# Determinantal Formulae for Matrices with Sparse Inverses, II: Asymmetric Zero Patterns

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ABSTRACT

In an earlier paper, formulae for det $A$ as a ratio of products of principal minors of $A$ were exhibited, for any given *symmetric* zero-pattern of $A^{-1}$. These formulae may be presented in terms of a spanning tree of the intersection graph of certain index sets associated with the zero pattern of $A^{-1}$. However, just as the determinant of a diagonal and of a triangular matrix are both the product of the diagonal entries, the symmetry of the zero pattern is not essential for these formulae. We describe here how analogous formulae for det $A$ may be obtained in the asymmetric-zero-pattern case by introducing a directed spanning tree. We also examine the converse question of determining all possible zero patterns of $A^{-1}$ which guarantee that a certain determinantal formula holds.

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## 1. INTRODUCTION

Let $A=(a_{ij})$ be an $n$-by-$n$ nonsingular matrix. It is well known that if $A^{-1}$ is diagonal, then

$$\det A=\prod_{i=1}^{n}a_{ii}. \tag{1.1}$$

If $A^{-1}$ happens to be tridiagonal, it is known [1] that

$$\det A=\frac{\prod_{i=1}^{n-1}\det\begin{bmatrix}a_{ii} & a_{i\,i+1}\\ a_{i+1\,i} & a_{i+1\,i+1}\end{bmatrix}}{\prod_{i=2}^{n-1}a_{ii}}. \tag{1.2}$$

However, if $A^{-1}$ is triangular (upper, lower, or essentially), it is a familiar fact that the formula for $\det A$ is the same as it is when $A^{-1}$ is diagonal, and it is known [1] that if $A^{-1}$ is (upper or lower) Hessenberg, then the formula for $\det A$ is the same as it is when $A^{-1}$ is tridiagonal. In [2] a broad class of determinantal formulae, generalizing the tridiagonal-inverse case, is demonstrated for various *symmetric* zero patterns in $A^{-1}$. As the triangular and Hessenberg cases indicate, symmetry of the inverse zero pattern is an unnecessary assumption for the same sort of determinantal formulae to hold. However, the determination of where nonzeros may asymmetrically occur for similar formulae to hold is not so simple as placing them all to one side of the diagonal. Generally, more complicated placements are possible, and only in certain circumstances is it possible to allow all entries on one side of the diagonal to be nonzero. We will describe the determinantal formulae in the spirit of [2] which are possible for various general asymmetric zero patterns in $A^{-1}$, and then make a detailed examination of the converse question of determining all possible zero patterns in $A^{-1}$ which guarantee that a given determinantal formula holds.

## 2. NOTATION, DEFINITIONS, AND PRIOR RESULTS

Throughout, we let $N=\{1,2,\ldots,n\}$, and let $A=(a_{ij})$ be an $n$-by-$n$ matrix. For nonempty index sets $\alpha,\beta\subseteq N$, we denote by $A(\alpha,\beta)$ that submatrix of $A$ lying in the rows indicated by $\alpha$ and the columns indicated by

$\beta$. When $\alpha$ and $\beta$ have the same number of elements we set $A_{\alpha,\beta} = \det A(\alpha,\beta)$ and $A_\alpha = A_{\alpha,\alpha}$.

We rely heavily upon some of the ideas of [2]; however, it is useful to restate the main result of [2] from a somewhat different point of view. Let $V_1,\dots,V_m$ be distinct subsets of $N$, and let $G_I$ be the intersection graph of $\{V_1,\dots,V_m\}$.

DEFINITION. A subgraph $G$ of $G_I$ is said to satisfy the *intersection property* if

$$\begin{array}{l} V_i \cap V_j \subseteq V_k \text{ whenever } V_k \text{ lies on a path from } V_i \text{ to } V_j \text{ in } G, \\ \quad\text{and} \\ V_i \cap V_j = \phi \text{ whenever } V_i \text{ and } V_j \text{ lie in distinct connected} \\ \quad\text{components of } G. \end{array} \tag{IP}$$

We also assume

$$\bigcup_{k=1}^{m} V_k = N \tag{2.1}$$

and

$$G_I \text{ is connected.} \tag{2.2}$$

THEOREM. *Let $A$ be an n-by-n matrix with inverse $B = (b_{ij})$. Let $V_1,\dots,V_m \subset N$ be index sets satisfying* (2.1) *and* (2.2), *and let $T$ be a spanning tree of $G_I$ satisfying* (IP). *If $b_{ij} = b_{ji} = 0$ whenever $\{i,j\}$ is contained in none of the index sets $V_1,\dots,V_m$, then*

$$\det A = \frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i,V_j\}\in\varepsilon(T)} A_{V_i\cap V_j}} \tag{2.3}$$

*($\varepsilon(T)$ denotes the edge set of $T$) provided the terms in the denominator are nonzero.*

REMARKS. It follows immediately that Equation (2.3) also holds if $T$ is replaced by a spanning forest $F$, because then $\det A = \prod_{T\subset F} A_T$, where the product is over all trees in $F$.

We also note that in equation (2.3), and throughout this paper, the entries of $A$ may be elements of any field.

## 3. EXAMPLES

We first consider two examples to illustrate certain ideas in our theorem concerning asymmetric zero patterns.

EXAMPLE 1. Let $A$ be a 6-by-6 matrix with tridiagonal inverse $B$. The formula for det $A$ mentioned in the introduction is

$$\det A = \frac{A_{\{1,2\}}A_{\{2,3\}}A_{\{3,4\}}A_{\{4,5\}}A_{\{5,6\}}}{a_{22}a_{33}a_{44}a_{55}}. \tag{3.1}$$

A proof [2] (which uses Jacobi's formula for minors of the inverse matrix $B$) is:

$$\begin{aligned}
&\frac{A_{\{1,2\}}A_{\{2,3\}}A_{\{3,4\}}A_{\{4,5\}}A_{\{5,6\}}}{a_{22}a_{33}a_{44}a_{55}}\\
&\quad= \frac{B_{\{3,4,5,6\}}B_{\{1,4,5,6\}}B_{\{1,2,5,6\}}B_{\{1,2,3,6\}}B_{\{1,2,3,4\}}}{(\det B)B_{\{1,3,4,5,6\}}B_{\{1,2,4,5,6\}}B_{\{1,2,3,5,6\}}B_{\{1,2,3,4,6\}}}\\
&\quad= \frac{B_{\{3,4,5,6\}}B_{\{1\}}B_{\{4,5,6\}}B_{\{1,2\}}B_{\{5,6\}}B_{\{1,2,3\}}B_{\{6\}}B_{\{1,2,3,4\}}}{B_{\{1\}}B_{\{3,4,5,6\}}B_{\{1,2\}}B_{\{4,5,6\}}B_{\{1,2,3\}}B_{\{5,6\}}B_{\{1,2,3,4\}}B_{\{6\}}}\det A\\
&\quad= \det A.
\end{aligned}$$

The second equality holds due to the tridiagonality of $B$. However, for this equality to hold, it suffices that one of each of the following pairs of matrices be the zero matrix:

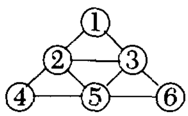
(i) $B(\{1\},\{3,4,5,6\})$ or $B(\{3,4,5,6\},\{1\})$,
(ii) $B(\{1,2\},\{4,5,6\})$ or $B(\{4,5,6\},\{1,2\})$,
(iii) $B(\{1,2,3\},\{5,6\})$ or $B(\{5,6\},\{1,2,3\})$,
(iv) $B(\{1,2,3,4\},\{6\})$ or $B(\{6\},\{1,2,3,4\})$.

In light of this, we note four of sixteen possible less sparse and nonsymmetric zero patterns for $B$ for which the second equality also holds:

$$
\text{Z1:}\ \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix},\quad
\text{Z2:}\ \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix},
$$

$$
\text{Z3:}\ \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{bmatrix},\quad
\text{Z4:}\ \begin{bmatrix} \times & \times & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & 0 \\ 0 & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \end{bmatrix}.
$$

EXAMPLE 2. Let $A$ be a 6-by-6 matrix whose inverse $B$ has the undirected graph




This $B$ has the symmetric zero pattern

$$
B = \begin{bmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & 0 & \times & \times \\ 0 & \times & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & \times & \times \\ 0 & 0 & \times & 0 & \times & \times \end{bmatrix}.
$$

(Diagonal entries are conventionally allowed to be nonzero [2]. Thus we draw no loops in the graph.) The following determinantal identity holds for $A$:

$$
\det A = \frac{A_{\{1,2,3\}} A_{\{2,3,5\}} A_{\{2,4,5\}} A_{\{3,5,6\}}}{A_{\{2,3\}} A_{\{2,5\}} A_{\{3,5\}}}. \tag{3.2}
$$

We proved this in [2] as follows:

$$\frac{A_{\{1,2,3\}}A_{\{2,3,5\}}A_{\{2,4,5\}}A_{\{3,5,6\}}}{A_{\{2,3\}}A_{\{2,5\}}A_{\{3,5\}}}$$
$$= \frac{B_{\{4,5,6\}}B_{\{1,4,6\}}B_{\{1,3,6\}}B_{\{1,2,4\}}}{(\det B)B_{\{1,4,5,6\}}B_{\{1,3,4,6\}}B_{\{1,2,4,6\}}}$$
$$= \frac{B_{\{4,5,6\}}B_{\{1\}}B_{\{4\}}B_{\{6\}}B_{\{1,3,6\}}B_{\{1,2,4\}}}{B_{\{1\}}B_{\{4,5,6\}}B_{\{4\}}B_{\{1,3,6\}}B_{\{1,2,4\}}B_{\{6\}}}\det A$$
$$= \det A.$$

The second equality is again due to the zero pattern of $B$. However, for the equality to hold it suffices that one of each of the following pairs of matrices be the zero matrix:

(i) $B(\{1\},\{4,5,6\})$ or $B(\{4,5,6\},\{1\})$,
(ii) $B(\{4\},\{1,3,6\})$ or $B(\{1,3,6\},\{4\})$,
(iii) $B(\{6\},\{1,2,4\})$ or $B(\{1,2,4\},\{6\})$.

We note three of eight possible less sparse and nonsymmetric zero patterns for $B$ for which the second equality also holds:

$$\text{Z1:}\quad \begin{bmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & 0 & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ 0 & 0 & \times & 0 & \times & \times \end{bmatrix},$$

$$\text{Z2:}\quad \begin{bmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ 0 & \times & 0 & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{bmatrix},$$

$$\text{Z3:}\quad \begin{bmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & 0 & \times & \times \\ \times & \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & \times & \times \end{bmatrix}.$$

This is an example, as mentioned in the introduction, for which it is not possible to allow all entries on one side of the diagonal to be nonzero. In fact, more than half (at least seven) of the original symmetrically placed twelve zeros must remain.

## 4. DIRECTED TREES

We now introduce certain notions which will enable us to graphically identify zero patterns of the inverse of a nonsingular matrix $A$ for which a determinantal formula of the type given by Equation (2.3) holds.

As in Section 2, let $V_1,\dots,V_m \subset N$ be index sets satisfying (2.1) and (2.2), and let $T$ be a spanning tree of $G_I$ satisfying the intersection property (IP).

Definition 4.1. Let $D$ be a directed tree on $\{V_1,\dots,V_m\}$ with a directed edge from $V_i$ to $V_j$ [denoted $(V_i,V_j)$] in $D$ only when $\{V_i,V_j\}\in \varepsilon(T)$. (Note that there are $2^{m-1}$ such $D$ for each $T$.) We call such a $D$ a directed tree related to $T$.

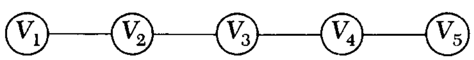
Definition 4.2. Let $B=(b_{ij})$ be an $n$-by-$n$ matrix. We say that $B$ has a nonzero pattern allowed by a related pair $(T,D)$ if whenever $b_{rs}\neq 0$, then either

(i) $\{r,s\}\subset V_k$ for some $k=1,\dots,m$, or

(ii) there is a (directed) path $(V_{i_1},V_{i_2},\dots,V_{i_l})$ in $D$ such that $r\in V_{i_1}$ and $s\in V_{i_l}$.

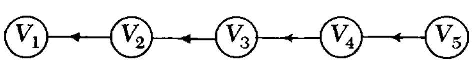
We now illustrate these definitions with the examples of Section 3.

Example 1. Taking $V_1=\{1,2\}$, $V_2=\{2,3\}$, $V_3=\{3,4\}$, $V_4=\{4,5\}$, $V_5=\{5,6\}$, the intersection graph of $V_1,\dots,V_5$ is already a spanning tree satisfying the intersection property (IP), namely

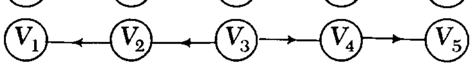



and Equation (2.3) becomes Equation (3.1) in this case. We list four of the sixteen possible directed trees $D$ related to $T$:

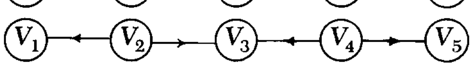
$D_1$:



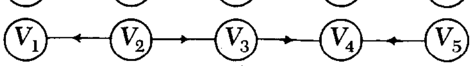

$D_2$:




$D_3$:

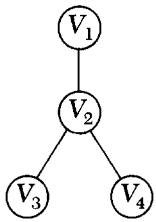



$D_4$:

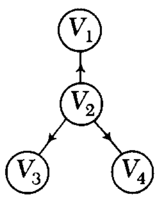


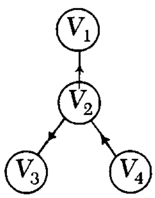
On applying Definition 4.2, as the reader may check, the nonzero patterns allowed by $(T, D_k)$, $k = 1,\dots,4$, are Z1, Z2, Z3, and Z4, respectively, of Example 1, Section 3.

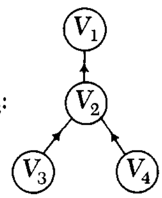
EXAMPLE 2. Taking $V_1 = \{1,2,3\}$, $V_2 = \{2,3,5\}$, $V_3 = \{2,4,5\}$, $V_4 = \{3,5,6\}$, the intersection graph of $V_1$, $V_2$, $V_3$, $V_4$ is the complete graph $K_4$. A spanning tree $T$ of $K_4$ satisfying the intersection property (IP) is




and for this tree, Equation (2.3) becomes Equation (3.2). Three of eight possible directed trees $D$ related to $T$ are

$D_1$: 


$D_2$: 


$D_3$: 


Applying definition 4.2 and comparing with example 2 of section 3, one sees that the nonzero patterns allowed by $(T, D_k)$, $k = 1,2,3$, are Z1, Z2, and Z3, respectively.

## 5. MAIN RESULTS

THEOREM 5.1. *Let $V_1,\dots,V_m \subset N$ be index sets satisfying* (2.1) *and* (2.2), *let $T$ be a spanning tree of the intersection graph of $V_1,\dots,V_m$ satisfying the intersection property (IP), and let $D$ be a directed tree related to $T$. If $A$ is a nonsingular n-by-n matrix whose inverse has a nonzero pattern* *allowed by* $(T, D)$, *then*

$$\det A = \frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i, V_j\} \in \varepsilon(T)} A_{V_i \cap V_j}}, \tag{5.1}$$

*provided the terms in the denominator are nonzero.*

*Proof.* By Jacobi's formula for minors of the inverse matrix,

$$\frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i, V_j\} \in \varepsilon(T)} A_{V_i \cap V_j}} = \frac{\prod_{k=1}^{m} B_{C_k}}{(\det B) \prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C_i \cup C_j}},$$

where $B = A^{-1}$ and $C_i = V_i^c$, $i = 1, \ldots, m$. We have used the fact that the number of edges in $\varepsilon(T)$ is $m - 1$ because $T$ is a tree. Since $\det A = (\det B)^{-1}$, we will be done if we show that

$$\frac{\prod_{k=1}^{m} B_{C_k}}{\prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C_i \cup C_j}} = 1. \tag{5.2}$$

For each node $V_i$ and any incident edge $\{V_i, V_j\} \in \varepsilon(T)$, let $T(i/j)$ be the component of the graph $T - \{V_i, V_j\}$ containing $V_i$ [note that $T(i/j)$ is a tree], let $V(i/j)$ be the node set of $T(i/j)$, and let

$$C(i/j) = \bigcap \{C_k : V_k \in V(i/j)\}. \tag{5.3}$$

We have simply split the tree $T$ into two trees by removing the edge connecting $V_i$ to $V_j$. $C(i/j)$ is the complement of the union of all node sets in that tree containing $V_i$. The idea of the proof is to decompose the minors in the denominator and numerator of (5.2) into smaller minors indexed by sets $C(i/j)$ to produce cancellation.

We first show that for all $\{V_i, V_j\} \in \varepsilon(T)$, $C_i \cup C_j$ can be written

$$C_i \cup C_j = C(i/j) \cup C(j/i). \tag{5.4}$$

Since $C_i \supset C(i/j)$ by definition, it is clear that $C_i \cup C_j \supset C(i/j) \cup C(j/i)$. Suppose that $r \in C_i \cup C_j$ and that $r \notin C(i/j) \cup C(j/i)$. Then $r \notin C_{k_1}$ for some $V_{k_1} \in V(i/j)$ and $r \notin C_{k_2}$ for some $V_{k_2} \in V(j/i)$. Then $r \in V_{k_1} \cap V_{k_2}$. Let $[V_{k_1}, V_i]$ be the path connecting $V_{k_1}$ and $V_i$ in $T(i/j)$, and let $[V_j, V_{k_2}]$ be the path connecting $V_j$ and $V_{k_2}$ in $T(j/i)$. Then $[V_{k_1}, V_i] \cup \{V_i, V_j\} \cup [V_j, V_{k_2}]$ is a path in $T$ connecting $V_{k_1}$ and $V_{k_2}$. Since $V_i$ and $V_j$ lie on this path, by the intersection property $V_{k_1} \cap V_{k_2} \subset V_i$ and $V_{k_1} \cap V_{k_2} \subset V_j$. This implies that $r \in V_i \cap V_j$ contradicting $r \in C_i \cup C_j = (V_i \cap V_j)^c$. This establishes the equality of the sets in (5.4).

The sets $C(i/j)$ and $C(j/i)$ are disjoint because

$$C(i/j) \cap C(j/i) = \bigcap \{C_k : V_k \in V(i/j) \cup V(j/i)\}$$

$$= \bigcap_{k=1}^{m} C_k = \left( \bigcup_{k=1}^{m} V_k \right)^c = N^c = \phi.$$

Corresponding to any edge $\{V_i, V_j\}$ in $T$, there is a directed edge, either $(V_i, V_j)$ or $(V_j, V_i)$, in $D$. *Suppose the directed edge is* $(V_i, V_j)$. Let $r \in C(i/j)$ and $s \in C(j/i)$. We claim that $b_{rs} = 0$. To see why, observe:

(1) $r \notin V_k$ for any $V_k \in V(i/j)$ and $s \notin V_k$ for any $V_k \in V(j/i)$, which implies that $\{r, s\} \not\subset V_k$ for $k = 1, \dots, m$.

(2) if there were a directed path $(V_{i_1}, \dots, V_{i_l})$ in $D$ with $r \in V_{i_1}$ and $s \in V_{i_l}$, then $V_{i_1} \in V(j/i)$ and $V_{i_l} \in V(i/j)$. But the path from $V_{i_1}$ to $V_{i_l}$ in $T$ must contain the edge $\{V_i, V_j\}$, so the directed path from $V_{i_1}$ to $V_{i_l}$ must contain the directed edge $(V_j, V_i)$. This is impossible by supposition.

Since $B = A^{-1}$ must have a nonzero pattern allowed by $(T, D)$, $b_{rs} = 0$ for $r \in C(i/j)$ and $s \in C(j/i)$. Interchanging the roles of $i$ and $j$, if the directed edge was $(V_j, V_i)$, gives $b_{rs} = 0$ for $r \in C(j/i)$ and $s \in C(i/j)$. In either case, it follows by Laplace's expansion that

$$B_{C_i \cup C_j} = B_{C(i/j) \cup C(j/i)} = B_{C(i/j)} B_{C(j/i)}$$

for all $\{V_i, V_j\} \in \varepsilon(T)$. Hence the denominator in Equation (5.2) may be written

$$\prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C_i \cup C_j} = \prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C(i/j)} B_{C(j/i)}. \tag{5.5}$$

We next show that for each node set $V_i$, $C_i$ can be written as the disjoint union

$$C_i = \bigcup \{ C(i/k) : \{ V_i, V_k \} \in \varepsilon(T) \}. \tag{5.6}$$

The containment $C_i \supset \bigcup\{ C(i/k) : \{ V_i, V_k \} \in \varepsilon(T)\}$ is clear from the definition of $C(i/k)$. Suppose that $r \in C_i$. Then for some $j \neq i$, $r \in V_j$. Let $[V_j, V_i]$ be the (unique) path connecting $V_j$ and $V_i$ in $T$, and let $V_p$ be the vertex in $[V_j, V_i]$ adjacent to $V_i$. We claim that $r \in C(i/p)$. For if not, then $r \notin C_l$ for some $V_l \in V(i/p)$. Thus $r \in V_j \cap V_l$, and since $V_i$ is on the path $[V_j, V_p] \cup \{ V_p, V_i \} \cup [V_i, V_l]$, by (IP) $r \in V_i$, contradicting $r \in C_i$. This establishes Equation (5.6).

To show that the union is disjoint, assume that $\{ V_i, V_{k_1} \}$ and $\{ V_i, V_{k_2} \}$ are in $\varepsilon(T)$ with $k_1 \neq k_2$ [if $\{ V_i, V_k \} \in \varepsilon(T)$ for only one $k$, there is nothing to show] and that $r \in C(i/k_1) \cap C(i/k_2)$. For some $j \neq i$, $r \in V_j$. Let $[V_j, V_i]$ be the path connecting $V_j$ and $V_i$ in $T$. Then either $V_{k_1} \notin [V_j, V_i]$ or $V_{k_2} \notin [V_j, V_i]$. Say $V_{k_1} \notin [V_j, V_i]$. Then $[V_j, V_i]$ is a path in $T(i/k_1)$, so $V_j \in V(i/k_1)$. But then, since $r \in C(i/k_1)$, we have $r \in C_j$, contradicting $r \in V_j$. Thus $C(i/k_1) \cap C(i/k_2) = \phi$. Hence the union in Equation (5.6) is disjoint.

We will now show that

$$B_{C_i} = \prod_{\{ k : \{ V_i, V_k \} \in \varepsilon(T) \}} B_{C(i/k)}. \tag{5.7}$$

Let $V_{k_1}, \ldots, V_{k_l}$ be the nodes adjacent to $V_i$ in $T$. We prove Equation (5.7) by showing that for $t = 2, \ldots, l$,

$$B_{\bigcup_{j=1}^{t} C(i/k_j)} = B_{C(i/k_t)} B_{\bigcup_{j=1}^{t-1} C(i/k_j)}. \tag{5.8}$$

Then, by induction, Equation (5.7) follows from Equations (5.6) and (5.8).

Corresponding to the edge $\{ V_i, V_{k_t} \}$ in $T$, there is a directed edge, either $(V_i, V_{k_t})$ or $(V_{k_t}, V_i)$, in $D$. *Suppose the directed edge is* $(V_i, V_{k_t})$. Let $r \in C(i/k_t)$ and $s \in \bigcup_{j=1}^{t-1} C(i/k_j)$. Then:

(1) $r \notin V_{j_1}$ for $V_{j_1} \in V(i/k_t)$ and $s \notin V_{j_2}$ for $V_{j_2} \in V(i/k_1)$. Thus $\{ r, s \} \not\subset V_j$ for $V_j \in V(i/k_t) \cup V(i/k_1)$. But $V(i/k_t) \cup V(i/k_1) = \{ V_1, \ldots, V_m \}$, so $\{ r, s \} \not\subset V_j$ for $j = 1, \ldots, m$.

(2) Since $r \notin V_j$ for any $V_j \in V(i/k_t)$, $r \in V_p$ for some $V_p \in V(k_t/i)$. Since $s \in C(i/k_u)$ for some $u = 1, \ldots, t-1$, $s \notin V_j$ for any $V_j \in V(i/k_u)$. Therefore $s \in V_q$ for some $V_q \in V(k_u/i)$. Let $[V_p, V_{k_t}]$ be the path from $V_p$ to $V_{k_t}$ in $T(k_t/i)$, and $[V_{k_u}, V_q]$ be the path from $V_{k_u}$ to $V_q$ in $T(k_u/i)$. Then $[V_p, V_{k_t}] \cup \{ V_{k_t}, V_i \} \cup \{ V_i, V_{k_u} \} \cup [V_{k_u}, V_q]$ is the unique path from $V_p$ to $V_q$ in $T$. Thus if there were a directed path from $V_p$ to $V_q$ in $T$, it would have to contain the directed edge $(V_{k_t}, V_i)$, which is impossible by supposition.

Since $B$ must have a nonzero pattern allowed by $(T, D)$, we have $b_{rs} = 0$ if $r \in C(i/k_t)$ and $s \in \bigcup_{j=1}^{t-1} C(i/k_j)$. Thus, an application of Laplace's expansion yields Equation (5.8). Similarly, if the directed edge is $(V_{k_t}, V_i)$, one can show that $b_{rs} = 0$ for $r \in \bigcup_{j=1}^{t-1} C(i/k_j)$ and $s \in C(i/k_t)$, and likewise arrive at Equation (5.8). This completes the verification of Equation (5.7).

Hence the numerator in Equation (5.2) may be written

$$\prod_{i=1}^{m} B_{C_i} = \prod_{i=1}^{m} \prod_{\{k : \{V_i, V_k\} \in \varepsilon(T)\}} B_{C(i/k)}. \tag{5.9}$$

The right-hand side of Equation (5.9) equals the right-hand side of Equation (5.5). Hence the left-hand sides of these equations are equal, which yields Equation (5.2), completing the proof of the theorem. ■

If any of the terms in the denominator of Equation (5.1) are zero, a trivial modification of the beginning of the proof yields the more general formula

$$\prod_{k=1}^{m} A_{V_k} = (\det A) \prod_{\{V_i, V_j\} \in \varepsilon(T)} A_{V_i \cap V_j}. \tag{5.10}$$

When $A^{-1}$ is (upper or lower) Hessenberg, Equation (1.2) of the introduction follows from Theorem 5.1 as a very special case. The examples in Section 3 are a small sample of the great variety of zero patterns $A^{-1}$ can have in the setting of Theorem 5.1. One can mechanically generate further examples by choosing appropriate index sets $V_1, \dots, V_m$ and tree pair $(T, D)$ and seeing what formulae for $\det A$ and zero patterns for $A^{-1}$ result.

We conclude this section by briefly noting that for any of the formulae (5.1) there are many corresponding formulae involving nonprincipal minors obtained simply by permuting rows and columns of the matrix $A$. For example, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A^{-1} = B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix},$$

and assume that $b_{22} = 0$. Since $b_{22}$ is a diagonal element of $A^{-1}$, Theorem 5.1 implies directly no formula for $\det A$. However, let

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

so that

$$(A')^{-1} = B' = \begin{bmatrix} b_{11} & b_{13} & b_{12} \\ b_{21} & b_{23} & b_{22} \\ b_{31} & b_{33} & b_{32} \end{bmatrix}.$$

If $a_{11} \neq 0$, let $V_1 = \{1,2\}$, $V_2 = \{1,3\}$, and let $D$ be the directed tree $(V_1) \longleftarrow (V_2)$. Since $b_{22} = 0$, $B'$ has a nonzero pattern allowed by $D$. Thus by Theorem 5.1,

$$\det A = -\det A' = -\frac{A'_{\{1,2\}} A'_{\{1,3\}}}{a'_{11}}$$

$$= -\frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}{a_{11}}.$$

Of course, any permutation which moves $b_{22}$ off the diagonal results in a similar formula.

## 6. THE CONVERSE QUESTION

We wish to address the question of a converse to Theorem 5.1. Specifically, does Theorem 5.1 essentially give all the zero patterns of $A^{-1}$ for which the determinantal formula (5.1) holds?

Before pursuing this question it is helpful to consider the case of index sets $V_1, V_2, \dots, V_m \subset N$ for which

$$\bigcup_{i=1}^{m} V_i = N \tag{6.1}$$

and

$$V_i \cap V_j = \phi \qquad \text{for} \quad i \neq j. \tag{6.2}$$

Since our preliminary observations are well understood, the proofs will be brief.

Definition 6.1. *Let* $V_1, \dots, V_m \subset N$ *be index sets satisfying* (6.1) *and* (6.2), *and let* $\sigma$ *be a permutation of* $\{1, \dots, m\}$. *If* $A$ *is an n-by-n matrix, we* *say that A has a nonzero pattern allowed by* $\sigma$ *if whenever* $a_{rs} \neq 0$, *then either*

(i) $\{r, s\} \subset V_k$ for some $k = 1, \dots, m$, or
(ii) for some $i$ and $j$ with $\sigma(i) > \sigma(j)$, we have $r \in V_{\sigma(i)}$ and $s \in V_{\sigma(j)}$.

For example, if $\sigma(i) = i$, $i = 1, \dots, m$, $A$ is a block lower triangular matrix, and if, in addition, each $V_i$ has one element, then $A$ is lower triangular.

THEOREM 6.2. *If A has a nonzero pattern allowed by* $\sigma$, *then*

$$\det A = \prod_{k=1}^{m} A_{V_k}. \tag{6.3}$$

*Proof.* Laplace's expansion. ■

LEMMA 6.3. *If A is nonsingular, then A has a nonzero pattern allowed by* $\sigma$ *if and only if* $A^{-1}$ *has a nonzero pattern allowed by* $\sigma$.

*Proof.* Permute to block triangular form. The result follows from the fact that the inverse of a block triangular matrix with respect to a given partition is block triangular with respect to the same partition. ■

THEOREM 6.4. *If* $A^{-1}$ *has a nonzero pattern allowed by* $\sigma$, *then*

$$\det A = \prod_{k=1}^{m} A_{V_k}.$$

*Proof.* This is a corollary of Lemma 6.3 and Theorem 6.2. ■

We now establish a converse to Theorem 6.2 (and hence 6.4 also). Of course, the identity (6.3) may hold for a particular matrix $A$ but have no implication for the zero pattern of $A$. For if $a_{ij}$ is any element outside the principal submatrices $A(V_1, V_1), \dots, A(V_k, V_k)$ whose cofactor is nonzero, one can vary the value of $a_{ij}$ to make Equation (6.3) hold. Thus, a natural idea for a converse to Theorem 6.2 is to hypothesize that the identity (6.3) holds for *all* matrices whose inverse has some fixed zero pattern. To examine this idea, it is helpful to consider the case $m = 2$. Then Theorem 6.2 reduces to the statement:

If $A(V_1, V_2)$ or $A(V_2, V_1)$ is the zero matrix, then $\det A = A_{V_1} A_{V_2}$.

However, a little reflection shows that there are other zero patterns for which the conclusion follows.

EXAMPLE 1. Let

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix}, \qquad V_1 = \{1,2\} \text{ and } V_2 = \{3\}.$$

Then

$$\det A = (a_{11}a_{22})a_{33} = A_{V_1}A_{V_2}.$$

In fact, the identity (6.3) will hold whenever $\det A$ is the product of the diagonal elements of $A$. To avoid trivial cases such as these, we make the following definitions.

DEFINITION 6.5. If $V_1,\dots,V_m \subset N$ are any index sets satisfying (6.1), we define the *profile* $P(V_1,\dots,V_m)$ of $V_1,\dots,V_m$ to be the set $\bigcup_{k=1}^{m}(V_k \times V_k)$. If $Z \subset N \times N$, we say that $Z$ lies outside the profile of $V_1,\dots,V_m$ if $Z \cap P(V_1,\dots,V_m) = \phi$.

DEFINITION 6.6. If $Z \subset N \times N$, we say that the $n$-by-$n$ matrix $A$ has a nonzero pattern allowed by $Z$ if $a_{rs} = 0$ for all $(r,s)$ in $Z$. We let $\mathscr{A}_Z$ be the set of all $n$-by-$n$ matrices $A$ with nonzero pattern allowed by $Z$.

The following theorem shows that if the principal submatrices $A(V_k, V_k)$, $k = 1,\dots,m$, are allowed to be arbitrary, then the identity (6.3) can only hold generically provided that there is a permutation $\sigma$ such that $A$ has a nonzero pattern allowed by $\sigma$.

THEOREM 6.7. Let $V_1,\dots,V_m \subset N$ be index sets satisfying (6.1) and (6.2), let $Z \subset N \times N$ lie outside the profile of $V_1,\dots,V_m$, and assume that $\det A = \prod_{k=1}^{m} A_{V_k}$ for all $A \in \mathscr{A}_Z$. Then there is a permutation $\sigma$ of $\{1,\dots,m\}$ such that $V_{\sigma(i)} \times V_{\sigma(j)} \subset Z$ whenever $\sigma(i) < \sigma(j)$.

*Proof.* The key step is to show that the hypotheses imply that for some $i \in \{1,\dots,m\}$, $V_i \times V_i^c \subset Z$. Suppose not. Pick $i_1 \in \{1,\dots,m\}$. There is an $i_2 \neq i_1$ such that $V_{i_1} \times V_{i_2} \not\subset Z$. Next, there is an $i_3 \neq i_2$ such that $V_{i_2} \times V_{i_3} \not\subset Z$. Continue this process until some $i_j$ occurs twice. Without loss of generality we may take this to be $i_1$. (Otherwise just delete $i_1, i_2, \ldots$ until the first and last $i_j$ are equal.) Thus we have a chain of sets $V_{i_1} \times V_{i_2}, V_{i_2} \times V_{i_3}, \ldots, V_{i_k} \times V_{i_1}$, with $i_1, i_2, \ldots, i_k$ distinct, none of which is contained in $Z$. Pick $(r_1, c_2) \in V_{i_1} \times V_{i_2}/Z$, $(r_2, c_3) \in V_{i_2} \times V_{i_3}/Z, \ldots, (r_k, c_1) \in V_{i_k} \times V_{i_1}/Z$. Since $r_1, \ldots, r_k$ are distinct, as are $c_1, \ldots, c_k$, and since $(r_s, c_s) \in V_{i_s} \times V_{i_s}$, $s = 1, \ldots, k$, we may select a permutation matrix $P$ such that $p_{r_1 c_2} = p_{r_2 c_3} = \cdots = p_{r_k c_1} = 1$ and all remaining ones in $P$ occur in the submatrices $P(V_i, V_i)$, $i = 1, \ldots, m$. Since the $P(V_i, V_i)$ are generally block matrices, it may be necessary to add ones to each. Then $P \in \mathscr{A}_Z$ and $\det P = \pm 1$. But $P_{V_{i_1}} = \cdots = P_{V_{i_k}} = 0$, so $\prod_{k=1}^{m} P_{V_k} = 0$, contradicting the hypothesis. Thus $Z \supset V_i \times V_i^c$ for some $i \in \{1, \ldots, m\}$.

We now complete the proof of Theorem 6.7. If $m = 2$, either $V_1 \times V_2 = V_1 \times V_1^c \subset Z$ or $V_2 \times V_1 = V_2 \times V_2^c \subset Z$, which is the assertion.

Proceeding by induction, assume the theorem is true for $m - 1$ and that $\det A = \prod_{k=1}^{m} A_{V_k}$ for all $A \in \mathscr{A}_Z$. For convenience take $V_1$ to be the set for which $V_1 \times V_1^c \subset Z$. Let $V = \bigcup_{k=2}^{m} V_k$. For any $A \in \mathscr{A}_Z$, let $\tilde{A}$ be the matrix defined by

$$\tilde{A}(V_1, V_1) = I_{|V_1|},$$

$$\tilde{A}(V_1, V) = 0,$$

$$\tilde{A}(V, V_1) = 0,$$

$$\tilde{A}(V, V) = A(V, V).$$

Then $\tilde{A} \in \mathscr{A}_Z$ also. Hence, by hypothesis, $\det \tilde{A} = \prod_{k=1}^{m} \tilde{A}_{V_k}$. But $\det \tilde{A} = \tilde{A}_{V_1} \tilde{A}_V = A_V$ by Laplace's expansion, and $\prod_{k=1}^{m} \tilde{A}_{V_k} = \prod_{k=2}^{m} A_{V_k}$. Therefore $A_V = \prod_{k=2}^{m} A_{V_k}$. Letting $\tilde{Z} = Z \cap (V \times V)$, and noting that $\mathscr{A}_{\tilde{Z}} = \{A(V, V) : A \in \mathscr{A}_Z\}$, we may apply the induction hypothesis to assert that there is a permutation $\sigma'$ of $\{2, \ldots, m\}$ such that $V_{\sigma'(i)} \times V_{\sigma'(j)} \subset \tilde{Z}$ whenever $\sigma'(i) < \sigma'(j)$. Define $\sigma$ by

$$\sigma(i) = \begin{cases} 1, & i = 1, \\ \sigma'(i), & 2 \leqslant i \leqslant m. \end{cases}$$

Then $V_{\sigma(i)} \times V_{\sigma(j)} \subset \tilde{Z} \subset Z$ whenever $\sigma(i) < \sigma(j)$, $2 \leqslant i,\ j \leqslant m$, and $V_{\sigma(1)} \times V_{\sigma(i)} = V_1 \times V_{\sigma(i)} \subset V_1 \times V_1^c \subset Z$ for $i = 2, \ldots, m$. Combining these, we see that $V_{\sigma(i)} \times V_{\sigma(j)} \subset Z$ whenever $\sigma(i) < \sigma(j)$, which completes the proof of the theorem. ∎

The assertion of Theorem 6.7 is, of course, equivalent to the statement that there is a permutation $\sigma$ of $\{1, \ldots, m\}$ such that $A$ has a nonzero pattern allowed by $\sigma$ (and in the nonsingular case, such that $A^{-1}$ has a nonzero pattern allowed by $\sigma$).

We now return to the question of a converse to Theorem 5.1. In this case we have a conjecture which we have not been able to prove. But we wish to describe some partial results. Several of the following ideas resemble those found in [3], where we established converses to some inequalities for positive definite matrices. We first establish the preliminary result:

Observation (Cancellation of indices). *Let $V_1,\dots,V_m \subset N$ be index sets satisfying* (6.1), *and let $G$ be a spanning subgraph of the intersection graph of $V_1,\dots,V_m$. For each $p \in N$, let $\alpha_p$ be the number of node sets $V_k$ containing $p$, and let $\beta_p$ be the number of edges $\{V_i, V_j\} \in \mathscr{E}(G)$ for which $V_i \cap V_j$ contains $p$. Let $Z \subset N \times N$ lie outside the profile of $V_1,\dots,V_m$, and assume that*

$$\prod_{k=1}^{m} A_{V_k} = (\det A) \prod_{\{V_i, V_j\} \in \mathscr{E}(G)} A_{V_i \cap V_j} \tag{6.4}$$

*for all nonsingular matrices $A$ for which $A^{-1} \in \mathscr{A}_Z$ (compare with* (5.10)). *Then $\alpha_p - \beta_p = 1$, $p = 1,\dots,n$.*

*Proof.* Let $\Lambda_p$ be the diagonal matrix with $\lambda \neq 0$ in the $p$th position and ones in the remaining diagonal positions. Then $\Lambda_p^{-1} \in \mathscr{A}_Z$, so by (6.4)

$$\lambda^{\alpha_p} = \lambda \lambda^{\beta_p},$$

from which the conclusion follows. ∎

Note from the proof that it is only necessary to assume that Equation (6.4) holds for all diagonal matrices. The following definition will make it convenient to state our conjecture.

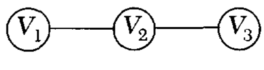
Definition 6.8. Given a related pair $T$, $D$, as in Section 4, let $Z(T, D)$ be the set of all $(r, s) \in N \times N$ satisfying neither (i) nor (ii) of definition 4.2; i.e., $Z(T, D)$ is the set of mandatory zeros if the matrix $A$ has a nonzero pattern allowed by $(T, D)$.

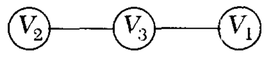
We must also deal with the possibility of identical determinantal formulae for distinct spanning trees associated with the node sets $V_1,\dots,V_m$, and satisfying the intersection property (IP), before stating our conjecture. As an example, let $A$ be a 4-by-4 matrix and let $V_1 = \{1,2\}$, $V_2 = \{1,3\}$, and

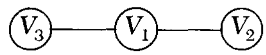
$V_3 = \{1,4\}$. Then the formula (2.3) becomes

$$\det A = \frac{A_{\{1,2\}}A_{\{1,3\}}A_{\{1,4\}}}{a_{11}^2} \tag{6.4$'$}$$

for each of the three trees

$T_1$:


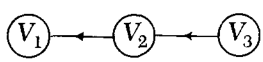

$T_2$:



$T_3$:



where there are four directed trees and accompanying matrices for each $T_i$. If $A^{-1}$ has the zero pattern

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \end{bmatrix},$$

and $a_{11} \neq 0$, then equation (6.4′) holds because this zero pattern corresponds to $Z(T_1, D_1)$, where $D_1$ is the directed tree




However, this zero pattern does not correspond to any of the $Z(T, D)$ when $T$ is $T_2$ or $T_3$. This motivates:

DEFINITION 6.8′. Let $V_1, \dots, V_m \subset N$ be index sets satisfying (6.1), and let $T_1$ and $T_2$ be distinct spanning trees of the intersection graph of $V_1, \dots, V_m$. We say $T_1$ and $T_2$ are equivalent if the two collections $\{V_i \cap V_j : \{V_i, V_j\} \in \varepsilon(T_1)\}$ and $\{V_i \cap V_j : \{V_i, V_j\} \in \varepsilon(T_2)\}$ are identical.

We conjecture the following converse to Theorem 5.1.

CONJECTURE 6.9. Let $V_1, \dots, V_m \subset N$ be index sets satisfying (6.1), and let $T$ be a spanning tree of the intersection graph of $V_1, \dots, V_m$. Let $Z \subset N \times N$ lie outside the profile of $V_1, \dots, V_m$, and assume that

$$\prod_{k=1}^{m} A_{V_k} = (\det A) \prod_{\{V_i, V_j\} \in \varepsilon(T)} A_{V_i \cap V_j} \tag{6.5}$$

for all nonsingular matrices $A$ for which $A^{-1} \in \mathscr{A}_Z$. Then $T$ satisfies the intersection property (IP). Furthermore there is a spanning tree $T'$ equivalent to $T$ and a directed tree $D$ related to $T'$ such that $Z \supset Z(T', D)$.

There is often no other spanning tree $T'$ of the intersection graph $G_I$ which is equivalent to $T$. This is the case in our subsequent examples, and hence we make no further mention of this issue.

Conjecture 6.9 says that the identity (6.5) can hold generically only under the hypotheses of Theorem 5.1.

*Partial proof.*

(i) We first prove that $T$ satisfies the intersection property (IP) by showing that if (IP) fails, then $\alpha_p - \beta_p \geqslant 2$ for some $p \in N$, violating the cancellation-of-indices observation made above. If (IP) fails, there are node sets $V_i$, $V_j$, and $V_l$, with $V_l$ between $V_i$ and $V_j$ and with $V_i \cap V_j \not\subseteq V_l$. Pick $p \in N$ such that $p \in V_i \cap V_j$ but $p \notin V_l$.

Remove from $T$ the node set $V_l$ and all edges incident with the node $V_l$ (i.e., of the form $\{V_l, V_k\}$). Thus, $p \notin V_l \cap V_k$ for any edge $\{V_l, V_k\}$ which is removed. This leaves two or more subtrees. Let $T_1$ be the subtree containing $V_i$, $T_2$ the one containing $V_j$, and $T_3, \ldots, T_q$ any remaining subtrees. For each of these trees $T_k$, let $F_k$ be the forest whose vertex set is the collection of all node sets $V_r$ in $T_k$ which contain $p$ and whose edge set is the collection of all edges $\{V_r, V_s\}$ for which $p \in V_r \cap V_s$. For each nonempty forest $F_k$,

$$|V(F_k)| \geqslant |\varepsilon(F_k)| + 1,$$

in which $V(F_k)$ is the vertex set and $\varepsilon(F_k)$ is the edge set of $F_k$. In particular, $F_1$ contains $V_i$ and $F_2$ contains $V_j$, so that this inequality holds at least for $k = 1, 2$. Thus, we have

$$\begin{aligned}\alpha_p - \beta_p &= \sum_{k=1}^{q} \left(|V(F_k)| - |\varepsilon(F_k)|\right) \\ &\geqslant \sum_{k=1}^{2} \left(|V(F_k)| - |\varepsilon(F_k)|\right) \geqslant 2,\end{aligned}$$

which contradicts the cancellation-of-indices observation. Thus $T$ satisfies (IP).

(ii) Secondly, we show the conjecture is true for $m = 2$. In this case, Equation (6.5) becomes

$$A_{V_1} A_{V_2} = (\det A) A_{V_1 \cap V_2}. \tag{6.6}$$

Letting $B = A^{-1}$ and $C_i = V_i^c$, $i = 1,2$, on application of Jacobi's formula for minors of the inverse matrix, we have

$$\frac{B_{C_1}}{\det B} \cdot \frac{B_{C_2}}{\det B} = \frac{1}{\det B} \frac{B_{C_1 \cup C_2}}{\det B},$$

or

$$B_{C_1 \cup C_2} = B_{C_1} B_{C_2}. \tag{6.7}$$

We now wish to use Equation (6.7) in the context of Theorem 6.7. Let $\tilde{Z} = Z \cap [(C_1 \cup C_2) \times (C_1 \cup C_2)]$, and let $\mathscr{A}_{\tilde{Z}}$ be the set of all matrices of order $|C_1 \cup C_2|$ with nonzero pattern allowed by $\tilde{Z}$. For each $B \in \mathscr{A}_Z$, let $\tilde{B} = B(C_1 \cup C_2, C_1 \cup C_2)$. Then $\{\tilde{B}: B \in \mathscr{A}_Z\} = \mathscr{A}_{\tilde{Z}}$. Since Equation (6.6) holds for all $A$ with $B = A^{-1} \in \mathscr{A}_Z$, Equation (6.7) holds for all $B \in \mathscr{A}_Z$. Therefore

$$\det \tilde{B} = B_{C_1 \cup C_2} = B_{C_1} B_{C_2}$$

for all $\tilde{B} \in \mathscr{A}_{\tilde{Z}}$.

Now, suppose $(r, s) \in \tilde{Z}$. Since $Z \cap P(V_1, V_2) = \phi$, $\tilde{Z} \cap P(V_1, V_2) = \phi$. If $(r, s) \in C_1 \times C_1$, then $r, s \notin V_1$, implying $r, s \in V_2$ or $(r, s) \in V_2 \times V_2 \subset P(V_1, V_2)$, which is contradictory. Likewise $(r, s) \in C_2 \times C_2$ is impossible. Thus if $(r, s) \in \tilde{Z}$ then $(r, s) \notin P(C_1, C_2)$. Take $N = C_1 \cup C_2$ in Theorem 6.7. Since $C_1 \cap C_2 = (V_1 \cup V_2)^c = N^c = \phi$, and $\tilde{Z} \subset N \times N$ lies outside the profile of $C_1, C_2$, all the hypotheses of Theorem 6.7 are satisfied for the case $m = 2$. We may therefore conclude that either $C_1 \times C_2 \subset \tilde{Z}$ or $C_2 \times C_1 \subset \tilde{Z}$, and hence that either $C_1 \times C_2 \subset Z$ or $C_2 \times C_1 \subset Z$.

In the first case, let $D$ be the directed tree on $\{V_1, V_2\}$ with directed edge $(V_1, V_2)$. Then $(r, s) \in Z(T, D)$ if and only if $r \notin V_1$ and $s \notin V_2$, i.e. $(r, s) \in C_1 \times C_2$. Thus $Z(T, D) = C_1 \times C_2 \subset Z$ as claimed. In the case $C_2 \times C_1 \subset Z$ choose the directed edge to be $(V_2, V_1)$. ∎

Although we have been unable to find a general proof for the conjecture, we wish to consider two elementary examples for $m = 3$. The first is a case where the proof can be successfully carried out. The second illustrates how a slight change can radically alter the complexity.

EXAMPLE 2. Let $N = \{1,2,3,4\}$, $V_1 = \{1,2\}$, $V_2 = \{2,3\}$, and $V_3 = \{3,4\}$. Let $Z$ lie outside the profile of $V_1, V_2, V_3$, and assume $A_{\{1,2\}} A_{\{2,3\}} A_{\{3,4\}} = (\det A) a_{22} a_{33}$ whenever $A^{-1} \in \mathscr{A}_Z$. By Jacobi's identity, $B = A^{-1}$ satisfies

$$B_{\{3,4\}} B_{\{1,4\}} B_{\{1,2\}} = B_{\{1,3,4\}} B_{\{1,2,4\}} \tag{6.8}$$

for all $B \in \mathscr{A}_Z$. Now replace $b_{22}$ by 1 and all other entries in the second row and column of $B$ by 0. The modified $B$ is still in $\mathscr{A}_Z$, and Equation (6.8) then reduces to

$$B_{\{3,4\}}B_{\{1,4\}}b_{11} = B_{\{1,3,4\}}B_{\{1,4\}}, \tag{6.9}$$

which must hold for all $B \in \mathscr{A}_Z$. This illustrates a general principle which can be applied to prove Conjecture 6.9. In an equation such as (6.8) any subscript or set of subscripts may be deleted from all index sets. Dividing Equation (6.9) by $B_{\{1,4\}}$ yields

$$b_{11}B_{\{3,4\}} = B_{\{1,3,4\}} \tag{6.10}$$

for all $B \in \mathscr{A}_Z$. [Note that if $B_{\{1,4\}}$ were zero, one could simply perturb $B$ in (6.8) by $\varepsilon I$ staying within $\mathscr{A}_Z$, reduce to (6.9), and then let $\varepsilon \to 0$.] By Theorem 6.7 ($m = 2$), either

$$\{1\}\times\{3,4\}\subset Z \quad \text{or} \quad \{3,4\}\times\{1\}\subset Z. \tag{6.11}$$

Deleting $\{3\}$ from (6.8) gives

$$b_{44}B_{\{1,4\}}B_{\{1,2\}} = B_{\{1,4\}}B_{\{1,2,4\}},$$

so that

$$B_{\{1,2\}}b_{44} = B_{\{1,2,4\}}$$

for all $B \in \mathscr{A}_Z$. Thus, again by Theorem 6.7,

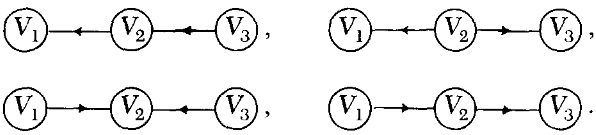
$$\{1,2\}\times\{4\}\subset Z \quad \text{or} \quad \{4\}\times\{1,2\}\subset Z. \tag{6.12}$$

Combining (6.11) and (6.12) shows that Z contains one of the zero patterns represented by the four matrices

$$\begin{bmatrix} \times & \times & 0 & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}, \quad \begin{bmatrix} \times & \times & 0 & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix},$$

$$\begin{bmatrix} \times & \times & \times & 0 \\ \times & \times & \times & 0 \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix}, \quad \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}.$$

In other words, $Z \supset Z(T, D)$ for one of the four directed trees




EXAMPLE 3. Now let $N = \{1,2,3,4,5\}$, $V_1 = \{1,2\}$, $V_2 = \{2,3,4\}$, and $V_3 = \{4,5\}$. Let $Z$ lie outside the profile of $V_1$, $V_2$, and $V_3$, and assume that

$$A_{\{1,2\}} A_{\{2,3,4\}} A_{\{4,5\}} = (\det A) a_{22} a_{44}$$

whenever $A^{-1} \in \mathscr{A}_Z$. By Jacobi's identity

$$B_{\{3,4,5\}} B_{\{1,5\}} B_{\{1,2,3\}} = B_{\{1,3,4,5\}} B_{\{1,2,3,5\}} \tag{6.13}$$

for all $B \in \mathscr{A}_Z$.

We apply the same strategy as before by deleting $\{2\}$. Then

$$B_{\{3,4,5\}} B_{\{1,5\}} B_{\{1,3\}} = B_{\{1,3,4,5\}} B_{\{1,3,5\}},$$

but there is no cancellation. Deleting $\{3\}$ in addition gives

$$B_{\{4,5\}} B_{\{1,5\}} b_{11} = B_{\{1,4,5\}} B_{\{1,5\}},$$

which reduces to

$$b_{11} B_{\{4,5\}} = B_{\{1,4,5\}}$$

on division by $B_{\{1,5\}}$. Thus

$$\{1\} \times \{4,5\} \subset Z \quad \text{or} \quad \{4,5\} \times \{1\} \subset Z. \tag{6.14}$$

Deleting $\{3,4\}$ from (6.13) gives the conclusion

$$\{1,2\} \times \{5\} \subset Z \quad \text{or} \quad \{5\} \times \{1,2\} \subset Z. \tag{6.15}$$

Deleting $\{1\}$ gives

$$B_{\{3,4,5\}} b_{55} B_{\{2,3\}} = B_{\{3,4,5\}} B_{\{2,3,5\}}$$

or

$$B_{\{2,3\}}b_{55} = B_{\{2,3,5\}}$$

on division by $B_{\{3,4,5\}}$. Thus

$$\{2,3\} \times \{5\} \subset Z \quad \text{or} \quad \{5\} \times \{2,3\} \subset Z. \tag{6.16}$$

Deleting $\{5\}$ yields the conclusion

$$\{1\} \times \{3,4\} \subset Z \quad \text{or} \quad \{3,4\} \times \{1\} \subset Z. \tag{6.17}$$

Deleting $\{3\}$ alone gives no conclusion. Thus it seems we have drawn all the conclusions we can from straightforward deletions. To illustrate a bad case, we take the first two alternatives in (6.14) and (6.15) and the second two in (6.16) and (6.17). This gives the zero pattern

$$\begin{bmatrix} \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ \times & 0 & 0 & \times & \times \end{bmatrix}, \tag{6.18}$$

and we have not yet shown that Z contains one of the four zero patterns predicted by Conjecture 6.9. These are

$$\begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}, \quad \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix},$$

$$\begin{bmatrix} \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix}, \quad \begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}.$$

However, there is great power in the hypothesis that (6.13) must hold for all $B \in \mathscr{A}_Z$. A further tedious analysis shows that (6.18) must indeed reduce to one of these four zero patterns. It is most disappointing that Example 3, which is only slightly different from Example 2, is so much more difficult to prove, and it is almost certain that this method is far too limited to prove Conjecture 6.9 if it is true. This example makes a proof even for $m = 3$ seem far out of reach. Ideally, in Example 3, one should prove that either

$$\{1\}\times\{3,4,5\}\subset Z \quad \text{or} \quad \{3,4,5\}\times\{1\}\subset Z \tag{6.19}$$

and

$$\{5\}\times\{1,2,3\}\subset Z \quad \text{or} \quad \{1,2,3\}\times\{5\}\subset Z, \tag{6.20}$$

and the conclusion would immediately follow as in Example 2. Unfortunately, we can see no way to do this in one step.

The following lemma shows that the problem of establishing (6.19) and (6.20) accurately represents the difficulty in the general case.

LEMMA 6.10. *Let $V_1,\dots,V_m \subset N$ be index sets satisfying* (6.1), *and let $T$ be a spanning tree of the intersection graph of $V_1,\dots,V_m$. Let $Z\subset N\times N$ lie outside the profile of $V_1,\dots,V_m$, and assume that for each $\{V_i,V_j\}\in\varepsilon(T)$ either $C(i/j)\times C(j/i)\subset Z$ or $C(j/i)\times C(i/j)\subset Z$ (see Definition* 5.3). *Then there is a directed tree $D$ related to $T$ such that $Z\supset Z(T,D)$.*

*Proof.* Define the directed tree $D$ as follows. For each $\{V_i,V_j\}\in\varepsilon(T)$, if $C(i/j)\times C(j/i)\subset Z$, let $(V_i,V_j)$ be a directed edge in $D$, and if $C(j/i)\times C(i/j)\subset Z$, let $(V_j,V_i)$ be a directed edge in $D$.

Suppose $(r,s)\in Z(T,D)$. Then $\{r,s\}\not\subset V_k$, $k=1,\dots,m$. Let $[V_{i_1},\dots,V_{i_l}]$ be a path in $T$ with $r\in V_{i_1}$ and $s\in V_{i_l}$. Then for some $k$, $1<k\leqslant l$, $(V_{i_k},V_{i_{k-1}})$ is a directed edge in $D$. This means that $C(i_k/i_{k-1})\times C(i_{k-1}/i_k)\subset Z$. We claim that $(r,s)\in C(i_k/i_{k-1})\times C(i_{k-1}/i_k)$. For if $r\notin C(i_k/i_{k-1})$, then $r\in V_j$ for every $j\in V(i_k/i_{k-1})$. In particular $r\in V_{i_l}$, so that $\{r,s\}\subset V_{i_l}$, a contradiction. Likewise, if $s\notin C(i_{k-1}/i_k)$, then $\{r,s\}\subset V_{i_1}$, a contradiction. Therefore $(r,s)\in C(i_k/i_{k-1})\times C(i_{k-1}/i_k)\subset Z$. Thus $Z(T,D)\subset Z$. ∎

One can easily check that, in Example 3, $C(2/1)=\{1\}$ and $C(1/2)=\{3,4,5\}$, while $C(2/3)=\{5\}$ and $C(3/2)=\{1,2,3\}$.

Starting with the identity (6.5),

$$\prod_{k=1}^{m} A_{V_k} = \det A \prod_{\{V_i,V_j\}\in\varepsilon(T)} A_{V_i\cap V_j},$$

and applying Jacobi's formula gives

$$\prod_{k=1}^{m} B_{C_k} = \prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C_i \cup C_j}.$$

Applying the same reasoning as used in Theorem 5.1 yields

$$\prod_{k=1}^{m} B_{\cup\{C(k/i):\{V_k, V_i\} \in \varepsilon(T)\}} = \prod_{\{V_i, V_j\} \in \varepsilon(T)} B_{C(i/j) \cup C(j/i)}.$$

Using the fact that this holds for all $B \in \mathscr{A}_Z$, it only (!) remains to show that

$$B_{C(i/j)} B_{C(j/i)} = B_{C(i/j) \cup C(j/i)}$$

for all $\{V_i, V_j\} \in \varepsilon(T)$. For then Theorem 6.7 ($m = 2$) and Lemma 6.10 imply Conjecture 6.9.

The hypothesis (6.5) in Conjecture 6.9 is a curious sort. It seems unmanageable enough to make the conjecture difficult to verify, while it is strong enough to make a potential counterexample nontrivial.

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