)$ to A^+ is an isometry for T^+.

Moreover, as Φ is a morphism of algebras with involutions $(M_n(k_s)^2, *) \to (A \otimes_k k_s, \sigma \otimes \mathrm{Id})$, it is also an isometry

$$\left(M_n(k_s)^2, T\right) \to \left(A \otimes_k k_s, (T_\sigma)_{k_s}\right).$$

Hence the induced morphism $\Phi^+\colon (M_n(k_s)^2)^+ \to A^+ \otimes_k k_s$ is an isometry between T^+ and $(T_\sigma^+)_{k_s}$. This proves that $\rho^*(c)\colon \gamma \mapsto \rho(\Phi^{-1} \circ {}^\gamma\Phi) = \Phi^{+-1} \circ {}^\gamma\Phi^+$ is a cocycle that represents the isometry class of T_σ^+.

The next result is the following:

PROPOSITION 7. (i) *We have the commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & PGL_n & \longrightarrow & \mathsf{A} & \longrightarrow & \mu_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tau} & & \\
1 & \longrightarrow & O^+(T^+) & \longrightarrow & O(T^+) & \longrightarrow & \mu_2 & \longrightarrow & 1.
\end{array}$$

(ii) *The map τ is given by $\tau(x) = x^{n(n-1)/2}$.*

Proof. To prove (i), we have to prove that $\rho(f_P)$ is of determinant 1 for any $P \in PGL_n$. But the $*$-invariant elements of M_n^2 are

$$\left(M_n^2\right)^+ = \left\{(X, X^t), X \in M_n\right\}.$$

Hence $((e_{ij}, e_{ji}))_{1 \le i, j \le n}$ is a basis of $(M_n^2)^+$. The matrix of $\rho(f_P)$ in this basis is $P \otimes (P^{-1})^t$. Hence we have $\det(\rho(f_P)) = 1$.

To prove part (ii), we must compute the determinant of $\rho(f_0)$. Let us order the basis of $(M_n^2)^+$ as follows $\{(e_{ii}, e_{ii})_{1 \le i \le n}, ((e_{ij}, e_{ji}),$

$(e_{ji}, e_{ij}))_{1 \leq i < j \leq n}\}$. Then, the matrix of $\rho(f_0)$ is

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & 1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}.$$

Hence $\det(\rho(f_0)) = (-1)^{n(n-1)/2}$, and this proves part (ii) of the proposition.

3.3. *Invariants of the Trace Form T_σ*

We can compute the invariants of the trace form T_σ. We obtain the following result:

PROPOSITION 8. *The determinant of T_σ is equal to $(-\alpha)^n$. Its Hasse invariant is trivial.*

To prove Proposition 8, we first prove the following lemma:

LEMMA 12. *We have $T_\sigma = \langle 1, -\alpha \rangle \otimes T_\sigma^+$.*

Let us assume first that α is different from 1. The map $A^+ \to A^-$, $x \mapsto \sqrt{\alpha} x$ induces an isometry between T_σ^- and $\langle -\alpha \rangle \otimes T_\sigma^+$. Now, if $\alpha = 1$, A^+ is the set $\{(x, x),\ x \in B\}$ and A^- is $\{(x, -x),\ x \in B\}$. So, the map $A^+ \to A^-, (x, x) \mapsto (x, -x)$ induces an isometry between T_σ^- and $\langle -1 \rangle \otimes T_\sigma^+$. Hence in both cases, we have $T_\sigma^- \simeq \langle -\alpha \rangle \otimes T_\sigma^+$. Since $T_\sigma = T_\sigma^+ \oplus T_\sigma^-$, this proves the lemma.

Since $\dim(T_\sigma^+) = n^2$, this already implies that the determinant of T_σ is $(-\alpha)^{n^2} = (-\alpha)^n \in k^*/k^{*2}$. In particular, it is trivial if n is even. Moreover, to compute the Hasse invariant of T_σ, it is actually enough to know the determinant of the form T_σ^+. It is given by the following:

LEMMA 13. *The determinant of T_σ^+ is $d(T_\sigma^+) = (-\alpha)^{n(n-1)/2}$.*

This can be easily deduced from Proposition 7. Indeed, diagram (i) induces

$$\begin{array}{ccc} H^1(k, A) & \xrightarrow{p^*} & k^*/k^{*2} \\ \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \tau^*} \\ H^1(k, O(T^+)) & \xrightarrow{\delta_1} & k^*/k^{*2}. \end{array}$$

Hence, we have $\delta_1(\rho^*([A, \sigma])) = \tau^*(p^*([A, \sigma]))$, i.e. $d(T_\sigma^+) = \tau^*(\alpha)d(T^+) = \alpha^{n(n-1)/2}d(T^+)$, and we only have left to prove that $d(T^+) = (-1)^{n(n-1)/2}$, which can be done by direct computation.

We end the proof of Proposition 8 applying Lemma 1(iv). We get $w_2(T_\sigma) = (n^2(n^2 - 1)/2)(-1, \alpha) + (n(n+1)/2)(\alpha, -\alpha) = 0$.

3.4. *Determinant Class Modulo* 2

In this section, we assume that n is even, $n = 2m$ (see Remark (i) below). To define a non-trivial invariant of the algebra with involution (A, σ), we consider the Hasse invariant of the restricted trace form T_σ^+. Precisely, we give the following definition:

DEFINITION 2. The determinant class modulo 2 of (A, σ) is $D(A, \sigma) = (m(m-1)/2)(-1, -\alpha) + m(\alpha, 2) + w_2(T_\sigma^+) \in \mathrm{Br}_2(k)$.

Remarks. (i) (See also [13].) If n is odd, $n = 2m + 1$, then we can give the same definition for the determinant class mod 2, but it is easy to check that it is always trivial. Indeed, this can be done by a direct computation in the split case (see proof of Proposition 10 below). In the general case, it is a consequence of Springer's theorem, since it is known that a maximal commutative subfield L of A is a splitting field for A, and has degree $n = 2m + 1$ over K.

(ii) In [8], Knus, Merkurjev, Rost, and Tignol defined an invariant of the algebra with involution (A, σ) which is a central simple algebra over k endowed with an involution of the first kind. One can deduce from their results that its class in the Brauer group of k is equal to $D(A, \sigma) + m(-1, \alpha)$. In particular, if m is even, then this class coincides with the determinant class mod 2.

The determinant class mod 2 can be easily computed in some special cases. Let us assume first that $\alpha = 1$. Then there exists a central simple algebra B of degree n over k such that $A = B \times B^0$ and σ is given by $\sigma(x, y) = (y, x)$. We then have the following:

PROPOSITION 9. *The determinant class* mod 2 *of* (A, σ) *is* $D(A, \sigma) = m[B]$.

Proof. In this case, the form T_σ^+ is isometric to the quadratic form $\mathrm{Trd}_B(x^2)$. The Hasse invariant of this form has been computed by Saltman (unpublished), Serre [23, annexe], Lewis and Morales [13], and Tignol [25]. They proved that $w_2(\mathrm{Trd}_B(x^2)) = (m(m-1)/2)(-1,-1) + m[B]$. Hence the determinant class mod 2 of (A, σ) is $D(A, \sigma) = m[B]$.

Let us assume now that $\alpha \neq 1$, and let $K = k(\sqrt{\alpha})$. We have the following:

PROPOSITION 10. *If A is the split algebra $M_n(K)$, and if σ is the adjoint involution σ_h with respect to some hermitian form h, then the determinant class* mod 2 *of (A, σ_h) is $D(A, \sigma_h) = (\alpha, d(h))$.*

Proof. Let B be a matrix of the hermitian form h, chosen as before in diagonal form. Its coefficients $b_1, \dots, b_n$ lie in k. Then, $M_n(K)^+$ consists of all matrices of the form $B^{-1}M + \sqrt{\alpha}B^{-1}N$, where M is a symmetric matrix, and N a skew-symmetric one. It is easy to check that the basis

$$\left(B^{-1}e_{ii}\right)_{1\le i\le n} \cup \left(B^{-1}(e_{ij}+e_{ji})\right)_{1\le i<j\le n} \cup \left(\sqrt{\alpha}B^{-1}(e_{ij}-e_{ji})\right)_{1\le i<j\le n}$$

of $M_n(K)^+$ is orthogonal for $T_{\sigma_h}^+$, and we get

$$T_{\sigma_h}^+ = \left\langle b_1^{-2}, \dots, b_n^{-2}\right\rangle \oplus \langle 1, -\alpha\rangle \otimes \psi,$$

where ψ is the form $\psi = \langle 2b_1^{-1}b_2^{-1}, 2b_1^{-1}b_3^{-1}, \dots, 2b_{n-1}^{-1}2b_{n-1}^{-1}\rangle$. Hence the Hasse invariant of $T_{\sigma_h}^+$ is $w_2(T_{\sigma_h}^+) = w_2(\langle 1, -\alpha\rangle \otimes \psi)$. Since the form ψ has dimension $N = n(n-1)/2 = m(2m-1)$ and determinant $2^m d(h)^{n-1}$, Lemma 1 gives $w_2(T_{\sigma_h}^+) = (N(N-1)/2)(-1,-\alpha) + (\alpha, 2^m d(h)^{n-1}) = (m(m-1)/2)(-1,-\alpha) + m(\alpha, 2) + (\alpha, d(h))$, and this proves the result.

Finally, we have the following:

PROPOSITION 11 (see also [8]). *Let us assume there exists a central simple algebra A_0 over k, endowed with an involution of the first kind σ_0, such that $A = A_0 \otimes_k K$ and $\sigma = \sigma_0 \otimes {}^-$. Then the determinant class* mod 2 *of (A, σ) is given by*

$$D(A,\sigma) = \begin{cases} (\alpha, d(\sigma_0)) + m[A_0] & \text{if } \sigma_0 \text{ is of orthogonal type}, \\ m(-1,\alpha) + m[A_0] & \text{if } \sigma_0 \text{ is of symplectic type}. \end{cases}$$

Proof. Since A^+ is the direct sum of A_0^+ and $\sqrt{\alpha}A_0^-$, we have a decomposition of T_σ^+ as

$$T_\sigma^+ = T_{\sigma_0}^+ \oplus \langle -\alpha\rangle \otimes T_{\sigma_0}^-.$$

We have seen that $T_{\sigma_0}^+$ and $T_{\sigma_0}^-$ both have determinant $2^m d(\sigma_0)$.

Let us assume that σ_0 is of orthogonal type. Then $T_{\sigma_0}^-$ has dimension $n(n-1)/2$. Hence by Lemma 1(i) and (iii), we get $w_2(T_\sigma^+) = w_2(T_{\sigma_0}^+) + w_2(T_{\sigma_0}^-) + (\alpha, d(\sigma_0)) + (m(m-1)/2)(-1,-\alpha) + m(\alpha, 2)$. Using Theorem 1, we see that $w_2(T_{\sigma_0}^+) + w_2(T_{\sigma_0}^-) = w_2(T_{\sigma_0}) - (-1, d(\sigma_0))$ is equal to $m[A_0]$, and this gives the result in the orthogonal case.

If σ_0 is of symplectic type, then its determinant is trivial, and $T_{\sigma_0}^-$ has dimension $n(n+1)/2$. Hence, again applying Lemma 1, we get $w_2(T_\sigma^+) = w_2(T_{\sigma_0}^+) + w_2(T_{\sigma_0}^-) + m(-1,-\alpha) + (m(m-1)/2)(-1,-\alpha) + m(\alpha, 2)$. Moreover Theorem 1 in that case gives $w_2(T_{\sigma_0}^+) + w_2(T_{\sigma_0}^-) = w_2(T_{\sigma_0}) = m(-1,-1) + m[A_0]$, and this ends the proof of this proposition.

3.5. *Cohomological Description of $D(A, \sigma)$*

In this section, if α is any element of k^*/k^{*2}, we will also denote by α the associated cocycle in $Z^1(k, \mu_2)$, that is, $\alpha_\gamma = \gamma(\sqrt{\alpha})/\sqrt{\alpha}$. We let ${}_\alpha\mu_n$ be the group μ_n twisted by the cocycle α. This means that Γ_k acts on ${}_\alpha\mu_n$ by

$$\gamma^* x = \begin{cases} {}^\gamma x & \text{if } \alpha_\gamma = 1 \\ {}^\gamma x^{-1} & \text{if } \alpha_\gamma = -1 \end{cases}, \qquad \text{for any } \gamma \in \Gamma_k.$$

Let (A, σ) be an algebra with involution satisfying (H), and let α be a generator of the extension $Z(A)/k$. With the isomorphism class of (A, σ), we associate an invariant $D(A, \sigma)$ which lies in the quotient of $H^2(k, {}_\alpha\mu_n)$ by the action of μ_2, and which we call the determinant class of (A, σ). This can be done for any value of the degree n.

If n is even, we have a non-trivial morphism ${}_\alpha\mu_n \to \mu_2$. Moreover, for any $x \in {}_\alpha\mu_n$, x and x^{-1} have the same image in μ_2. Hence, we have a morphism $H^2(k, {}_\alpha\mu_n)/\mu_2 \to H^2(k, \mu_2)$. The main result of this section is the following:

THEOREM 2. *If n is even, then the morphism $H^2(k, {}_\alpha\mu_n)/\mu_2 \to H^2(k, \mu_2)$ maps the determinant class $D(A, \sigma)$ to the determinant class* mod 2.

I thank J.-P. Serre for the useful comments and suggestions he made concerning this section.

In order to explain how $D(A, \sigma)$ is defined and to prove this theorem, we need a few preliminaries.

3.5.1. *Classification of Algebras with a Fixed Center*

We have seen in Proposition 5 that isomorphism classes of algebras with involution (A, σ) satisfying (H) and such that $Z(A)$ is split are classified by the quotient of $H^1(k, PGL_n)$ by the action of μ_2.

Let $\alpha \in k^*/k^{*2}$, $\alpha \neq 1$, and let $K = k(\sqrt{\alpha})$. Following [23, I.5], to classify isomorphism classes of algebras with involution (A, σ) satisfying (H) and such that $Z(A) = K$ we have to twist the group PGL_n by a cocycle. Precisely, if G is an algebraic group on which μ_2 acts, we denote by ${}_\alpha G$ the group G twisted by the cocycle α. If the action of μ_2 on G is denoted by $x \mapsto \epsilon.x$, then the action of Γ_k on ${}_\alpha G$ is given by ${}^{\gamma *}x = \alpha_\gamma.{}^\gamma x$ for any $\gamma \in \Gamma_k$. In particular, Γ_k acts on ${}_\alpha \mathsf{A}$ by

$$
{}^{\gamma *}f = \begin{cases} {}^\gamma f & \text{if } \alpha_\gamma = 1, \\ f_0^{-1} \circ {}^\gamma f \circ f_0 & \text{if } \alpha_\gamma = -1. \end{cases}
$$

Moreover, for any $P \in PGL_n$, we have $f_0^{-1} \circ f_P \circ f_0 = f_{(P^{-1})^t}$. Hence the action of Γ_k on ${}_\alpha PGL_n$ is given by

$$
{}^{\gamma *}P = \begin{cases} \gamma(P) & \text{if } \alpha_\gamma = 1, \\ \left(\gamma(P)^{-1}\right)^t & \text{if } \alpha_\gamma = -1. \end{cases}
$$

We now have the following:

PROPOSITION 12. *There is a one-to-one correspondence between the quotient of $H^1(k, {}_\alpha PGL_n)$ by the action of μ_2 and isomorphism classes of algebras with involution (A, σ) satisfying* (H) *and such that $Z(A) = k[X]/(X^2 - \alpha)$.*

Proof. If $\alpha = 1$, this has already been proved in Proposition 5. Let us assume now that $\alpha \neq 1$. The proof we give here is explained in a more general context in [23, I.5.5, Corollaire 2]. We denote by σ_α the involution defined on the algebra $M_n(K)$ by $\sigma_\alpha(M) = \overline{M}^t$. Let ϕ be the natural isomorphism $M_n(K) \otimes_K k_s \to M_n(k_s)$, let Φ be the morphism $M_n(k_s)^2 \to A \otimes_k k_s$ described in Lemma 9, and let $a \in Z^1(k, \mathsf{A})$ be the cocycle representing the isomorphism class of $(M_n(K), \sigma_\alpha)$ defined by $a(\gamma) = \Phi^{-1} \circ {}^\gamma \Phi$. The cocycle a is given by

$$
a(\gamma) = \begin{cases} \mathrm{Id} & \text{if } \alpha_\gamma = 1 \\ f_0 & \text{is } \alpha_\gamma = -1. \end{cases}
$$

Hence a is the image of the cocycle α under the map $\mu_2 \to \mathsf{A}$ defined by $1 \mapsto \mathrm{Id}$ and $-1 \mapsto f_0$. From now on, we will also denote by α the cocycle a.

From the exact sequence $1 \to PGL_n \to \mathsf{A} \to \mu_2 \to 1$, we deduce

$$
1 \to {}_\alpha PGL_n \to {}_\alpha \mathsf{A} \to {}_\alpha \mu_2 \to 1,
$$

which induces

$$H^1(k, {}_\alpha PGL_n) \to H^1(k, {}_\alpha \mathsf{A}) \xrightarrow{p^*} H^1(k, {}_\alpha \mu_2).$$

Exactly as in the proof of Proposition 5, the quotient of $H^1(k, {}_\alpha PGL_n)$ by the action of μ_2 is in bijection with the kernel of p^*.

Since α is a cocycle with values in A, the cohomology sets $H^1(k, \mathsf{A})$ and $H^1(k, {}_\alpha \mathsf{A})$ are in one-to-one correspondence [23, I.5.3, Proposition 35 bis]. Hence $H^1(k, {}_\alpha \mathsf{A})$ classifies isomorphism classes of algebras with involution (A, σ) satisfying (H). For the same reason, and since μ_2 is abelian, translation by α is a bijection $H^1(k, \mu_2) \to H^1(k, {}_\alpha \mu_2)$. Hence the base point of $H^1(k, {}_\alpha \mu_2)$ is α, and the kernel of p^* is in one-to-one correspondence with isomorphism classes of algebras with involution (A, σ) such that $p^*([A, \sigma]) = \alpha$, i.e., such that $Z(A) = k(\sqrt{\alpha})$. This proves the result.

We then have the following:

Proposition 13. *There exists a morphism*

$$H^1(k, {}_\alpha PGL_n)/\mu_2 \to H^1\big(k, O^+\big(T^+_{\sigma_\alpha}\big)\big),$$

which maps the isomorphism class of (A, σ) *to the isometry class of* T^+_σ.

Proof. From diagram (i) of Proposition 7, we deduce

$$\begin{array}{ccccccccc} 1 & \longrightarrow & {}_\alpha PGL_n & \longrightarrow & {}_\alpha \mathsf{A} & \xrightarrow{p} & {}_\alpha \mu_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \rho & & \downarrow \tau & & \\ 1 & \longrightarrow & {}_{\rho^*(\alpha)}O^+(T^+) & \longrightarrow & {}_{\rho^*(\alpha)}O(T^+) & \longrightarrow & {}_{\rho^*(\alpha)}\mu_2 & \longrightarrow & 1. \end{array}$$

Moreover, since the cocycle α represents the isomorphism class of the algebra with involution $(M_n(K), \sigma_\alpha)$, $\rho^*(\alpha)$ represents the isometry class of the trace form $T^+_{\sigma_\alpha}$. Hence the twisted groups ${}_{\rho^*(\alpha)}O(T^+)$ and ${}_{\rho^*(\alpha)}O^+(T^+)$ are respectively isomorphic to the orthogonal group and the special orthogonal group of the form $T^+_{\sigma_\alpha}$.

Let a be a cocycle in $Z^1(k, {}_\alpha \mathsf{A})$. If $a \in \operatorname{Ker}(p^*)$, that is, $p^*(a) = \alpha$, then $\rho^*(a)$ will be a cocycle with values in $O^+(T^+_{\sigma_\alpha})$, representing in $H^1(k, O^+(T^+_{\sigma_\alpha}))$ the isometry class of T^+_σ. Hence the restriction of ρ to the kernel of p^* induces a morphism $H^1(k, {}_\alpha PGL_n)/\mu_2 \to H^1(k, O^+(T^+_{\sigma_\alpha}))$, and this proves Proposition 13.

3.5.2. *Definition of* $\mathsf{D}(A, \sigma)$

We can now give the definition of $\mathsf{D}(A, \sigma)$. From the exact sequence

$$1 \to {}_\alpha \mu_n \to {}_\alpha SL_n \to {}_\alpha PGL_n \to 1,$$

we deduce a connecting map $\Delta_2\colon H^1(k, {}_\alpha PGL_n) \to H^2(k, {}_\alpha \mu_n)$, which induces a map $D\colon H^1(k, {}_\alpha PGL_n)/\mu_2 \to H^2(k, {}_\alpha \mu_n)/\mu_2$, where the action of μ_2 on $H^2(k, {}_\alpha \mu_n)$ is induced by its action on ${}_\alpha \mu_n$.

DEFINITION 3. The determinant class of (A, σ) is the image under D of the isomorphism class of (A, σ).

3.5.3. *Proof of Theorem* 2

Let us assume now that n is even. Since we have a morphism ${}_\alpha PGL_n \to O^+(T^+_{\sigma_\alpha})$, we also have a morphism between the corresponding simply connected covers ${}_\alpha SL_n \to Spin(T^+_{\sigma_\alpha})$ (see again [4, Proposition 2.24(i), p. 262]). Hence we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & {}_\alpha \mu_n & \longrightarrow & {}_\alpha SL_n & \longrightarrow & {}_\alpha PGL_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin(T^+_{\sigma_\alpha}) & \longrightarrow & O^+(T^+_{\sigma_\alpha}) & \longrightarrow & 1. \end{array}$$

It induces

$$\begin{array}{ccc} H^1(k, {}_\alpha PGL_n) & \xrightarrow{\Delta_2} & H^2(k, {}_\alpha \mu_n) \\ \downarrow & & \downarrow{\scriptstyle \tau^*} \\ H^1(k, O^+(T^+_{\sigma_\alpha})) & \xrightarrow{\delta_2} & H^2(k, \mu_2). \end{array}$$

By Proposition 13, the morphism $H^1(k, {}_\alpha PGL_n) \to H^1(k, O^+(T^+_{\sigma_\alpha}))$ induces a morphism $H^1(k, {}_\alpha PGL_n)/\mu_2 \to H^1(k, O^+(T^+_{\sigma_\alpha}))$ that maps the isomorphism class of (A, σ) to the isometry class of T^+_σ. Moreover, since μ_2 acts trivially on μ_2, the morphism $H^2(k, {}_\alpha \mu_n) \to H^2(k, \mu_2)$ also induces a morphism $H^2(k, {}_\alpha \mu_n)/\mu_2 \to H^2(k, \mu_2)$. Hence we actually have the diagram

$$\begin{array}{ccc} H^1(k, {}_\alpha PGL_n)/\mu_2 & \xrightarrow{D} & H^2(k, {}_\alpha \mu_n)/\mu_2 \\ \downarrow & & \downarrow{\scriptstyle \tau^*} \\ H^1(k, O^+(T^+_{\sigma_\alpha})) & \xrightarrow{\delta_2} & H^2(k, \mu_2). \end{array}$$

The commutativity of this diagram now gives

$$w_2(T^+_\sigma) = w_2\big(T^+_{\sigma_\alpha}\big) + \tau^*(D(A, \sigma)).$$

The Hasse invariant of $T^+_{\sigma_\alpha}$ is easy to compute. Indeed, the σ_α-invariant elements of $M_n(K)$ are matrices of the type $M + \sqrt{\alpha}N$, where M is a symmetric matrix, and N a skew-symmetric one, both with coefficients in k, and one can check that the basis $(e_{ii})_{1 \le i \le n} \cup (e_{ij} + e_{ji})_{1 \le i < j \le n} \cup \sqrt{\alpha}(e_{ij} - e_{ji})_{1 \le i < j \le n}$ of $M_n(K)^+$ is orthogonal for the trace form. We get

$$T^+_{\sigma_\alpha} = n\langle 1 \rangle \oplus \frac{n(n-1)}{2}\langle 2, -2\alpha \rangle.$$

Hence $w_2(T^+_{\sigma_\alpha}) = (m(m-1)/2)(-1, -\alpha) + m(\alpha, 2)$, and we now have

$$D(A, \sigma) = \tau^*(\mathsf{D}(A, \sigma)).$$

To prove Theorem 2, we only have left to prove that the map τ is non-trivial. Let t_1 and t_2 be two indeterminates, and let $L = k(t_1, t_2)$. It is enough to prove that τ is non-trivial over the field L.

This can be done as follows. Let Q_0 be the quaternion algebra (t_1, t_2), defined over L, let σ_0 be its canonical involution, and let us consider the algebra with involution $(Q = Q_0 \otimes_L L(\sqrt{\alpha}), \sigma = \sigma_0 \otimes {}^-)$. By Proposition 11, its determinant class mod 2 is given by $D(Q, \sigma) = (-1, \alpha) + (t_1, t_2)$. Hence, over $L(\sqrt{\alpha}), D(Q, \sigma)_{L(\sqrt{\alpha})} = (t_1, t_2)_{L(\sqrt{\alpha})}$ is a division quaternion algebra. This proves that over L, $D(Q, \sigma) = \tau^*(\mathsf{D}(Q, \sigma))$ is not trivial, and Theorem 2 is now proved.

Remark. Let B be a central simple algebra of degree n over k, and let us consider the algebra $B \times B^0$ endowed with the involution σ defined by $\sigma(x, y) = (y, x)$. Its trace form T^+_σ is isometric to the form $\mathrm{Trd}_B(x^2)$.

In that case, $\alpha = 1$ and $H^2(k, {}_\alpha\mu_n)$ is the n-part of the Brauer group of k, $\mathrm{Br}_n(k)$. Moreover, we have $\mathsf{D}(A, \sigma) = [B] \in \mathrm{Br}_n(k)/\mu_2$. Hence the formula given in [23, annexe; 13, 25] for the Hasse invariant of the trace form $\mathrm{Trd}_B(x^2)$ is a particular case of Theorem 2.

4. DECOMPOSABILITY

In this section, we assume that A is a central simple algebra. We recall the following definition:

DEFINITION 4. The algebra with involution (A, σ) is said to be decomposable if σ stabilizes a non-trivial subalgebra A_1 of A, with $Z(A_1) = Z(A)$.

Indeed, if σ stabilizes A_1, it also stabilizes the centralizer A_2 of A in A_1. Let us denote by σ_i the restriction of σ to A_i for $i = 1, 2$. We then have $A = A_1 \otimes_K A_2$ and $\sigma = \sigma_1 \otimes \sigma_2$.

It is a classical problem to study the decomposability of an algebra with involution, and the invariants may be a useful tool in that context. In particular, for involutions of the first kind, Knus, Parimala, and Sridharan have proved the following:

PROPOSITION 14 [9]. *Let A be a central simple algebra of degree* 4, *and σ an involution of the first kind on A. Then (A, σ) is decomposable if and only if the determinant of σ is trivial.*

In the second kind case, we now have the following result:

PROPOSITION 15. *Let A be a central simple algebra of degree* 4, *and σ an involution of the second kind on A. If (A, σ) is decomposable, then its determinant class* mod 2, *$D(A, \sigma)$, is trivial.*

Proof. To prove this result, we use the following lemma:

LEMMA 14 (Albert [1] or [24]). *Let Q be a quaternion algebra over K, endowed with an involution of the second kind σ. Then there exists a quaternion algebra Q_0 defined over k such that $Q = Q_0 \otimes_k K$ and $\sigma = \tau \otimes {}^-$, where τ is the canonical involution on Q_0.*

To prove Proposition 15, let us assume now that the algebra with involution (A, σ) is decomposable. Since A is of degree 4, it actually decomposes as a tensor product of two quaternion algebras. Hence, because of Albert's lemma, there exist two quaternion algebras Q_1 and Q_2 over k such that $A = (Q_1 \otimes_k Q_2) \otimes_k K$, and $\sigma = (\tau_1 \otimes \tau_2) \otimes {}^-$, where τ_i is the canonical involution on Q_i. Since $\tau_1 \otimes \tau_2$ is of orthogonal type, by Proposition 11, we get that the determinant class mod 2 of (A, σ) is $D(A, \sigma) = (\alpha, d(\tau_1 \otimes \tau_2)) + 2[Q_1 \otimes Q_2]$. Moreover, the involution $\tau_1 \otimes \tau_2$ is decomposable, and hence has trivial determinant, which proves the result.

Another natural question when we study the decomposability of algebras with involution is the existence of indecomposable involutions. In the first kind case, Amitsur, Rowen, and Tignol have constructed an indecomposable involution on a biquaternion division algebra [3]. In the second kind case, examples of indecomposable involutions have been constructed on degree 4 algebras that are either split or similar to a quaternion algebra [16]. Here, using the existence of indecomposable involutions of the first kind, we construct an indecomposable involution of the second kind on a biquaternion division algebra. Namely, we have the following result:

PROPOSITION 16. *Let (A_0, σ_0) be a biquaternion division algebra, endowed with an indecomposable involution of the first kind and of orthogonal type. Then the algebra $A = (A_0 \otimes_k k(t)) \otimes_{k(t)} k(\sqrt{t})$ is a biquaternion divi-*

sion algebra, and the involution $\sigma = (\sigma_0 \otimes \mathrm{Id}) \otimes {}^-$ *is an indecomposable involution of the second kind on* A.

Proof. First of all, since $\sqrt{t}$ is an indeterminate, the morphism $k \to k(\sqrt{t})$ induces an injective morphism $\mathrm{Br}(k) \to \mathrm{Br}(k(\sqrt{t}))$ that preserves indices [15]. Hence $A = A_0 \otimes_k k(\sqrt{t})$ is a biquaternion division algebra.

Moreover, by Proposition 11, we can compute the determinant class mod 2 of (A, σ). We get the following: $D(A, \sigma) = (d, t)_{k(t)}$, where d is the determinant of the involution $\sigma_0 \otimes \mathrm{Id}$. But the determinant of σ_0 is non-trivial, since σ_0 is indecomposable. Hence d, which is the image of $d(\sigma_0)$ under the morphism $k^*/k^{*2} \to k(t)^*/k(t)^{*2}$ is also non-trivial. Now, one can check [15, pp. 383–385] that $(t, d)_{k((t))}$ is the twisted Laurent series algebra $k(\sqrt{d})((x, {}^-))$, where $x^2 = t$. Hence it is a division algebra, and a fortiori $(t, d)_{k(t)} \neq 0 \in \mathrm{Br}_2(k(t))$. Hence $D(A, \sigma)$ is non-trivial, and by Proposition 15, this proves the result.

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