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Generic covers branched over $\{x^n = y^m\}$

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Abstract

In this paper the authors study generic covers of $\mathbb{C}^2$ branched over $\{x^n + y^m\} = 0$ s.t. the total space is a normal analytic surface.

They found a complete description of the monodromy of the cover in terms of the monodromy graphs and an almost complete description of the local fundamental groups in case $(n, m) = 1$.

For the general case, they give explicit descriptions of base changes in terms of monodromy graphs; they describe completely the embedded resolution graphs in the case $n|m$. Via these base changes every cover is a quotient of such a cover.

Keywords: Normal surface singularities; Branched covering spaces; Monodromy

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0. Introduction

In this paper we study normal singularities of complex analytic surfaces. Recall that (see, e.g., [11]) the singularities of a normal analytic variety form an analytic subvariety of codimension at least 2. So, a normal curve is automatically smooth, and a normal surface is automatically a surface with isolated singularities. The converse holds only for hypersurfaces (see [5]).

Then, in case of dimension 2, in order to study germs of normal analytic surfaces we can consider analytic surfaces with just one singular point.

Recall that, by Weierstrass preparation theorem, in a suitable neighborhood of every point of an analytic surface there exists a holomorphic function to a disc which is an analytic cover branched over a curve (see [6]). Moreover, a generic function like this one is a "generic" cover, i.e., a branched cover of degree d such that the fiber over a smooth point

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of the branching curve has $d-1$ points (see [12]). Every element in the fundamental group of the set of regular values of this map, induces a permutation of the d points of the fiber over the base point, thus a homomorphism from this group to $\mathcal{S}_d$, called "the monodromy of the cover". The "generic" condition means that for each geometric loop (i.e., a loop around a smooth point of the curve) its monodromy is a transposition.

This property can be usefully applied to study singularities; in fact, given a curve C contained in a disc Δ (or in $\mathbb{C}^2$), and a homomorphism $\mu : \pi_1(\Delta - C)$ (respectively $\pi_1(\mathbb{C}^2 - C)) \to \mathcal{S}_d$, s.t. the images of the geometric loops are transpositions, there exists a unique normal surface S and a generic cover from S to Δ (respectively $\mathbb{C}^2$) with C as branch locus and μ as monodromy (for an explicit construction see, e.g., [12]; unfortunately, this construction is quite involved, so it does not give directly a satisfying description of normal singularities).

So, in order to classify generic covers $\pi : S \to \mathbb{C}^2$ of degree d, with S normal, branched over some curve C, we need to classify only the generic monodromies $\mu : \pi_1(\mathbb{C}^2 - C) \to \mathcal{S}_d$.

In this paper we restrict to the case where the branching curve has (up to analytic equivalence) the equation $\{x^n = y^m\}$. This is a very particular case, but, by the classification of singularities of plane curves given by Puiseux (see [4]), it seems to be the natural starting point.

In Section 1 we state some well known expressions of the fundamental group of the disc minus our curves, via generators and relations, and we give a combinatorial bound for the degree of the cover.

In Section 2 we prove that our family of covers is stable under base change with maps of type $f_{a,b}(x, y) = (x^a, y^b)$; we represent the monodromy of a generic cover of degree d branched on the curve $\{x^n = y^m\}$ by a graph with d vertices and n labeled edges and we describe the action of a base change as above over these graphs.

In Section 3, we restrict ourselves to the case $(n, m) = 1$, and we give a complete classification of the graphs associated to these covers. In particular, we prove the following

Theorem 0.1. *The monodromy graphs for generic covers $\pi : S \to \mathbb{C}^2$ of degree $d \geqslant 3$ branched over the curve $\{x^n = y^m\}$, with $(n, m) = 1$, are the following*:

(1) *"Polygons" with d vertices, valence n/d (or m/d) and increment j, with $(j, d) = 1$, $j < d/2$, $j(d-j)|m$ (respectively $j(d-j)|n$). Moreover, d must divide n (respectively m).*

(2) *"Double stars" of type $(j, d-j)$ and valence $n/j(d-j)$ (or $m/j(d-j)$), with $(j, d) = 1$, $j < d/2$, $j(d-j)|n$ (respectively $j(d-j)|m$). Moreover, d must divide m (respectively n).*

The base change induced by the map $f(x, y) = (y, x)$ in $\mathbb{C}^2$ takes graphs of type 1 in graphs of type 2, and vice versa.

For the definition of "polygons" and "double stars" see Definitions 3.2 and 3.3.

So, in order to classify generic covers, we get the following

Corollary 0.2. *If $(n,m)=1$ then the generic covers $\pi : S \to \mathbb{C}^2$ of degree greater than 3 branched over $\{x^n = y^m\}$, S normal surface, are classified by the disjoint union of the sets*

$$\left\{(j,d) \mid d \geqslant 3,\ d|n,\ (j,d)=1,\ j<\tfrac{1}{2}d,\ j(d-j)|m\right\}$$

and

$$\left\{(j,d) \mid d \geqslant 3,\ d|m,\ (j,d)=1,\ j<\tfrac{1}{2}d,\ j(d-j)|n\right\}.$$

In both cases, d is the degree of the cover.

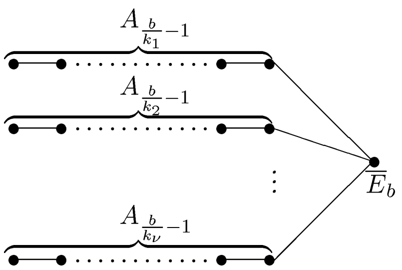
All these graphs correspond to some cover also if $(n,m)>1$, but they do not give a complete classification. We found also explicit equations for the singularities of S in the case $j=1$.

In Section 4 we compute the local fundamental group of the surfaces associated to some of the graphs constructed in Section 3. This gives partial answer to the smoothness problem (is S smooth?); moreover it provides an useful tool in the proof of Theorem 0.3.

In Section 5 we describe completely the embedded resolution graphs of all the possible singularities in case $m=bn$ for some b, using, to simplify the calculations, the equation $\{x^n + y^{bn} = 0\}$ for the branching locus; we prove the following

Theorem 0.3. *Let $\pi : S \to \mathbb{C}^2$ be a d-sheeted generic cover branched over the curve $\{x^n + y^{bn} = 0\}$, S normal.*

S has a resolution which is the plumbing variety of the following normal crossing configuration of smooth curves:



where the vertex $\overline{E}_b$ has genus $(n-d-\nu+2)/2$ and self-intersection $-\nu$.

Moreover, S is smooth $\Leftrightarrow$ the monodromy graph is a tree; this can occur only if d divides b.

Remark that by the results about base change of Section 2, all the possible surfaces under consideration are the quotient of one of these singularities by the action of a finite group.

1. Fundamental groups and maps

Let $C_{n,m}$ be the curve in $\mathbb{C}^2$ defined by the equation $x^n = y^m$.

The aim of this section is to compute some useful presentations for the fundamental group of $\mathbb{C}^2 \setminus C_{n,m}$ and derive the first necessary conditions for the existence of generic covers of given degree branched over $C_{n,m}$.

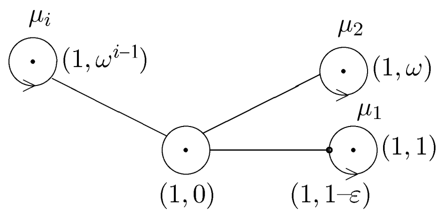
Let $\beta, \bar{\mu}_j$ be the paths $\beta(t) = (1, t\varepsilon + (1-t)(1-\varepsilon))$, $\bar{\mu}_j(t) = (1, \varepsilon \mathrm{e}^{2\pi \mathrm{i} t j/m})$ for $t \in [0,1]$ and if $z \in \mathbb{C}^*$ and $\lambda = (\lambda_1, \lambda_2)$ is a path in $\mathbb{C}^2$ let $z(\lambda) = (\lambda_1, z\lambda_2)$.

Let $\mu_1, \ldots, \mu_m$ be the geometric basis of $\pi_1(\{x = 1\} \setminus C_{n,m})$ with $(1, 1-\varepsilon)$ $(0 < \varepsilon < 1/2)$ as base point given by

$$\mu_1 = \left(1, \varepsilon \mathrm{e}^{\mathrm{i}(2\pi t + \pi)} + 1\right)_{t \in [0,1]},$$

$$\mu_j = \beta \cdot \bar{\mu}_j \cdot \omega^{j-1}\left(\beta^{-1} \cdot \mu_1 \cdot \beta\right) \cdot \bar{\mu}_j^{-1} \cdot \beta^{-1}$$

for $j = 2, \ldots, m$, where $\omega = \mathrm{e}^{2\pi \mathrm{i}/m}$.



Proposition 1.1. *The fundamental group of* $\mathbb{C}^2 \setminus C_{n,m}$ *admits the three equivalent presentations*

(1) $\pi_1(\mathbb{C}^2 \setminus C_{n,m}) = \langle \mu_1, \ldots, \mu_m \mid \mu_i = M\mu_{i+n}M^{-1},\ i = 1, \ldots, m\rangle$, *where* $M = \mu_1 \cdots \mu_n$ *and the indices are taken to be cyclical* (mod m);

(2) $\pi_1(\mathbb{C}^2 \setminus C_{n,m}) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_i = \overline{\Gamma}\gamma_{i+m}\overline{\Gamma}^{-1},\ i = 1, \ldots, n\rangle$, *where* $\overline{\Gamma} = \gamma_1 \cdots \gamma_m$ *and all indices are taken to be cyclical* (mod n);

(3) $\pi_1(\mathbb{C}^2 \setminus C_{n,m}) = \langle \gamma_1, \ldots, \gamma_n \mid \overline{\Gamma} = \gamma_i \cdots \gamma_{m+i-1},\ i = 2, \ldots, n\rangle$, *where all indices are taken to be cyclical* (mod n).

Proof. Applying Zariski–Van Kampen theorem (see [14,13,9]) to the projection $\phi_x : \mathbb{C}^2 \to \{y = 1\}$, $(x, y) \mapsto (x, 1)$, we get the first presentation.

Let $M_j = \mu_1 \cdots \mu_j$ (with cyclical indices (mod m) and $M_0 = 1$) and define $\gamma_i = M_{i-1}\mu_i^{-1}M_{i-1}^{-1}$ for $i = 1, \ldots, n$.

Since $\mu_i = \Gamma_{i-1}\gamma_i^{-1}\Gamma_{i-1}^{-1}$ for $i = 1, \ldots, m$, where $\Gamma_h = \gamma_1 \cdots \gamma_h$ (with cyclical indices (mod n) and $\Gamma_0 = 1$), $\gamma_1, \ldots, \gamma_n$ are a new set of generators and rewriting the relations of the first presentation in terms of the γ's we get the second one.

The third presentation is easily obtained from the second one. $\square$

Call $\mu_1, \ldots, \mu_m$ the standard generators of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ and $\gamma_1, \ldots, \gamma_n$ the minimal standard generators of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$.

From these presentations it is immediate to verify that, setting $\Gamma = \gamma_1 \cdots \gamma_n$

$$\Gamma = M^{-1} \quad \text{and} \quad \Gamma^{m/(n,m)} = \overline{\Gamma}^{n/(n,m)} \text{ is in the center.}$$

Observe that if we apply Zariski–Van Kampen theorem to the projection $\phi_y : \mathbb{C}^2 \to \{x = 1\}$, $\phi_y(x, y) = (1, y)$, and proceed as in Proposition 1.1, we may take as generators of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ a geometric basis $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$ of $\pi_1(\{y = 1\} \setminus C_{n,m})$ with relations

$$\tilde{\mu}_i = \tilde{\mu}_1 \cdots \tilde{\mu}_m \tilde{\mu}_{i+m} (\tilde{\mu}_1 \cdots \tilde{\mu}_m)^{-1}$$

for $i = 1, \ldots, n$ and cyclical indices (mod n).

Note also that this is the same as calculating the fundamental group of the complement of $C_{m,n} = \{x^m = y^n\}$ via ϕ_x.

Proposition 1.2. *With the above notations*

$$\tilde{\mu}_i = \gamma_i.$$

Proof. Let $f_{n,m} : \mathbb{C}^2_{x,y} \to \mathbb{C}^2_{\xi,\eta}$ be the map $(\xi, \eta) = f_{n,m}(x, y) = (x^n, y^m)$.

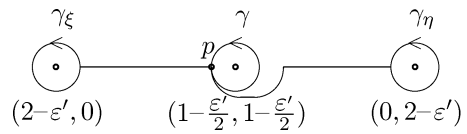
$$f_{n,m}|_{\mathbb{C}^2 \setminus (\{xy=0\} \cup C_{n,m})} : \mathbb{C}^2 \setminus \big(\{xy = 0\} \cup C_{n,m}\big) \to \mathbb{C}^2 \setminus \big\{\xi\eta(\xi - \eta) = 0\big\}$$

is a covering.

Take $(1, 1 - \varepsilon)$ as base point in $\mathbb{C}^2 \setminus (\{xy = 0\} \cup C_{n,m})$ and take as generators of $\pi_1(\mathbb{C}^2 \setminus (\{xy = 0\} \cup C_{n,m}))$ the standard generators of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ $\mu_1, \ldots, \mu_m$, $\mu_x = \bar{\mu}_m = (1, (1 - \varepsilon)\mathrm{e}^{2\pi \mathrm{i} t})$ (loop around the x-axes) and $\mu_y = (\mathrm{e}^{2\pi \mathrm{i} t}, 1 - \varepsilon)$ (loop around the y-axes) for $t \in [0, 1]$.

Observe that if we quotient $\pi_1(\mathbb{C}^2 \setminus (\{xy = 0\} \cup C_{n,m}))$ by the subgroup normally generated by μ_x and μ_y we obtain $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ and that we can do the same thing with the $\tilde{\mu}$'s as generators.

In the plane $\xi + \eta = 1 + (1 - \varepsilon)^m = 2 - \varepsilon'$ take as generators of $\pi_1(\mathbb{C}^2 \setminus \{\xi\eta(\xi - \eta) = 0\})$ γ, γ_ξ, γ_η as shown in the figure below where the line $\xi + \eta = 2 - \varepsilon'$ is identified with $\mathbb{C}$ via the η coordinate and $p = (1, 1 - \varepsilon')$.



γ, γ_ξ, γ_η are related by the equations $\gamma_\xi \gamma_\eta \gamma = \gamma \gamma_\xi \gamma_\eta = \gamma_\eta \gamma \gamma_\xi$.

Since $(f_{n,m})_*$ is injective, we can identify $\pi_1(\mathbb{C}^2 \setminus (\{xy = 0\} \cup C_{n,m}))$ with its image obtaining:

$$\begin{aligned}
&\mu_j = \gamma_\xi^{j-1} \gamma \gamma_\xi^{-j+1} \quad \text{for } j = 1, \ldots, m,\\
&\mu_x = \gamma_\xi^m,\\
&\mu_y = \gamma_\eta^n,\\
&\tilde{\mu}_j = \gamma_\eta^{-j+1} \gamma^{-1} \gamma_\eta^{j-1} \quad \text{for } j = 1, \ldots, n,\\
&\tilde{\mu}_x = \gamma_\xi^{-m},\\
&\tilde{\mu}_y = \gamma_\eta^{-n}.
\end{aligned}$$

Note now that

$$\gamma_\eta^{-1}\gamma^{-1}\gamma_\eta = \gamma\gamma_\xi\gamma^{-1}\gamma_\xi^{-1}\gamma^{-1} = \mu_1\mu_2^{-1}\mu_1^{-1},$$

thus

$$\begin{aligned}\tilde{\mu}_j &= \gamma_\eta^{-j+1}\gamma^{-1}\gamma_\eta^{j-1}\\ &= (\gamma\gamma_\xi)^{j-1}\gamma^{-1}(\gamma\gamma_\xi)^{-j+1}\\ &= \mu_1\cdots\mu_{j-1}\mu_j^{-1}(\mu_1\cdots\mu_{j-1})^{-1}\end{aligned}$$

as we wanted. □

Now we look for which $d \geqslant 3$ there exists a surjective homomorphism ρ from the fundamental group of $\mathbb{C}^2 \setminus \{x^n = y^m\}$ in $\mathcal{S}_d$ sending the geometric generators in transpositions, i.e., a normal surface S and a d-sheeted generic covering $\pi : S \to \mathbb{C}^2$ branched over $\{x^n = y^m\}$.

For every $\sigma \in \mathcal{S}_d$ call $\nu(\sigma)$ the number of orbits of σ, ν the value of this function for $\rho(\Gamma)$. ν is the number of cycles of a permutation constructed multiplying n transpositions in $\mathcal{S}_d$, so $n + d + \nu$ is even.

Call $a_1, \ldots, a_\nu$ the length of the ν cycles of the monodromy of Γ. Being $\Gamma^{m/(m,n)}$ central and $d \geqslant 3$, the order of $\rho(\Gamma)$ divides $m/(m,n)$; then every a_i does.

Let $\mathcal{D}_{mn} \subset \mathbb{N}$, be the set of (positive) divisors of $m/(m,n)$, and consider all the possible ways to write d as sum of elements of this set.

Let $\mathcal{K}_{mn}^d$ be the set of all the possible "lengths" of this sums, where the "length" of a sum, is the number of integers we are adding.

Now define the function

$$\Lambda(m,n,d) = \inf\bigl\{\nu \in \mathcal{K}_{mn}^d \mid n + d + \nu \text{ is even}\bigr\}.$$

Λ gives a lower bound for the number of orbits of $\rho(\Gamma)$; in fact it is the minimal number of orbits for permutations with order that divides $m/(m,n)$ ($\nu \in \mathcal{K}_{mn}^d$) and with the same parity of $\rho(\Gamma)$ (product of n transpositions).

Now define

$$\chi(m,n,d) = n + 2 - d - \Lambda(m,n,d).$$

We have the following

Proposition 1.3. *Let (S,φ) be a generic cover branched on $\{x^n = y^m\}$ of degree $d > 2$. Then $\chi(m,n,d) \geqslant 0$.*

Moreover, if n divides m, the converse holds.

Proof. For the first part of the proposition we prove, by induction on n, that for the product of a transitive set of n transpositions in $\mathcal{S}_d$, the number of orbits ν must be $\nu \leqslant n + 2 - d$. In fact, if $n = 1$ then $d = 2$ and $\nu = 1$, and there is nothing to prove.

If $n > 1$, consider the first $n - 1$ transpositions $\rho(\gamma_i)$, and let ν' be the number of orbits of their product; $\nu' = \nu + 1$ or $\nu' = \nu - 1$.

If these $n-1$ transpositions generate $\mathcal{S}_d$ then $\nu \leqslant \nu' + 1 \leqslant n - 1 + 2 - d + 1 = n + 2 - d$.

Otherwise they generate $\mathcal{S}_{d-k} \times \mathcal{S}_k$, and the last transposition "connects" two different orbits of their product. Suppose that in $\mathcal{S}_k$ there are exactly g among the first $n-1$ transpositions, then their product has by induction $\nu'' \leqslant g + 2 - k$ orbits and the product of the other $n-1-g$ transpositions has $\nu' - \nu'' \leqslant n - 1 - g + 2 - (d-k)$ orbits.

So $\nu = \nu' - 1 \leqslant g + 2 - k + n - 1 - g + 2 - (d-k) - 1 = n + 2 - d$.

By definition $\Lambda(m,n,d) \leqslant \nu$ and we get the result.

For the converse, remark that in this case the fundamental group is generated by $\gamma_1, \dots, \gamma_n$ with the only relation that $\overline{\Gamma}$ is central. So it is sufficient to exhibit a set of n transpositions $\sigma_1, \dots, \sigma_n$ s.t. their product is the identity of $\mathcal{S}_d$.

Now assume $\chi(m,n,d) \geqslant 0$, i.e., $\Lambda(m,n,d) \leqslant n - d + 2$.

Then there exist $a_1, \dots, a_{\Lambda(m,n,d)} \in \mathbb{N}$ such that:

(1) $\sum a_i = d$,

(2) $\forall i$ a_i divides $m/(m,n)$,

(3) $n + d + \Lambda(m,n,d)$ is an even number.

Choose the following transpositions:

$$\sigma_i = \begin{cases} (1, i+1), & 1 \leqslant i \leqslant d-1, \\ \left(1, d+1 - \sum_{1}^{i-d+1} a_k\right), & d \leqslant i \leqslant d + \Lambda - 2, \\ (1,2), & d + \Lambda - 1 \leqslant i \leqslant n. \end{cases}$$

Of course this choice verifies our condition, then it describes a d-sheeted generic cover branched on the curve $\{x^n = y^m\}$. □

In general these functions are not so simple to compute. The following holds:

Remark 1.4.

$$\chi(m,n,d+1) \geqslant \chi(m,n,d) - 2.$$

Proof. Let $\Lambda = \Lambda(m,n,d)$. If $\Lambda = +\infty$ there is nothing to prove. Otherwise, let $a_1, \dots, a_\Lambda \in \mathcal{D}_{mn}$ realizing the minimum as in the definition of Λ.

Then $a_1, \dots, a_\Lambda, a_{\Lambda+1} = 1 \in \mathcal{D}_{mn}$, with $n + d + 1 + \Lambda + 1$ even and $\sum_1^{\Lambda+1} a_i = d + 1$ which implies $\Lambda(m,n,d+1) \leqslant \Lambda(m,n,d) + 1$.

But then

$$\begin{aligned} \chi(m,n,d+1) &= n + 2 - d - 1 - \Lambda(m,n,d+1) \\ &\geqslant n - d - \Lambda(m,n,d) = \chi(m,n,d) - 2. \quad \square \end{aligned}$$

Remark that if $m = n$, m even, $\Lambda(m,m,d) = d$, i.e., there exist generic covers if and only if $m + 2 - 2d \geqslant 0$, i.e., $d \leqslant m/2 + 1$ (if $m = n$, m odd, there are no generic covers for $d > 2$; in fact this is true for every n, m s.t. mn is odd, see next section for details).

2. Fiber products, monodromies and graphs

Let $\pi : S \to \mathbb{C}^2$ be a d-sheeted generic cover branched over $\{x^n = y^m\}$ with S an irreducible surface with an isolated singularity in $P = \pi^{-1}(0,0)$ and let ρ be its monodromy.

Consider the map $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ and let $\overline{S} = S \times_{\mathbb{C}^2} \mathbb{C}^2$ be the fiber product.

We get the following commutative diagram:

$$\begin{tikzcd} \overline{S} \arrow[r, "\psi"] \arrow[d, "\bar{\pi}"] & S \arrow[d, "\pi"] \\ \mathbb{C}^2 \arrow[r, "f_{a,b}"] & \mathbb{C}^2 \end{tikzcd}$$

Proposition 2.1. *$\overline{S}$ as above is an analytic surface with an isolated singularity in $\overline{P} = \overline{\pi}^{-1}(0,0)$.*

$\overline{\pi}$ is a d-sheeted generic cover branched over $\{x^{an} = y^{bm}\}$ and its monodromy $\bar{\rho}$ is the composition $\rho \circ (f_{a,b})_*$.

Proof. By definition of fiber product we get immediately that $\overline{S}$ is analytic, and $\overline{\pi}$ a d-sheeted generic cover branched over $\{x^{an} = y^{bm}\}$.

The two maps π and $f_{a,b}$ are coverings of $\mathbb{C}^2$ whose branching loci intersect just in the origin. This easily implies that $\overline{S}$ is smooth outside $\overline{P}$.

Now consider the homomorphism $(f_{a,b})_* : \pi_1(\mathbb{C}^2 \setminus \{x^{an} = y^{bm}\}) \to \pi_1(\mathbb{C}^2 \setminus \{x^n = y^m\})$. Of course

$$(f_{a,b})_*(\gamma_i) = \gamma_{[i]} \quad 0 < i \leqslant an,$$

where $[i]$ is the remainder class of i mod n, and if we choose the correct enumeration for the points in the fibers over the two base points, we have that $\forall\, 0 < i \leqslant an$, γ_i and $(f_{a,b})_*(\gamma_i)$ act in the same way and this holds for any $\gamma \in \pi_1(\mathbb{C}^2 \setminus \{x^{an} = y^{bm}\})$. This proves the second part of the theorem. □

Theorem 2.2. *In the above hypotheses, if moreover S is a normal surface, then $\overline{S}$ is normal too.*

Proof. First note that we can assume $b = 1$. Define

$$A = \mathbb{C}(S) = \frac{\mathbb{C}\{x', y, z_1, \dots, z_n\}}{I}, \qquad \overline{A} = \mathbb{C}(\overline{S}) = \frac{\mathbb{C}\{x, y, z_1, \dots, z_n\}}{\psi^* I}$$

and denote the quotient fields by

$$Q = \mathrm{Quot}(A), \qquad \overline{Q} = \mathrm{Quot}(\overline{A}).$$

ψ^* injects A in $\overline{A}$ and Q in $\overline{Q}$ ($\psi^*(x') = x^a$); so, we can consider $\overline{A}$ and $\overline{Q}$ as extensions of A and Q, respectively.

Recall that, by definition, S normal means that A is integrally closed in Q, i.e., for any $f \in Q$ such that $\exists\, p \in A[t]$ monic with $p(f) = 0 \Rightarrow f \in A$. We must check that the same property holds for $\overline{A}$ in $\overline{Q}$.

Let $f \in \overline{Q}$, $p = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in \overline{A}[t]$ monic such that $p(f) = 0$.

$\omega = \mathrm{e}^{2\pi \mathrm{i}/a}$ acts on $\overline{A}$ (respectively $\overline{Q}$) via the natural map

$$\omega\big(f(x, y, z_1, \dots, z_n)\big) = f(\omega x, y, z_1, \dots, z_n)$$

and we have that Fix(ω)$= A$ (respectively Q).

For every l such that $0 \leqslant l < a$ define

$$f_l = \sum_{0 \leqslant i < a} \omega^i(f)\omega^{il}.$$

Remark that

$$\omega(f_l) = \omega\left(\sum_{0 \leqslant i < a} \omega^i(f)\omega^{il}\right) = \sum_{0 \leqslant i < a} \omega^{i+1}(f)\omega^{il} = \frac{f_l}{\omega^l}.$$

In particular, if $g_l = f_l x^l$, $g_l \in Q$.

$\forall\, 0 \leqslant i, l < a$ define $p_l^i = t^n + \omega^i(a_{n-1})\omega^{il}t^{n-1} + \cdots + \omega^i(a_0)\omega^{nil}$; $p_l^i(\omega^i(f)\omega^{il}) = 0$.

Let $q_l = \prod p_l^i$, $h_1^l, \dots, h_{an}^l$ its roots in some suitable extension of $\overline{Q}$ and let $r_l = \prod_{1 \leqslant i_1 < \cdots < i_a \leqslant an}(t - h_{i_1}^l - \cdots - h_{i_a}^l)$. By the fundamental theorem of symmetric functions $r_l \in \overline{A}[t]$ and $r_l(f_l) = 0$.

If $r_l = t^N + b_{N-1}t^{N-1} + \cdots + b_N$, let $s_l = t^N + x^l b_{N-1}t^{N-1} + \cdots + x^{lN} b_N$; we have $s_l(g_l) = 0$, and $s_l \in \overline{A}[t]$; up to multiplying s_l by some suitable polynomial, we can assume $s_l \in A[t]$, so $g_l \in A$.

But f_l is a weakly holomorphic function on $\overline{S}$, so f_l is holomorphic in the smooth points of $\overline{S}$. If $l = 0$, $f_0 = g_0$ and $f_0 \in A \subset \overline{A}$. If $l > 0$, g_l is 0 in $\overline{S} \cap \{x = 0\}$, i.e., is 0 as function in $S \cap \{x' = 0\}$. Thus, g_l/x' is a holomorphic function on $S \setminus \mathrm{Sing}(S)$, then g_l/x' is a holomorphic function on S (see [11, Chapter 6, Proposition 4]), i.e., $g_l/x^a \in \overline{A}$.

So

$$f_l = \frac{g_l}{x^l} = \frac{g_l}{x^a}x^{a-l} \in \overline{A}.$$

We conclude the proof noting that

$$f = \frac{1}{a}\sum_{0 \leqslant l < a} f_l. \quad \square$$

Corollary 2.3. *If $\pi : S \to \mathbb{C}^2$ is a d-sheeted generic cover branched over $\{x^n = y^m\}$ with S a normal surface singular in $P = \pi^{-1}(0,0)$, ρ is its monodromy, and $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ is the map defined above then $\overline{\pi} : \overline{S} = S \times_{\mathbb{C}^2} \mathbb{C}^2 \to \mathbb{C}^2$ is the normal d-sheeted generic cover associated to the monodromy $\overline{\rho} = \rho \circ (f_{a,b})_*$.*

Remark that base change via $f_{a,b}$ induces a partial ordering among generic covers; call a generic cover minimal if it cannot be induced by other covers via one of these base changes.

Proposition 2.4. *$(f_{a,b})_*:\pi_1(\mathbb{C}^2\setminus\{x^{an}=y^{bm}\})\to\pi_1(\mathbb{C}^2\setminus\{x^n=y^m\})$ and $\psi_*:\pi_1(\overline{S}\setminus\{\overline{P}\})\to\pi_1(S\setminus\{P\})$ are surjective.*

Proof. $(f_{a,b})_*$ is surjective since $(f_{a,b})_*(\gamma_i)=\gamma_{[i]}$ for $0<i\leqslant an$, so it sends the minimal standard generators onto the minimal standard generators.

Call $R=\pi^{-1}(C_{n,m})$, $\overline{R}=\overline{\pi}^{-1}(C_{an,bm})$; $\pi_1(S\setminus R)$ is the subgroup of $\pi_1(\mathbb{C}^2\setminus C_{n,m})$ of those γ such that $\rho(\gamma)(1)=1$, and the same holds for $\pi_1(\overline{S}\setminus\overline{R})$ in $\pi_1(\mathbb{C}^2\setminus C_{an,bm})$ and $\overline{\rho}$. Since $(f_{a,b})_*$ is surjective and $\overline{\rho}=\rho\circ(f_{a,b})_*$, $\psi_*:\pi_1(\overline{S}\setminus\overline{R})\to\pi_1(S\setminus R)$ is surjective too.

Considering the following commutative diagram

$$\begin{tikzcd}
\pi_1(\overline{S}\setminus\overline{R}) \arrow[r, two heads, "\psi_*"] \arrow[d, two heads] & \pi_1(S\setminus R) \arrow[d, two heads] \\
\pi_1(\overline{S}\setminus\{\overline{P}\}) \arrow[r, "\psi_*"] & \pi_1(S\setminus\{P\})
\end{tikzcd}$$

we obtain that $\psi_*:\pi_1(\overline{S}\setminus\{\overline{P}\})\to\pi_1(S\setminus\{P\})$ is surjective. □

Our aim is to classify all homomorphisms $\rho:\pi_1(\mathbb{C}^2\setminus C_{n,m})\to\mathcal{S}_d$ (up to inner automorphisms) whose image is transitive and such that $\tau_i=\rho(\gamma_i)$ is a transposition (i.e., homomorphisms representing normal surfaces, see Section 0). Observe that if the second condition holds, the image of ρ is transitive if and only if ρ is surjective.

The case $d=2$ is trivial since ρ is unique and gives the double cover of $\mathbb{C}^2$ branched over $C_{n,m}$ obtained projecting on $\{z=0\}$ the surface in $\mathbb{C}^3$

$$z^2=x^n-y^m$$

(note that a hypersurface with isolated singularities is normal (see [5])), so suppose $d\geqslant 3$.

Since

$$\gamma_i=\overline{\Gamma}^n\gamma_i\overline{\Gamma}^{-n}$$

for all i (actually $\overline{\Gamma}^{n/(n,m)}$), it follows that $\rho(\overline{\Gamma}^n)$ must be in the center of $\mathcal{S}_d$ which is trivial if $d\geqslant 3$, so it must be $\rho(\overline{\Gamma}^n)=1$.

Now, if n and m are both odd, $\rho(\overline{\Gamma}^n)$ is an odd permutation and thus it cannot be equal to 1, so nm must be even.

A surjective homomorphism $\rho':\mathcal{F}_r\to\mathcal{S}_d$ from a free group $\mathcal{F}_r$ with r generators $g_1,\dots,g_r$ such that the image of each generator is a transposition, can be represented by a connected graph with d labeled vertices and r labeled edges in the following way: take a vertex for every $l=1,\dots,d$ and if $\rho'(g_h)=(i,j)$ connect the vertex i to the vertex j with the edge labeled h. Note that the same graph with the numeration of the vertices suppressed represents ρ' up to inner automorphisms of $\mathcal{S}_d$.

A permutation $\sigma\in\mathcal{S}_d$ acts on the set of graphs with d labeled vertices and r labeled edges in the following way: if N is such a graph then if the edge labeled l in N connects the vertices h and k then the edge labeled l in $\sigma(N)$ connects the vertices $\sigma(h)$ and $\sigma(k)$.

Since our presentations have the peculiar forms

$$\gamma_i = \overline{\Gamma}\gamma_{i+m}\overline{\Gamma}^{-1}, \qquad \mu_i = M\mu_{i+n}M^{-1}$$

we can interpret our monodromy ρ as a map $\rho' : \mathcal{F}_n$ (respectively $\mathcal{F}_m$) $\to \mathcal{S}_d$ such that

$$\tau_i = \rho'(\overline{\Gamma})\tau_{i+m}\rho'(\overline{\Gamma})^{-1} \quad \text{(respectively } \tau_i = \rho'(M)\tau_{i+n}\rho'(M)^{-1}\text{)},$$

i.e., by a connected graph N with d vertices and n (respectively m) labeled edges such that if we act on N by $\rho'(\overline{\Gamma})$ (respectively $\rho'(M)$) for a (fixed) numeration of the vertices, then the edge labeled by j is transformed into the edge labeled by $j+m$ with cyclical indices (mod n) (respectively $j+n$ with cyclical indices (mod m)).

By sake of simplicity, from now on we consider only the presentation of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ in terms of minimal standard generators and so graphs with n edges.

Observe that in order to have a connected graph it must be $d \leqslant n+1$.

So, a generic cover branched over $\{x^n = y^m\}$ is defined by a graph with d vertices and n labeled edges, and an integer m.

Remark that we have proved that $(f_{a,b})_*$ acts on the graphs substituting the edge labeled j with a edges labeled $j+sn$ for $0 \leqslant s < a$, and multiplying m by b. So we restrict ourselves to "minimal" monodromy graphs, i.e., graphs associated to minimal covers.

Note that we are interested in graphs with labeled edges up to a cyclical permutation of the edges as we can see from the third presentation in Proposition 1.1.

3. Generic covers branched over $\{x^n = y^m\}$ with n, m relatively prime

The aim of this section is to classify all edge labeled graphs corresponding to generic monodromies $\rho : \pi_1(\mathbb{C}^2 \setminus C_{n,m}) \to \mathcal{S}_d$ in the case n and m are relatively prime and $d \geqslant 3$.

Definition 3.1. Given a graph N, p a vertex of N and L an edge of N, let the valence of p be the number of edges of N having p as an end point, and let the valence of L be the number of edges of N with the same end points as L.

Definition 3.2 (*Polygons*). Let a polygon with d vertices, valence a and increment j, with j and d relatively prime, be a graph with $n = ad$ labeled edges of valence a, d vertices of valence $2a$ and such that $\forall s, t$ the edges labeled s and t have

(1) two vertices in common if and only if $s - t = \lambda d$,
(2) one vertex in common if and only if $s - t = \lambda d + j$ or $s - t = \lambda d - j$,
(3) no vertices in common otherwise.

Definition 3.3 (*Double stars*). Let a double star of type (j,k), with j and k relatively prime, and valence a be a graph with $n = ajk$ labeled edges of valence a, $d = j + k$ vertices of which j of valence ak and k of valence aj and such that $\forall s, t$ the edges labeled s and t have

(1) two vertices in common if and only if $s - t = \lambda jk$,
(2) one vertex in common if and only if $s - t = \lambda jk + \mu j$ or $s - t = \lambda jk + \mu k$,
(3) no vertices in common otherwise.

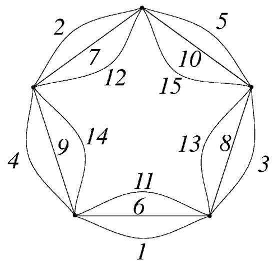


Fig. 1. A polygon with 5 vertices, valence 3 and increment 2.

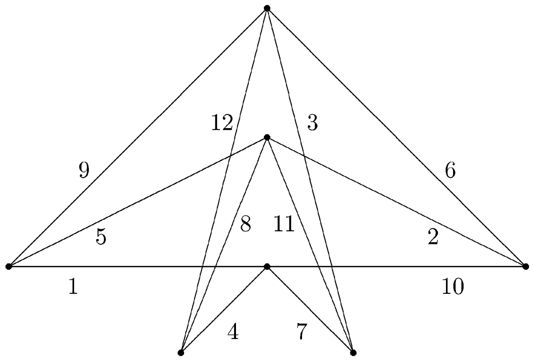


Fig. 2. A double star of type $(3,4)$ and valence 1.

Observe that between each vertex of valence aj and each vertex of valence ak there are exactly a edges and that there are no edges between vertices of the same valence.

Recall that, in both cases, the labeling of the edges is cyclical (mod n).

The main result of this section is the following

Theorem 3.4. *The graphs which correspond to generic monodromies* $\rho:\pi_1(\mathbb{C}^2\setminus C_{n,m})\to \mathcal{S}_d$ *for* $d\geqslant 3$ *with* $(n,m)=1$, *are the following*:

(1) *Polygons with* d *vertices, valence* n/d (*or valence* m/d) *and increment* j, *with* $(j,d)=1$, $j<d/2$, $j(d-j)|m$ (*respectively* $j(d-j)|n$). *Moreover,* d *must divide* n (*respectively* m).

(2) *Double stars of type* $(j,d-j)$ *and valence* $n/j(d-j)$ (*or valence* $m/j(d-j)$), *with* $(j,d)=1$, $j<d/2$, $j(d-j)|n$ (*respectively* $j(d-j)|m$). *Moreover,* d *must divide* m (*respectively* n).

The base change induced by the map $f(x,y)=(y,x)$ *in* $\mathbb{C}^2$ *takes graphs of type* 1 *in graphs of type* 2, *and vice versa.*

Let N be the graph representing the monodromy ρ. N has n edges and d vertices.

Since n and m are relatively prime, the inner action of the subgroup generated by $\overline{\Gamma}$ on the γ_i is transitive, and so is the action of $\rho(\overline{\Gamma})$ (and its powers) on the edges of N.

Thus all edges have the same valence v and $v|n$.

Call a petal the set of all edges between two fixed vertices. If $v \geqslant 2$ then choose two edges, say i and $i+h$, in the same petal. With a suitable power of $\overline{\Gamma}$ you can send the edge i in the edge $i+h$, but if so, since then the set of the vertices of the petal is fixed, the petal remain fixed (as a set), so the edges $i+2h$ (image of the edge $i+h$), $i+3h, \ldots,$ are in the petal.

If another edge, say $i+k$, is in the same petal but $h \nmid k$, then all edges $i+(h,k)$, $i+2(h,k), \ldots,$ are in the petal.

Thus, in general, the v edges in a petal are labeled i, $i+n/v$, $i+2n/v, \ldots, i+(v-1)n/v$.

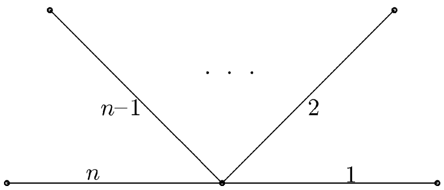
Observe that you can retrieve the numeration of the whole graph once you know a suitable numeration for the case of edges of valence 1, so assume $v=1$. (This corresponds to studying minimal covers in the sense of Section 2.)

Since $\rho'(\overline{\Gamma})$ acts by conjugation, it transforms relations between the $\rho'(\gamma_i)$'s in relations of the same form between the $\rho'(\overline{\Gamma})\rho'(\gamma_i)\rho'(\overline{\Gamma})^{-1}$'s, thus intersecting edges go to intersecting edges and non intersecting edges go to non intersecting edges. Moreover, the valence of a vertex is maintained, thus there are only two possible valences for the vertices (possibly equal) and each edge has vertices of both valences.

Call an end a vertex of valence 1, and a leaf an edge with an end as vertex.

Lemma 3.5. *If N has a leaf then N is a double star of type $(1, d-1)$ and valence 1; moreover $d|m$.*

Proof. If N has a leaf then every other edge is a leaf. In this case $d=n+1$, there is a vertex of valence n and there is only one possible numeration of the edges, i.e., N is a double star of type $(1,n)$ and valence 1.



In order to calculate m we must construct $\overline{\Gamma}$. Let $\Gamma_{r,s} = (\gamma_1 \cdots \gamma_n)^r \gamma_1 \cdots \gamma_s$ for $r \geqslant 0$, $0 \leqslant s < n$ and act on the vertices of the edge labeled 1 first by $\Gamma_{0,1} = \gamma_1$, then by $\Gamma_{0,2} = \gamma_1\gamma_2, \ldots, \Gamma_{1,0} = \gamma_1 \cdots \gamma_n$, $\Gamma_{1,1} = \gamma_1 \cdots \gamma_n \gamma_1$ and so on until for $\Gamma_{\bar{r},\bar{s}}$ the two vertices coincide with the vertices of another edge, say the edge $1+k$. Then check if $k=\bar{s}$ and if $\rho'(\Gamma_{\bar{r},\bar{s}})$ sends the edge i in the edge $i+\bar{s}$. If this is the case then $\overline{\Gamma} = \Gamma^{\alpha}_{\bar{r},\bar{s}}$ and $m = \alpha(n\bar{r}+\bar{s})$.

It is easy to see that in this case $\bar{r}=1$, $\bar{s}=1$, so $\overline{\Gamma} = (\gamma_1 \cdots \gamma_n \gamma_1)^{\alpha}$, $d|m$, and N gives a homomorphism $\rho : \pi_1(\mathbb{C}^2 \setminus C_{d-1,\alpha d}) \to \mathcal{S}_d$. $\square$

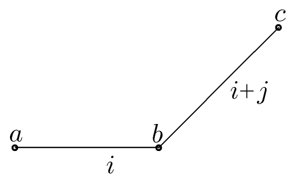


Fig. 3.

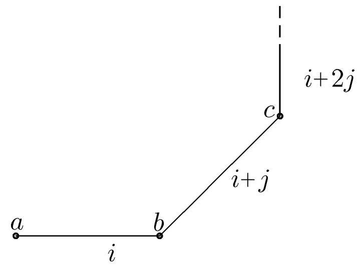


Fig. 4.

So you get a d-sheeted covering of $\mathbb{C}^2$ ramified on $x^{a(d-1)} = y^{bd}$ and it can be realized as the projection on $\{z = 0\}$ of the normal surface in $\mathbb{C}^3$

$$z^d - dy^b z + (d-1)x^a = 0.$$

Note that this surface is smooth $\Leftrightarrow a = 1$.

We may now assume that N has no ends (no leaves) and edges of valence 1.

Consider two edges of N, say i and $i + j$, with a (single) vertex in common and label their vertices as in Fig. 3.

Suppose that the power of $\rho'(\overline{\Gamma})$ that takes the edge i to the edge $i + j$ acts on the vertices sending $a \mapsto b \mapsto c$.

Lemma 3.6. *In the above case N is a polygon with d vertices, valence* 1 *and increment j; moreover $j(d-j)|m$.*

Proof. With the same power of $\rho'(\overline{\Gamma})$ that takes the edge i to the edge $i + j$, the edge $i + j$ is sent to the edge $i + 2j$ and this must contain the vertex c (Fig. 4).

Proceeding in this way we obtain a sequence of edges labeled $i + kj$ such that two edges have in common (only) one vertex if and only if they are labeled $i + hj$ and $i + (h+1)j$ for some h. Eventually one of these edges must have a vertex in common with the edge i and this must be the vertex a, thus closing the circle (if not there would be an edge of valence at least 2, see Fig. 5). This last edge must have the label $i - j$.

Observe that every edge must belong to a circle like this one with the same j.

Suppose now that there is another edge, say with the label $i + l$, containing the vertices b and d (Fig. 6).

Again a suitable power of $\rho'(\overline{\Gamma})$ takes the edge i to the edge $i + l$ and suppose $b \mapsto d$.

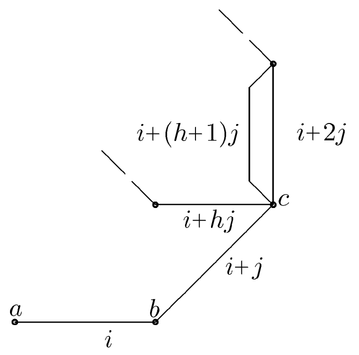


Fig. 5.

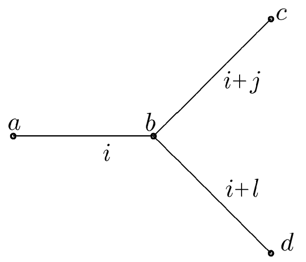


Fig. 6.

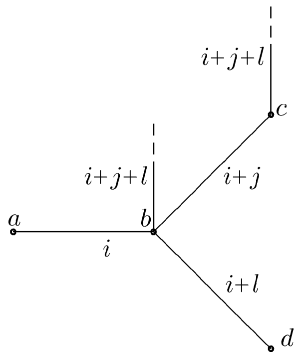


Fig. 7.

Then the power of $\rho'(\overline{\Gamma})$ which sends the edge i to the edge $i+j+l$ must send the vertex a in both the vertices c and d, a contradiction.

Otherwise, suppose b fixed for the power of $\rho'(\overline{\Gamma})$ which sends the edge i in the edge $i+l$; then the edge $i+j$ is sent to the edge $i+j+l$ which must contain the vertex b too. Moreover the edge $i+j+l$ must contain the vertex c as one can see acting with the power of $\rho'(\overline{\Gamma})$ that takes the edge i to the edge $i+j$ (and so the edge $i+l$ to the edge $i+l+j$). This contradicts our assumption that all edges have valence 1 (Fig. 7).

So the only possibility is that there is only one circle with $n = d$ edges labeled consecutively $1, 1 + j, \ldots, 1 + hj, \ldots, 1 + d - j$ with $1 \leqslant j < d/2$ and $(j, d) = 1$, i.e., a polygon with d vertices, valence 1 and increment j.

To calculate m, observe that one of the vertices of the edge 1 moves once every j steps, i.e., moves once applying γ_1, twice applying $\gamma_1 \cdots \gamma_{1+j}$ and so on, while the other vertex moves once every $n - j$ steps in the opposite direction. Under this condition they will be again consecutive vertices every l steps with

$$\left[\frac{l-1}{j}\right] + \left[\frac{l-1}{d-j}\right] + 2 \equiv 0 \ (\mathrm{mod}\ d),$$

and they are the vertices of the edge labeled

$$1 + \left(\left[\frac{l-1}{j}\right] + 1\right) j = 1 + \left(\left[\frac{l-1}{d-j}\right] + 1\right)(d - j).$$

We require this edge to be the edge labeled $l + 1$, so

$$l = \left(\left[\frac{l-1}{j}\right] + 1\right) j + kd$$

with $k \geqslant 0$, thus

$$kd = 1 + \left\{\frac{l-1}{j}\right\} j - j < 1$$

and it must be $k = 0$.

Thus, $j | l$ and the same argument shows that $(d - j) | l$.

The minimal l verifies

$$\left[\frac{l-1}{j}\right] + \left[\frac{l-1}{d-j}\right] + 2 = d,$$

thus $l = j(d - j)$.

Now $j(d - j) | m$.

In fact, since the graph does not change if we cyclically permute the edges, we have that acting on the edge labeled 2 by $\gamma_2 \cdots \gamma_{l+1}$ we obtain the edge labeled $2 + l$.

If $j \neq 1$ we get the same result if we act by $\gamma_1 \cdots \gamma_l$ (the edge labeled 1 (respectively $l + 1$) does not intersect the edge labeled 2 (respectively $2 + l$)), thus the edge labeled i is sent to the edge labeled $l + i$.

If $j = 1$ it is immediate to verify that $\forall i$, acting on the edge labeled i by $\gamma_1 \cdots \gamma_l$ we obtain the edge labeled $i - 1$. $\square$

So, as before, we get (allowing edges of valence a) a d-sheeted cover branched over

$$x^{ad} = y^{bj(d-j)}$$

with $(j, d) = 1$.

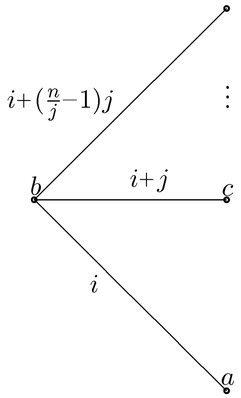


Fig. 8.

We may now assume that for any couple of intersecting edges, say i and $i+j$, the suitable power of $\rho'(\overline{\Gamma})$ that sends the edge i to the edge $i+j$ leaves the common vertex b fixed.

Lemma 3.7. *Under this assumption, N is a double star af type $(j, d-j)$ and valence* 1; *moreover $d|m$.*

Proof. In this case, all edges labeled $i+hj$ pass through the vertex b (or, they generate a star with increment j and vertex b).

As in the case of petals, if the edge $i+k$ contains the vertex b and $j\nmid k$ then all edges in the form $i+h(j,k)$ are in the star, so if a star contains l edges, then they are numbered $i+h(n/l)$ and every edge of the graph is in such a star with the same l.

Observe that all edges that have a vertex b in common generate a star with an increment that divides n and vertex b (see Fig. 8).

Since we assume that the graph has no leaves, then $j \neq 1$ and there must be another edge containing the vertex a.

This edge is labeled, say, $i+k$ with $(k, j) = 1$, and, since we assume edges of valence 1, it does not contain the vertex b.

Its other vertex A cannot be in common with the edge $i+hj$ otherwise the power of $\rho'(\overline{\Gamma})$ that sends i to $i+hj$ would send the edge $i+k$ to the edge $i+hj+k$ not fixing the common vertex (Fig. 9).

Thus, also the edges containing the vertex a generate a star and we may suppose that it has increment k, i.e., that its edges are labeled $i+h'k$ with $h' = 0, 1, \ldots, n/k-1$.

The same happens at the vertices of the edges $i+hj$ other than b, while in the vertex A there is a star with increment j (edges labeled $i+k+hj$ with $h = 0, 1, \ldots, n/j-1$).

The free vertices of this star coincide with the free vertices of the star with vertex b. The same happens with all the other edges of the star with vertex a (Fig. 10).

Comparing the edges we come to kj edges and $k+j$ vertices forming j stars with increment j and k edges, and k stars with increment k and j edges. Since there is only

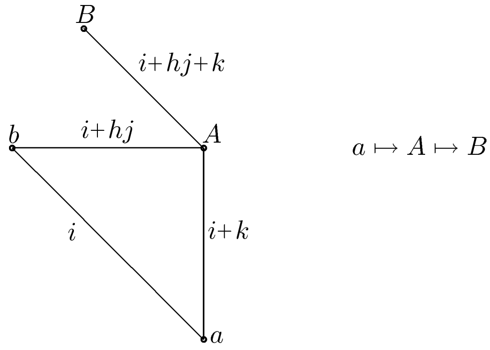


Fig. 9.

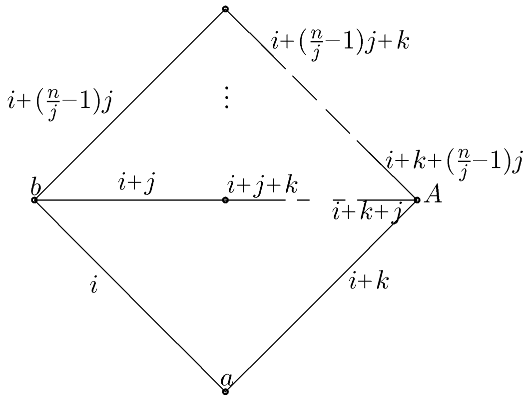


Fig. 10.

one way to number a star when the label of a single edge is known (as for petals), there is only one possible numeration of the edges, i.e., we get a double star of type (j,k) and valence 1.

We can immediately see that $j+k|m$. $\square$

Allowing petals of valence a and choosing $j<k$, we get a d-sheeted cover branched over

$$x^{aj(d-j)}=y^{bd}$$

with $2\leqslant j\leqslant d/2$ and $(j,d)=1$.

Proof of Theorem 3.4. The three lemmas give the complete classification of the graphs as in the statement; so we have only to understand the action of $f:\mathbb{C}^2\to\mathbb{C}^2$ defined by $f(x,y)=(y,x)$ on these graphs.

This map sends the minimal standard generators (γ_i) for $\pi_1(\mathbb{C}^2\setminus C_{n,m})$ into the standard generators (μ_i) for $\pi_1(\mathbb{C}^2\setminus C_{m,n})$, and vice versa.

Being (see the proof of the Propositions 1.1 and 1.2) $\mu_i = \Gamma_{i-1}\gamma_i^{-1}\Gamma_{i-1}^{-1}$, this isomorphism can be seen on the graphs in this way: act on the edge i by $\rho(\gamma_{i-1}\cdots\gamma_1)$ (the edge 1 remains fixed).

If you do this on $m = aj(d-j)$ consecutive edges of a polygon with d vertices, valence a and increment j you get a double star of type $(j, d-j)$ and valence a and if you do this on $n = ad$ consecutive edges of a double star of type $(j, d-j)$ and valence a you get a polygon with d vertices, valence a and increment j. □

By the bijection between monodromy graphs and generic covers, we get Corollary 0.2.

4. Local fundamental groups

Let $\pi : S \to \mathbb{C}^2$ be the cover branched over $C_{n,m}$ constructed in the previous section from the polygon with n vertices, valence 1 and increment α. In order to see if S is singular (note that away from $P = \pi^{-1}((0,0))$ S is smooth) we must check whether $\pi_1(S \setminus \{P\})$ is trivial or not (see [10]), so we must calculate $\pi_1(S \setminus \{P\})$.

Let $\beta = n - \alpha$, $b = \alpha\beta/m$. Recall that $(\alpha, \beta) = 1$, b is an integer and we may assume $\alpha < \beta$.

Let

$$R_i = \gamma_i\gamma_{i+1}\cdots\gamma_{i+m-1},$$

where all indices are taken cyclical (mod n).

Then the relations defining $\pi_1(\mathbb{C}^2 \setminus \{x^n = y^m\})$ may be written as

$$R_i = R_{i+1}$$

for $i = 1, \dots, n-1$.

In order to compute $\pi_1(S \setminus \{P\})$ consider $\pi|_{S\setminus R} : S \setminus R \to \mathbb{C}^2 \setminus C_{n,m}$, where $R = \pi^{-1}(C_{n,m})$. This is an unramified cover, and we can identify $\pi_1(S \setminus R)$ with the subgroup of $\pi_1(\mathbb{C}^2 \setminus C_{n,m})$ given by those elements γ such that $\rho(\gamma)(1) = 1$.

We apply the Reidemeister–Shreier method (see [8]) to the Shreier set of left cosets

$$L_j = \gamma_1\gamma_{1+\alpha}\gamma_{1+2\alpha}\cdots\gamma_{1+(j-1)\alpha}$$

for $j = 0, \dots, n-1$ ($L_0 = e$).

A set of generators for $\pi_1(S \setminus R)$ is given by the following elements

$$A_i = \gamma_1\gamma_{1+\alpha}\cdots\gamma_{1+(i-2)\alpha}\gamma_{1+(i-1)\alpha}^2\gamma_{1+(i-2)\alpha}^{-1}\cdots\gamma_{1+\alpha}^{-1}\gamma_1^{-1}$$

for $i = 1, \dots, n-1$;

$$B_{j,i} = L_j\gamma_{1+i\alpha}L_j^{-1}$$

for $i \neq j-1, j$, and $j = 0, \dots, n-1$.

$$C = \gamma_1\gamma_{1+\alpha}\cdots\gamma_{1+(n-2)\alpha}\gamma_{1+(n-1)\alpha},$$

$$D = \gamma_{1+(n-1)\alpha}\gamma_{1+(n-2)\alpha}^{-1}\cdots\gamma_{1+\alpha}^{-1}\gamma_1^{-1}$$

while a set of defining relators is given by rewriting in terms of the above generators the following

$$L_j R_i R_{i+1}^{-1} L_j^{-1}$$

for all choices of i and j.

Observe that $\pi_1(S \setminus \{P\})$ is obtained from $\pi_1(S \setminus R)$ by adding the relations

$$A_i = B_{i,j} = CD = e$$

for all choices of i and j (they represent loops around all the components of R).

Thus $\pi_1(S \setminus \{P\})$ is generated by C and to determine its order we have to go through the rewriting process for the relations.

Theorem 4.1. *With the above notations if $\alpha > 1$, $\pi_1(S \setminus \{P\}) = \mathbb{Z}/b\mathbb{Z}$.*

Proof. First observe that $\alpha > 1 \Rightarrow \alpha + \beta < \alpha\beta$.

Consider $L_j R_i$ and write $R_i = \gamma_i \cdots \gamma_{i+m-1} = \lambda_1 \cdots \lambda_n R_i'$ where $\lambda_l = \gamma_{i+l-1}$.

Let h, k be integers such that $\lambda_h = \gamma_{1+(j-1)\alpha}$, $\lambda_k = \gamma_{1+j\alpha}$.

Since if $i \neq 1 + \beta$

$$L_j \gamma_i = \begin{cases} B_{j,i} L_j & \text{if } i \neq 1 + (j-1)\alpha, 1 + j\alpha, \\ A_j L_{j-1} & \text{if } i = 1 + (j-1)\alpha, \\ L_{j+1} & \text{if } i = 1 + j\alpha, \end{cases}$$

while

$$L_j \gamma_{1+\beta} = \begin{cases} B_{j,1+\beta} L_j & \text{if } j \neq 0, n-1, \\ C L_0 & \text{if } j = n-1, \\ D L_{n-1} & \text{if } j = 0, \end{cases}$$

then if $h > k$

$$L_j R_i = \begin{cases} H L_{j+1} \lambda_{k+1} \cdots \lambda_n R_i' & \text{if } j \neq n-1, \\ H C L_0 \lambda_{k+1} \cdots \lambda_n R_i' & \text{if } j = n-1 \end{cases}$$

while if $h < k$;

$$L_j R_i = \begin{cases} H L_{j-1} \lambda_{h+1} \cdots \lambda_n R_i' & \text{if } j \neq 0, \\ H D L_{n-1} \lambda_{h+1} \cdots \lambda_n R_i' & \text{if } j = 0 \end{cases}$$

with H a word in the A's and B's.

Observe that in the first case we pass first from L_j to L_{j+1} and then to L_{j+2}, L_{j+3} and so on (cyclic indices (mod n)) every α steps, i.e.,

$$\begin{aligned}
L_j R_i &= K_1 L_{j+1} \lambda'_1 \cdots \lambda'_s \\
&= K_1 K_2 L_{j+2} \lambda'_{1+\alpha} \cdots \lambda'_s \\
&= K_1 K_2 K_3 L_{j+3} \lambda'_{1+2\alpha} \cdots \lambda'_s \\
&\vdots
\end{aligned}$$

where K_h is a word in the A's, B's and C and K_h contains C if and only if $j+h \equiv 0 \pmod n$. If this happens we say that rewriting $L_j R_i$ the coset index increases.

Analogously, in the second case we pass first from L_j to L_{j-1} and then to L_{j-2}, L_{j-3} and so on every β steps and the corresponding K_h is a word in the A's, B's and D and K_h contains $D \Leftrightarrow j-h \equiv n-1 \pmod n$. In this case we say that the coset index decreases.

Note that for a fixed i there are only α indices j for which the coset index increases, namely those indices such that $1+j\alpha = i, i+1, \ldots, i+\alpha-1$.

Suppose that rewriting $L_j R_i$ the coset index increases and write $1+j\alpha = i+c-1$ with $0 < c \leqslant \alpha$. Then deleting all the words in the A's and B's

$$\begin{aligned}
L_j R_i &= L_{j+1} \gamma_{i+c} \cdots \gamma_{i+m-1} \\
&= L_{j+2} \gamma_{i+c+\alpha} \cdots \gamma_{i+m-1} \\
&\vdots \\
&= C L_0 \gamma_{i+c+(n-j-1)\alpha} \cdots \gamma_{i+m-1} \\
&= C^2 L_0 \gamma_{i+c+(2n-j-1)\alpha} \cdots \gamma_{i+m-1} \\
&= C^t L_l \gamma_{i+c+m-\alpha} \cdots \gamma_{i+m-1} = C^t L_l
\end{aligned}$$

(if $c = \alpha$ there are no γ's in the last line) for a suitable l and where

$$t = \left[\frac{m-\alpha-(n-j-1)\alpha}{\alpha n}\right] + 1 = \left[\frac{b\beta-(\alpha+\beta)+j}{\alpha+\beta}\right] + 1 = \left[\frac{b\beta+j}{\alpha+\beta}\right].$$

On the other hand, suppose that rewriting $L_j R_i$ the coset index decreases and write $1+ja = i+c'-1$ with $\alpha < c' \leqslant n$. Then again

$$\begin{aligned}
L_j R_i &= L_{j-1} \gamma_{i+c'-\alpha} \cdots \gamma_{i+m-1} \\
&= L_{j-2} \gamma_{i+c'-\alpha+\beta} \cdots \gamma_{i+m-1} \\
&\vdots \\
&= D L_0 \gamma_{i+c'-\alpha+j\beta} \cdots \gamma_{i+m-1} \\
&= D^2 L_0 \gamma_{i+c'-\alpha+(n+j)\beta} \cdots \gamma_{i+m-1} \\
&= D^{t'} L_{l'} \gamma_{i+c'+m-\alpha-\beta} \cdots \gamma_{i+m-1} = D^{t'} L_{l'}
\end{aligned}$$

(if $c' = n$ there are no γ's in the last line) for a suitable l' and where

$$t' = \left[\frac{m-\beta-j\beta}{\beta n}\right] + 1 = \left[\frac{b\alpha-j-1}{\alpha+\beta}\right] + 1.$$

Observe that if the coset index increases rewriting both $L_j R_i$ and $L_j R_{i+1}$ then $1 < c \leqslant \alpha$ and

$$
\begin{aligned}
L_j R_{i+1} &= L_{j+1}\gamma_{i+c}\cdots\gamma_{i+m}\\
&= L_{j+2}\gamma_{i+c+\alpha}\cdots\gamma_{i+m}\\
&\;\vdots\\
&= CL_0\gamma_{i+c+(n-j-1)\alpha}\cdots\gamma_{i+m}\\
&= C^2L_0\gamma_{i+c+(2n-j-1)\alpha}\cdots\gamma_{i+m}\\
&= C^tL_l\gamma_{i+c+m-\alpha}\cdots\gamma_{i+m} = C^tL_l,
\end{aligned}
$$

that is rewriting a relation $L_jR_iR_{i+1}^{-1}L_j^{-1}$ for which the coset index increases for both L_jR_i and L_jR_{i+1} yields the trivial relation $C^t = C^t$.

The same thing happens if the coset index decreases rewriting both L_jR_i and L_jR_{i+1}, i.e., if $\alpha < c' \leqslant n$ and we get $D^{t'} = D^{t'}$.

In the case $c = 1$, (respectively $c' = \alpha + 1$) rewriting L_jR_i the coset index increases (respectively decreases) and rewriting L_jR_{i+1} the coset index decreases (respectively increases) so we get $C^t = D^{t'}$, i.e., $C^{t+t'} = 1$.

Observe that $t + t' = b$, in fact write

$$
\begin{aligned}
b\beta &= r(\alpha+\beta) + s,\\
b\alpha &= r'(\alpha+\beta) + s',
\end{aligned}
$$

with $0 \leqslant s, s' < \alpha + \beta$. Adding the above equations we get

$$b(\alpha+\beta) = (r + r')(\alpha+\beta) + s + s',$$

i.e., $(\alpha+\beta)|(s + s')$ which implies $s + s' = 0$ or $s + s' = \alpha + \beta$.

In the first case we have $s = s' = 0$ (this is true if and only if $(\alpha+\beta)|b$) and $b\beta + j = r(\alpha+\beta) + j$, $b\alpha - j - 1 = r'(\alpha+\beta) - j - 1$ which implies $t = r$, $t' = r' - 1 + 1\ \forall j$ so that $t + t' = r + r' = b$.

In the other case $(s, s' \neq 0)$ we have

$$b\beta + j = r(\alpha+\beta) + s + j, \quad b\alpha - j - 1 = r'(\alpha+\beta) + s' - j - 1$$

and $s + j \geqslant \alpha + \beta \Leftrightarrow s' - j - 1 = \alpha + \beta - s - j - 1 < 0$ so we have only two possibilities $t + t' = r + r' + 1$ or $t + t' = r + 1 + r' - 1 + 1$ that is $t + t' = b$.

Summing up $\pi_1(S \setminus \{P\}) = \langle C \mid C^b = 1\rangle \equiv \mathbb{Z}/b\mathbb{Z}$. $\square$

Consider now the $n = \alpha + \beta$-sheeted generic cover $\overline{\pi} : \overline{S} \to \mathbb{C}^2$ branched over $C_{a(\alpha+\beta),b\alpha\beta}$ corresponding to the polygon with $\alpha + \beta$ vertices, valence a and increment α.

This surface is obtained from S via fiber product with $f_{a,1}$ (see Section 2).

Corollary 4.2. *If $a = b = 1$ $\overline{S}$ is smooth. If $b \neq 1$ $\overline{S}$ is singular.*

Proof. If $\alpha\beta < \alpha + \beta$, i.e., $\alpha = 1$ the result follows observing that in this case $\overline{S} \subset \mathbb{C}^3$ has equation

$$z^{n+1} - (n+1)x^a z + ny^b = 0,$$

so suppose $\alpha + \beta < \alpha\beta$.

If $a = 1$ then, by the above proposition, $\pi_1(\overline{S} \setminus \{\overline{P}\})$ is trivial if and only if $b = 1$.

If $a > 1$ then $\pi_1(S \setminus \{P\}) \neq 0$, so, by Proposition 2.4, $\pi_1(\overline{S} \setminus \{\overline{P}\}) \neq 0$, and $\overline{S}$ is singular. $\square$

5. Generic covers branched on $\{x^n + y^{bn} = 0\}$

In this section S is a normal surface, $\pi : S \to \mathbb{C}^2$ a d-sheeted ($d \geqslant 3$) generic cover branched over the curve $C = \{x^n + y^{bn} = 0\}$. In particular, b and n cannot be both odd, so bn is even.

Let $\widetilde{S}$ be a resolution of the (isolated) singularity of S obtained from π using the standard algorithm (see, e.g., [7]).

We get the following diagram:

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\pi''"] \arrow[d, "\tilde{\pi}"] & S \arrow[d, "\pi"] \\ T \arrow[r, "\pi'"] & \mathbb{C}^2 \end{tikzcd}$$

with π' a sequence of ordinary blow-up's, $\tilde{\pi}$ proper, π'' a resolution of S. $\widetilde{S}$ is the plumbing variety of a normal crossing configuration of smooth curves, i.e., of the compact components of $\widetilde{C} = \tilde{\pi}^{-1}(\pi'^{-1}(C))$. We look for the dual graph of this configuration (see, e.g., [7]).

Recall that, as in the previous sections,

$$\pi_1(\mathbb{C}^2 \setminus C) = \langle \gamma_1, \ldots, \gamma_n \mid \Gamma^b \gamma_i \Gamma^{-b} \gamma_i^{-1} \rangle,$$

where $\Gamma = \gamma_1 \cdots \gamma_n$, with γ_i geometric loops around C supported on the line $\{y = 1\}$ such that Γ is a loop around all the points of C on this line.

Consider the monodromy $\rho : \pi_1(\mathbb{C}^2 \setminus C) \to \mathcal{S}_d$, and let ν be the number of orbits of $\sigma = \rho(\Gamma)$, $k_1, \ldots, k_\nu$ the cardinalities of these orbits.

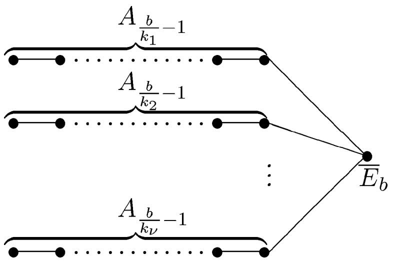
Remark that every labeled graph with d vertices and n edges represents the monodromy of a cover branched over $\{x^n + y^{bn} = 0\}$ for a suitable b; more precisely for every b multiple of the order of the permutation corresponding to the ordered product of the edges.

Definition 5.1 (*Strings*). Call a string the dual graph of a normal crossing configuration of curves which is a tree and such that no vertex is contained in more than two edges, i.e., a graph like the following:

Call a string of type A_k a string with k vertices such that every vertex corresponds to a smooth rational curve with self-intersection -2.

In this section we prove the following result:

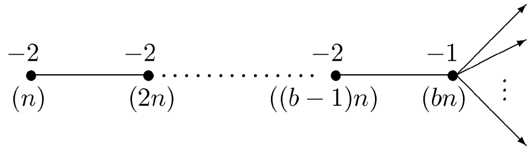
Theorem 5.2. *In the above hypothesis and notations, S has a* (*minimal*) *resolution which is the plumbing variety of the following normal crossing configuration of smooth curves*:



where the vertex $\overline{E}_b$ has genus $(n-d-\nu+2)/2$ and self-intersection $-\nu$. Moreover, S is smooth $\Leftrightarrow$ the monodromy graph is a tree; *this can occur only if σ is a d-cycle, so d divides b.*

First, we need the embedded resolution graph of $C \subset \mathbb{C}^2$, i.e., the dual graph of $C' = {\pi'}^{-1}(C)$, and the geometric loops around the irreducible components of this curve.

Lemma 5.3. *The embedded resolution graph of $\{x^n + y^{bn} = 0\}$ in $\mathbb{C}^2$ is the following*:



where the vertices correspond to exceptional divisors, the number over each vertex is its self-intersection and the numbers (jn) are the multiplicities of zero of the function $f' = {\pi'}^* f$, $f = x^n + y^{bn}$; *the arrowhead vertices are exactly n, the number of irreducible components of C.*

Let E_j be the irreducible curve in T corresponding to the vertex of multiplicity jn in our graph. Let $p_j = E_j \cap E_{j+1}$, $P = (1,1) \in \mathbb{C}^2$.

$\forall j$ there exists a neighborhood U_j of p_j in T, and local coordinates (ξ, η) in U_j, such that $C' \cap U_j = (E_j \cap U_j) \cup (E_{j+1} \cap U_j) = \{\xi\eta = 0\}$, and $P' = {\pi'}^{-1}(P) \in U_j$ with coordinates $(1,1)$. Choosing P' as base point for $\pi_1(T \setminus C')$, the natural geometric loops in $U_j \setminus C'$ around E_j $(\{(\mathrm{e}^{\mathrm{i}\theta}, 1)\})$ and and E_{j+1} $(\{(1, \mathrm{e}^{\mathrm{i}\theta})\})$ are respectively Γ^i and Γ^{i+1} under the isomorphism ${\pi'}_* : \pi_1(T \setminus C') \to \pi_1(\mathbb{C}^2 \setminus C)$.

Proof. We are looking for C' and the monodromy around one of its irreducible components. Choose as base point the point $P = (1,1)$ which is not contained in C for all b and n.

Consider the following lines in $\mathbb{C}^2$:

$$\begin{aligned} \Pi' &= \{y = 1\}, \\ \Pi &= \{x = y\}, \qquad (1) \\ \Pi'' &= \{x = 1\}. \end{aligned}$$

Remark that the intersection of these lines is exactly the point P.

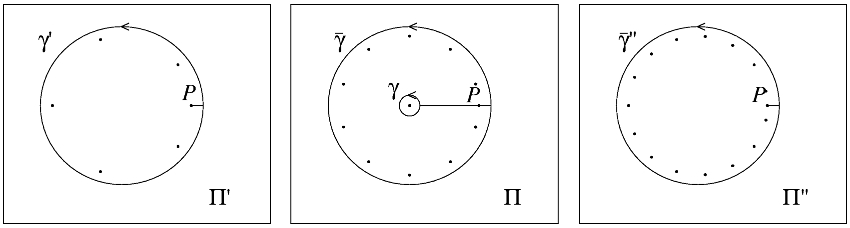


Fig. 11.

Let $\gamma' = \Gamma$ (respectively $\bar{\gamma}$, $\bar{\gamma}''$) be a loop in Π' (respectively Π, Π'') around all points of C and let γ be a loop in Π around $(0,0)$ as in Fig. 11.

$C \cap \Pi$ is given by the equations $\{x = y,\ x^n(1 + x^{(b-1)n}) = 0\}$, i.e., the origin (with multiplicity n) and $(b-1)n$ distinct points on the unitary circle.

The pencil of lines $\Pi_\lambda = \{x = 1 - \lambda + \lambda y\}$, for $0 \leqslant \lambda \leqslant 1$ defines an homotopy in $\mathbb{C}^2 \setminus C$ between $\bar{\gamma}'' \in \Pi''$ and $\bar{\gamma} \in \Pi$ (observe that $\forall \lambda$ C cuts on Π_λ n points (with multiplicity) and $P \in \Pi_\lambda$).

Now consider the lines $\overline{\Pi}_\lambda = \{y = \lambda\}$, for $\varepsilon \leqslant \lambda \leqslant 1$, with $0 \leqslant \varepsilon \leqslant 1$.

$$C \cap \overline{\Pi}_\lambda = \left\{(\lambda^b \mathrm{e}^{(2r+1)\mathrm{i}\pi/n}, \lambda),\ 0 \leqslant r \leqslant n-1\right\}.$$

Thus, considering the paths $\alpha = \{(1-t, 1-t)\}_{t \in [0,1-\varepsilon]}$, $\beta_l = \{(\varepsilon, \varepsilon + (l-1)\varepsilon t)\}_{t\in[0,1]}$ and $\tau = \{(\varepsilon \mathrm{e}^{\mathrm{i}t}, 0)\}_{t\in[0,2\pi]}$, $\forall\, 0 < \varepsilon < 1$, γ' is homotopic in $\mathbb{C}^2 \setminus C$ to $\alpha\beta_0\tau\beta_0^{-1}\alpha^{-1}$.

$\beta_\lambda(\varepsilon\mathrm{e}^{\mathrm{i}t}, \lambda\varepsilon\mathrm{e}^{\mathrm{i}t})_{t\in[0,2\pi]}\beta_\lambda^{-1}$ for $0 \leqslant \lambda \leqslant 1$ defines an homotopy in $\mathbb{C}^2 \setminus C$ between $\beta_0\tau\beta_0^{-1}$ and $(\varepsilon\mathrm{e}^{\mathrm{i}t}, \varepsilon\mathrm{e}^{\mathrm{i}t})_{t\in[0,2\pi]}$, so $\gamma' = \gamma$ in $\pi_1(\mathbb{C}^2 \setminus C)$.

C' is obtained by blowing up recursively $\mathbb{C}^2$ in points of our curves (and his complete transform). So, the complementary of the complete transform does not change after every blow-up. With a slight abuse of notation, we do not change name to paths after every blow-up.

The first step is to blow up $\mathbb{C}^2$ in the origin. Recall that we get a complex manifold obtained pasting two charts biholomorphic to $\mathbb{C}^2$, respectively V_1 and U_1, with projections on $\mathbb{C}^2$ given (in coordinates) respectively by:

$$(V_1) \qquad (x, y) \mapsto (xy, y),$$

$$(U_1) \qquad (x, y) \mapsto (x, xy).$$

The complete transform of C in these two charts is, in (V_1), $y^n(x^n + y^{(b-1)n}) = 0$, and in (U_1), $\{x^n(1 + x^{(b-1)n}y^{bn}) = 0\}$. Remark that the last one is smooth (after reduction). Then the singularities of the complete transform of our curve are in (V_1).

$V_1 \cong \mathbb{C}^2$, so we can compute the lines Π_1', Π_1, Π_1'' (i.e., the lines in (V_1) given by the Eq. (1)). Note that the inverse image of P in (V_1) has coordinates $(1,1)$.

We claim that in (V_1) we have the following situation (see Fig. 12).

In fact, from the explicit equation of the projection, Π_1' is exactly Π', while Π_1'' is exactly Π. Rewriting the homotopies we find that the loop around the points cut by C_1, the strict transform of C, on Π_1 is $\overline{\gamma}$. We have to find the homotopy class of the loop around

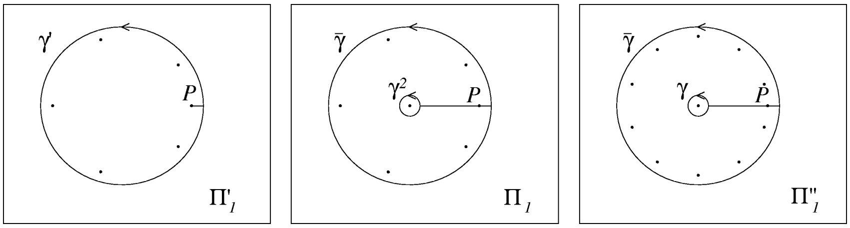


Fig. 12.

the origin in Π_1, that is of $\eta = \alpha(\varepsilon e^{it}, \varepsilon e^{it})_{t\in[0,2\pi]}\alpha^{-1}$ where α is a real positive path (i.e., $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i(t) \in \mathbb{R}^+$) from $(1,1)$ to $(\varepsilon, \varepsilon)$, as a loop in $\mathbb{C}^2 \setminus (C_1 \cup \{y = 0\})$.

Since $|y| \leqslant 1 \Rightarrow |x| \leqslant 1$ and $|x| \leqslant |y|$ then, by the homotopy $(\lambda\varepsilon e^{it}, \varepsilon e^{it})_{t\in[0,2\pi]}$, with $1 \leqslant \lambda \leqslant 2$, $\eta \sim \eta' = \beta(2\varepsilon e^{it}, \varepsilon e^{it})_{t\in[0,2\pi]}\beta^{-1}$ with β a real positive path from $(1,1)$ to $(2\varepsilon, \varepsilon)$.

Now

$$\bigl(2\varepsilon e^{i(1+\lambda)t}, \varepsilon e^{(1-\lambda)t}\bigr)_{t\in[0,\pi]} \cup \bigl(2\varepsilon e^{i(2\pi\lambda+(1-\lambda)t)}, \varepsilon e^{i((2t-2\pi)\lambda+(1-\lambda)t)}\bigr)_{t\in[\pi,2\pi]}$$

for $0 \leqslant \lambda \leqslant 1$ defines a homotopy $\eta' \sim \beta(2\varepsilon e^{it}, \varepsilon)_{t\in[0,2\pi]}(2\varepsilon, \varepsilon e^{it})_{t\in[0,2\pi]}\beta^{-1}$.

Moreover

$$\bigl(\bigl(2\varepsilon + \lambda(1-\varepsilon)\bigr)e^{it}, \varepsilon + \lambda(1-\varepsilon)\bigr)_{t\in[0,2\pi]} \quad \text{for } 0 \leqslant \lambda \leqslant 1$$

gives $\beta(2\varepsilon e^{it}, \varepsilon)_{t\in[0,2\pi]}\beta^{-1} \sim \beta'((1+\varepsilon)e^{it}, 1)_{t\in[0,2\pi]}\beta'^{-1}$ with β' a real positive path in $\{y = 1\}$ from $(1,1)$ to $(1+\varepsilon, 1)$ and

$$\bigl(\lambda + (1-\lambda)2\varepsilon, \varepsilon e^{it}\bigr)_{t\in[0,2\pi]} \quad \text{for } 0 \leqslant \lambda \leqslant 1$$

gives $\beta(2\varepsilon, \varepsilon e^{it})_{t\in[0,2\pi]}\beta^{-1} \sim \beta''(1, \varepsilon e^{it})_{t\in[0,2\pi]}\beta''^{-1}$ with β'' a real positive path in $\{x = 1\}$ from $(1,1)$ to $(1, \varepsilon)$.

Thus the loop around the origin in Π_1 is just γ^2.

Now we conclude the proof by induction.

The inductive hypothesis is that after k blow-up's in the origin (of the V-charts), the complete transform of C has equations, $y^{kn}(x^n + y^{(b-k)n}) = 0$ in V_k, and $x^{kn}y^{(k-1)n}(1 + x^{(b-k)n}y^{(b-k+1)n}) = 0$ in U_k; moreover the complete transform of C in U_k^c has as dual graph a string of type A_{k-1}. One extremal component of this configuration intersects (transversally in the origin of U_k) the new exceptional divisor, and on V_k our lines are as in Fig. 13.

Call E_l the exceptional curve of the lth blow-up, or, with abuse of notation, its strict transform in every other blow-up.

Remark that we have proven the inductive hypotheses for $k = 1$.

Blow up the origin of V_k. Π'_{k+1} and Π''_{k+1} are the same as Π'_k and Π_k and the paths in Π_k can be computed as in the previous step. The new equations in V_{k+1} and in U_{k+1} are obtained simply from the old equations in V_k (and the equations of the blow-up). Moreover E_k is contained in V_{k+1}^c; we are blowing up a point $q_k \in E_k$ so the

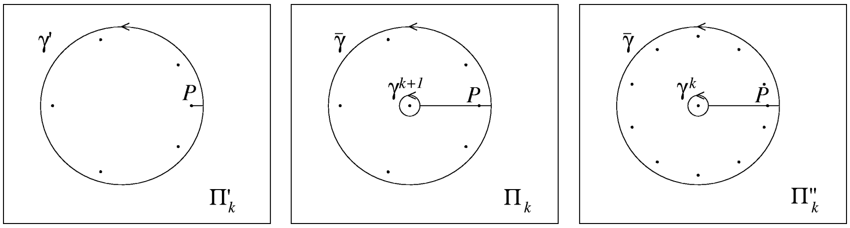


Fig. 13.

self-intersection of E_k pass from -1 to -2, and it intersects transversally E_{k-1}, since $q_k \notin \bigcup_{j=1}^{k-1} E_j$. $\bigcup_{j=1}^{k} E_j$ is a normal crossing configuration of type A_k and E_{k+1} intersects transversely E_k.

Remark that U_k (with the given coordinates) is the chart we are looking for around $p_k = E_k \cap E_{k+1}$. $\square$

The dual graph of the standard resolution of the singularity of S depends only on b, n, d and the conjugacy class of $\sigma = \rho(\Gamma)$ in $\mathcal{S}_d$ (this follows easily from the explicit construction of S from C and ρ, see, e.g., [7,12]).

From now on, we call multiplicity of a curve in $\widetilde{S}$ the multiplicity of $f' \circ \tilde{\pi}$; $\forall i$, $\overline{E}_i = \tilde{\pi}^{-1}(E_i)$.

Lemma 5.4. *$\overline{E}_b$ is an irreducible compact connected curve of multiplicity bn and genus $(n-d-\nu+2)/2$. Moreover, $\tilde{\pi}|_{\overline{E}_b}$ has degree d.*

Proof. $\{\sigma_i\} = \{\rho(\gamma_i)\}$ is a family of transpositions generating $\mathcal{S}_d$, and if $d \geqslant 3$, $\sigma^b = 1$.

We have remarked before that σ^b is the monodromy of a geometric loop around E_b, then for a small neighborhood V of a generic point of E_b, $\tilde{\pi}^{-1}(V)$ has d connected components, and $\tilde{\pi}|_{\tilde{\pi}^{-1}(V)}$ is a cover of degree d.

$\tilde{\pi}|_{\overline{E}_b}$ is a branched cover of degree d (a priori non connected), and the multiplicity of $f' \circ \tilde{\pi}$ on $\overline{E}_b$ is exactly the multiplicity of f on E_b, i.e., bn. The branching points are the intersection points of E_b with the other branches of the configuration, i.e., every branch of $\overline{C}'$, the strict transform of C by π', and E_{b-1}, so we must consider $n+1$ points.

A geometric loop λ in $\overline{E}_b$ around such a point, acts on the d sheets in the same way as a small perturbation λ' of λ (λ' a loop in $T \setminus C' \cong \mathbb{C}^2 \setminus C$).

The geometric loops around the points of intersection of E_b with $\overline{C}'$ are the γ_i's, which act transitively on the fiber, so $\overline{E}_b$ is irreducible.

Since σ_i is a transposition, the branching index of $\tilde{\pi}|_{\overline{E}_b}$ in these points is 1.

On the other hand, a geometric loop around $p_{b-1} = E_b \cap E_{b-1}$, has monodromy $\rho(\Gamma^{b-1}) = \sigma^{-1}$, which is in the same conjugacy class as σ in $\mathcal{S}_d$, so the branching index in this point is exactly $d-\nu$.

Thus, by Hurwitz formula, $\chi(\overline{E}_b) = d\chi(E_b) - n - (d-\nu) = d + \nu - n$. $\square$

Lemma 5.5. *$\overline{E}_{b-1}$ has ν connected components, irreducible of genus 0, $\overline{E}^i_{b-1}$, of multiplicity $(b-1)nk_i$. $\tilde{\pi}^{-1}(p_{b-1})$ is a set of ν distinct points. Moreover, $(\overline{E}_b)^2 = -d$.*

Proof. A geometric loop around E_{b-1} is Γ^{b-1}.

Thus, $\tilde{\pi}|_{\overline{E}_{b-1}}$ is a branched cover of degree ν. Now, we have just two branching points; but a small perturbation of a geometric loop around p_{b-1} in $\overline{E}_{b-1}$ acts as the identity, and a small perturbation of a geometric loop around p_{b-2} in $\overline{E}_{b-1}$ acts as $\rho(\Gamma^{b-2}) = \sigma^{-2} = (\sigma^{-1})^2$, and they both do not connect the orbits of σ^{-1}. Thus $\tilde{\pi}|_{\overline{E}_{b-1}}$ is a cover (unramified) of degree ν with ν connected components, i.e., $\overline{E}_{b-1}$ is the disjoint union of ν curves, $\overline{E}^i_{b-1}$, biholomorphic to E_{b-1} (and thus of genus 0). Moreover $\tilde{\pi}$ is ramified of index $k_i - 1$ over $\overline{E}^i_{b-1}$, and this gives us the multiplicities.

Let U be a neighborhood of p_{b-1} with local coordinates (ξ, η) such that the curve $(E_b \cup E_{b-1}) \cap U$ has equation $\xi\eta = 0$. Then the fundamental group of the complement of this curve in U is $\mathbb{Z}^2$, generated by Γ^b and Γ^{b-1}.

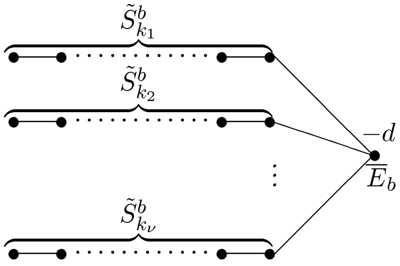
The connected analytic covers of $U \setminus (E_b \cup E_{b-1})$, are then classified by the subgroups of $\mathbb{Z}^2$. $\tilde{\pi}^{-1}(U)$ has ν connected components V_i, each associated to an orbit Λ_i of σ, and $\tilde{\pi}|_{V_i}$ is a cover of degree exactly the cardinality k_i of Λ_i associated to the lattice in $\mathbb{Z}^2$ generated by $(0, k_i)$ and $(1, 0)$.

Thus, locally the cover is $(\xi, \eta) \mapsto (\xi^{k_i}, \eta)$, and (see [7]), $\tilde{\pi}^{-1}(E_b \cap E_{b-1})$ is formed by ν points, $\overline{E}^i_{b-1} \cap \overline{E}_b$.

In order to compute $(\overline{E}_b)^2$ we need only to note that the intersection product $\overline{E}_b \cdot (f' \circ \tilde{\pi}) = 0$. Then

$$(\overline{E}_b)^2 = -\frac{dn + \sum (b-1)nk_i}{bn} = -\frac{dbn}{bn} = -d. \quad \square$$

Lemma 5.6. *The resolution graph of S is*:



where the $\widetilde{S}^b_{k_i}$ are strings which depend only on b and k_i.

Proof. $\forall\ i \leqslant b-1$, consider $\overline{E}_i$. As in the previous lemmas, the connected components of $\overline{E}_i$ are in canonical bijection with the orbits of the subgroup of $\mathcal{S}_n$ generated by $\sigma^i, \sigma^{i-1}, \sigma^{i+1}$. But these are the orbits of σ, then $\overline{E}_i$ has ν components, and each component of $\overline{E}_i$ intersects only the components of $\overline{E}_{i-1}$ and $\overline{E}_{i+1}$ associated to the same orbit (or an extremal vertex of a string dominating p_i or p_{i-1}, see [7]). In particular the $S^b_{k_i}$ are strings, and by construction they depend only on b and k_i. $\square$

Call now $\widetilde{T}_i$ a tubular neighborhood of $\widetilde{S}^b_{k_i}$ in $\widetilde{S}$, $S^b_{k_i}$ the string obtained by $\widetilde{S}^b_{k_i}$ recursively contracting all the possible exceptional curves of the first kind (smooth rational curves with self-intersection -1), $l^b_{k_i} = (\overline{E}_b)^2_{\widetilde{S}^b_{k_i}} - (\overline{E}_b)^2_{S^b_{k_i}}$ (i.e., how many times in this contraction we contract a vertex near $\overline{E}_b$), $s^b_{k_i}$ the number of vertices of $S^b_{k_i}$.

Lemma 5.7.
(1) $S^b_{k_i}$ *are strings of type* $A_{s^b_{k_i}}$;
(2) $l^b_{k_i} = k_i - 1$;
(3) $s^b_{k_i} = (b/k_i)(s^{k_i}_{k_i} + 1) - 1$;
(4) $s^b_1 = b - 1$.

Proof. First we prove (4). In this case $k_i = 1$ and $\overline{E}^i_j \to E_j$ is $1:1$ $\forall j$, so $\widetilde{S}^b_1$ is a string isomorphic to the string it dominates, i.e., a string of type A_{b-1} and $S^b_1 = \widetilde{S}^b_1$.

Observe that k_i divides b $\forall i$, since $\sigma^b = 1$, and that for k_i fixed, we must only prove the parts (1) and (2) for $b = k_i$; in fact we can compute the k_i-string for any b starting from the string for $b = k_i$, by the following argument.

First we consider $\overline{E}^i_{\lambda k_i}$.

A geometric loop around $E_{\lambda k_i}$ acts as the identity on $\widetilde{T}_i$, so $\overline{E}^i_{\lambda k_i}$ dominates $E_{\lambda k_i}$ as a k_i-sheeted cover totally ramified over two branching points, namely, the intersections of $E_{\lambda k_i}$ with $E_{\lambda k_i + 1}$ and $E_{\lambda k_i - 1}$.

So, $\overline{E}^i_{\lambda k_i}$ is rational by Hurwitz formula and the multiplicities of $\overline{E}^i_{\lambda k_i}$, $\overline{E}^i_{\lambda k_i+1}$, $\overline{E}^i_{\lambda k_i-1}$ are respectively $n\lambda k_i$, $n\lambda k_i(k_i+1)$, $n\lambda k_i(k_i-1)$.

In a neighborhood of $E_{\lambda k_i} \cap E_{\lambda k_i \pm 1}$, the cover restricted to $\widetilde{T}_i$ is associated to the lattice $(0, k_i)$, $(1, 0)$, so (see [7]) the fibers over these points are made up of a finite number of points, and the part of the string that dominates $E_{\lambda k_i}$ is only a vertex, corresponding to a rational curve. Its self-intersection (in $\widetilde{S}^b_{k_i}$) can be computed using multiplicities, and we obtain

$$-\frac{n\lambda k_i(k_i+1) + n\lambda k_i(k_i-1)}{n\lambda k_i} = -2k_i.$$

The other part of the string, i.e., the parts that dominate the substrings between $E_{\lambda k_i+1}$ and $E_{(\lambda+1)k_i-1}$, are exactly the $\widetilde{S}^{k_i}_{k_i}$.

In fact, the irreducible components of $\tilde{\pi}^{-1}(C')$ which dominate the $E_{\lambda k_i + j}$ and $E_{\lambda k_i + j} \cap E_{\lambda k_i + j + 1}$ (and the normal bundles in $\widetilde{S}$) in this substrings are constructed in the same way as the components that dominate E_j, $E_j \cap E_{j+1}$ for $b = k_i$, except for $E_{\lambda k_i + 1}$. In this case the construction is different from the one for E_1 (E_0 is not defined), but a generic loop around $E_{\lambda k_i}$, acts as the identity, and the result follows.

Thus we get b/k_i strings connected by $b/k_i - 1$ rational vertices with self-intersection $-2k_i$. All the substrings $\widetilde{S}^{k_i}_{k_i}$ contract to $S^{k_i}_{k_i}$.

Remark that, by construction, the strings $\widetilde{S}^b_{k_i}$ are symmetric (because σ e σ^{-1} are conjugate). Then after the contractions the self-intersection of $\overline{E}_{\lambda k_i}$ becomes $-2k_1+2l^{k_i}_{k_i} = -2k_i+2(k_i-1) = -2$, l_{k_i} does not depends on b and

$$s^b_{k_i} = \frac{b}{k_i} s^{k_i}_{k_i} + \frac{b}{k_i} - 1$$

(in particular, if $b > k_i$, $s^b_{k_i} > 0$).

Thus we prove (1) and (2) in the case $b = k_i$ by induction on b.

For $b = 1$ then $k_i = 1$ and the result follows by (4).

Now assume (1) and (2) true if $\bar{b} < b$. We know that $\forall b$ there exists a smooth cover of degree b generically branched on the curve $\{x^{b-1} + y^{b(b-1)} = 0\}$, i.e., on a curve of the class under consideration ($n = b - 1$).

In this case we can easily check that σ is an b-cycle, i.e., $\nu = 1$. In fact, otherwise by the inductive hypothesis we would have a tree with at least two non empty branches without exceptional curves, and then the graph cannot be contracted (i.e., the surface cannot be smooth).

Now, using the previous considerations, we find that the resolution graph of this surface has the form:

where all the vertices without decoration (if they exist) are non contractible.

We know a priori that this graph must be contractible. In particular the last vertex must have self-intersection -1 (no other vertex can), i.e., $l_{k_i} = k_i - 1$, and it must be rational. Moreover the string must have only smooth rational curves with self-intersection -2. □

Proof of Theorem 5.2. By Lemmas 5.7 and 5.6, we must only check that, if $b = k_i$ then $s^{k_i}_{k_i} = 0$, i.e., $S^{k_i}_{k_i}$ is empty.

Recall that by Theorem 4.1 we know $\pi_1(S \setminus \{p\})$ for some special monodromy graphs. In that theorem, the branching curves were $\{x^{\alpha+\beta} = y^{l(\alpha\beta)}\}$, for fixed α, β, with $(\alpha, \beta) = 1$; in that case we have $d = \alpha + \beta$, $\nu = 2$, $k_1 = \alpha$, $k_2 = \beta$, $\pi_1(S \setminus \{p\}) = \mathbb{Z}/l\mathbb{Z}$.

Assume now $\alpha = n - 1$, $\beta = 1$, $l = n$. We get one of our curves, with $b = n - 1$.

By the previous lemmas, the configuration is a string of $k = s^{n-1}_{n-1} + s^{n-1}_1 + 1$ rational curves with self-intersection -2. So S has a singularity of type A_k (see [1]).

The local fundamental group of such a singularity is a cyclic group of order equal to the number of the vertices plus 1. We know that this order is $l = n$, then we have $n - 1$ vertices and by Lemma 5.7, $s^{n-1}_1 = n - 2$, so $s^{n-1}_{n-1} = l - s^{n-1}_1 - 2 = 0$.

Moreover, if σ has $\nu > 1$ orbits then the graph has no -1 curves and S cannot be smooth.

So, if S is smooth, σ is a d-cycle and $\sigma^b = 1 \Rightarrow d|b$. Moreover, if S is smooth, the genus of $\overline{E}_b$ must be 0, i.e., $n - d + \nu = 0$, $n = d - \nu = d - 1$.

Conversely, if $n = d - 1$ the genus of $\overline{E}_b$ is 0 and the monodromy graph has d vertices and $n = d - 1$ edges, i.e., it is a tree. Thus σ is transitive, $\nu = 1$, and the resolution graph is a string of rational curves with self-intersection -2 except for the last one which has self-intersection -1. So S is smooth. □

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