HISTORIA MATHEMATICA **24** (1997), 361–375
ARTICLE NO. HM962146

The Origin of the Method of Steepest Descent

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The method of steepest descent, also known as the saddle-point method, is a natural development of Laplace's method applied to the asymptotic estimate of integrals of analytic functions. Mathematicians have often attributed the method of steepest descent to the physicist Peter Debye, who in 1909 worked it out in an asymptotic study of Bessel functions. Debye himself remarked that he had borrowed the idea of the method from an 1863 paper of Bernhard Riemann. The present article offers a detailed historical analysis of the creation of the method of steepest descent. We show that the method dates back to Cauchy and that, 25 years before Debye, the Russian mathematician Pavel Alexeevich Nekrasov had already used this technique and extended it to more general cases. © 1997 Academic Press

La méthode de la descente la plus rapide, que l'on appelle actuellement méthode du col, représente un développement naturel de la méthode de Laplace pour l'estimation asymptotique des intégrales des fonctions analytiques. La méthode du col est généralement liée au nom du physicien Peter Debye, qui a utilisé en 1909 les idées que Riemann a présentées dans son article de 1863 afin de donner une analyse asymptotique des fonctions de Bessel. Le présent travail donne une étude bien detaillée de la création de la méthode du col. Tout d'abord nous montrons que la méthode du col remonte aux travaux de Cauchy et ensuite que le mathématicien russe Pavel Alexeevich Nekrasov a appliqué cette méthode un quart de siècle avant Debye et l'a étendue aux cas généraux. © 1997 Academic Press

Метод наибыстрейшего спуска,который часто называют методом перевала,является естественным развитием метода Лапласа для асимптотической оценки интегралов от аналитических функций. Математики часто связывают метод перевала с именем физика П.Дебая,который разработал его в 1909 году для асимптотического анализа функций Бесселя. Сам Дебай отметил,что идея метода была им заимствована из работы Б.Римана 1863 года.

Целью настоящей статьи является детальный анализ истории создания метода перевала.Во-первых,будет показано,что идея метода восходит к О.Коши и,во-вторых,что за 25 лет до Дебая русский математик П.А.Некрасов использовал этот метод и распространил его на более общий случай. © 1997 Academic Press

MSC 1991 subject classifications: 01A55, 01A60, 41A60.
KEY WORDS: method of steepest descent, Laplace's method, estimate of integrals of analytic functions, saddle-point, asymptotic expansions, Lagrange series, Bessel functions.

0315-0860/97 $25.00

1. INTRODUCTION

The method of steepest descent, also known as the saddle-point method, is a natural development of Laplace's method applied to the estimation of integrals of analytic functions. Laplace estimated an integral of the form

$$\mathcal{I}_n = \int_a^b f^n(x)g(x)\,dx = \int_a^b e^{nu(x)}g(x)\,dx, \qquad f(x) > 0, \tag{1}$$

as $n \to \infty$ in the following way (cf., for example, [1]). Assume that the function $u(x) = \ln f(x)$ attains its maximum value at the unique point $c \in (a, b)$ and that the conditions $u'(c) = 0$, $u''(c) < 0$ hold at this point, while $g(x)$ is a continuous function with $g(c) \neq 0$. Then as $n \to \infty$, we obtain the estimate

$$\begin{aligned}\mathcal{I}_n &= \int_{c-\varepsilon_n}^{c+\varepsilon_n} e^{nu(x)}g(x)\,dx + \int_a^{c-\varepsilon_n} e^{nu(x)}g(x)\,dx + \int_{c+\varepsilon_n}^b e^{nu(x)}g(x)\,dx \\ &\sim e^{nu(c)} \cdot g(c) \int_{c-\varepsilon_n}^{c+\varepsilon_n} e^{(nu''(c)/2)(x-c)^2}\,dx \\ &\sim e^{nu(c)} \cdot g(c) \int_{-\infty}^{\infty} e^{n(u''(c)/2)(x-c)^2}\,dx = \sqrt{\frac{2\pi}{-nu''(c)}}\, g(c)e^{nu(c)}.\end{aligned} \tag{2}$$

Here, ε_n tends to 0 and is chosen so that, on the one hand, the integral over the exterior of the neighborhood $(c - \varepsilon_n, c + \varepsilon_n)$ is infinitely small in comparison with the integral over the neighborhood itself and, on the other hand,

$$u(x) - u(c) = \frac{u''(c)}{2}(x-c)^2 + R_2(x) \sim \frac{u''(c)}{2}(x-c)^2,$$

for all $x \in [c - \varepsilon_n, c + \varepsilon_n]$. The restrictions imposed above on the functions $u(x)$ and $g(x)$ are not particularly important in the application of Laplace's method—if the maximum value of the function is attained at several points of the interval (a, b), the estimate of the integral $\mathcal{I}_n$ can be provided by a suitable sum of estimates of the form (2), and if the maximum value is attained at an endpoint of the interval $[a, b]$ and $u' = 0$ at that endpoint, we obtain half of (2) as the estimate of the integral $\mathcal{I}_n$.[1] Finally, if the first nonzero derivative at the point c is $u^{(2k)}(c) < 0$, the same method can be used to reduce the estimate to the integral

$$\int_{-\infty}^{\infty} e^{n(u^{(2k)}(c)/(2k)!)(x-c)^{2k}}\,dx,$$

which can easily be expressed in terms of the gamma function.

However, when the function $u(x)$ in the integral (1) is complex valued, Laplace's method no longer works (as we shall show below in our analysis of the work of Cauchy). Actually, the problem of estimating integrals of the form (1) with complex integrands taken over some contour in the complex plane arose in the work of

[1] When $u' \neq 0$ at the endpoint, the estimate is even simpler.

Laplace and Cauchy. Examples of such problems are provided by the estimation of the Taylor coefficients

$$\frac{1}{2\pi i}\int_C \frac{\varphi(z)}{(z-a)^{n+1}}\,dz,$$

or the coefficients of the Lagrange series for the inverse function

$$\frac{1}{2\pi i}\int_C \frac{\varphi^n(z)}{nz^n}\,dz$$

as $n \to \infty$. A similar problem arises when we estimate the inverse Laplace transform

$$\frac{1}{2\pi i}\int_{h-i\infty}^{h+i\infty} e^{xz}\varphi(z)\,dz$$

as $x \to \infty$.

The method of steepest descent was developed over several decades in order to estimate integrals of the form

$$\mathscr{I}_n = \int_C e^{nu(z)}g(z)\,dz \tag{3}$$

as $n \to \infty$. The essence of this method is as follows: we deform the contour of integration C in the domain of analyticity of the functions $u(z)$ and $g(z)$ without changing its endpoints if it is nonclosed in such a way that the following conditions hold:

(i) the contour C passes through a point z_0, called a *saddle point*, at which $u'(z_0) = 0$;

(ii) the condition $\operatorname{Im} u(z) \equiv \operatorname{Im} u(z_0)$ holds on at least some neighborhood of the point z_0 on the contour C;

(iii) for points $z \neq z_0$ on the contour C the condition $\operatorname{Re} u(z) < \operatorname{Re} u(z_0)$ holds.

For ease of exposition we add the further condition

(iv) $g(z_0) \neq 0$, $u''(z_0) \neq 0$, and z_0 is not an endpoint of the contour C.

Let C_ε be the portion of the contour in a neighborhood of the point z_0 on which condition (ii) holds. Then

$$\mathscr{I}_n = \int_{C_\varepsilon} e^{nu(z)}g(z)\,dz + \int_{C\setminus C_\varepsilon} e^{nu(z)}g(z)\,dz = \mathscr{I}_n' + \mathscr{I}_n''.$$

In the first integral, we introduce the natural arc-length parameter $z = z(t)$, $t \in [-\varepsilon, \varepsilon]$, $z(0) = z_0$; we then have

$$\mathscr{I}_n' = e^{nu(z_0)}g(z_0)\int_{-\varepsilon}^{\varepsilon} \frac{e^{n[u(z(t))-u(z_0)]}g(z(t))}{g(z_0)}\,z'(t)\,dt.$$

By condition (ii), we have Im $\{u[z(t)] - u(z_0)\} \equiv 0$, and we can estimate the first integral by Laplace's method, which yields

$$\mathcal{I}'_n \sim \sqrt{\frac{2\pi}{n|u''(z_0)|}}\, g(z_0) e^{nu(z_0)+i\alpha},$$

where $e^{i\alpha} = z'(0)$. As for the second integral, the following expression gives a rough estimate of it:

$$|\mathcal{I}''_n| \leq A e^{n[\operatorname{Re} u(z_0)-\delta]}, \qquad \delta > 0,$$

that is,

$$\mathcal{I}''_n = o(\mathcal{I}'_n) \qquad \text{and} \qquad \mathcal{I}_n \sim \mathcal{I}'_n.$$

We remark further that the contour C (defined by the equation $z = z(t)$), since it passes through the point z_0 at which the condition Im $u(z_0) \equiv$ Im $u(z)$ holds, has the following property: if in three-dimensional space we construct the graph of the absolute value $|f(z)| = e^{\operatorname{Re} u(z)}$, and on that surface we consider the curve $|f(z(t))|$, which projects to the contour C, this curve will have the property that, at each point, it is directed along the line of fastest decrease of $|f(z)|$. In other words, a fluid flowing over the surface of the absolute value from this point will follow precisely the curve $|f(z(t))|$ in both directions. This fact accounts for the name "method of steepest descent." We note further that, by the maximum-modulus principle, the surface $|f(z)|$ has a saddle point at z_0, which accounts for the terminology "saddle-point method."

To conclude this section, we consider some modifications of the method of steepest descent. If the first nonzero derivative at the saddle point z_0 is of order m, that is, $u''(z_0) = \cdots = u^{(m-1)}(z_0) = 0$ and $u^{(m)}(z_0) \neq 0$, there will be m lines of steepest descent instead of two (one ascending to z_0 and one descending to it), and the contour C must be drawn along two of these lines. In this case, as in the method of Laplace, the estimate reduces in the final analysis to a gamma function. If the largest value of Re $u(z)$ is attained at an endpoint of the curve, the contour C must again be drawn along a line of steepest descent, but the estimate will be simpler, since it reduces to computing the integral

$$\int_0^\infty e^{-n|u'(z_0)|t}\, dt.$$

For the following discussion, it is essential to remark that it is not necessary to draw the contour precisely along a line of steepest descent (which may be technically difficult) in order to obtain an asymptotic estimate of the integral $\mathcal{I}_n$. When condition (iv) holds for the contour, we obtain the same estimate if we draw the contour C in a neighborhood of a saddle point z_0 along a segment of the tangent to the curve of steepest descent at the point z_0. Its parametric equation is $z - z_0 = te^{i\alpha}$, and the condition $(u''(z_0)/2)\, e^{2i\alpha} < 0$ holds. Moreover, the contour C can in general be drawn along any segment of the line $z - z_0 = te^{i\alpha}$, on which the condition $\mathrm{Re}\,[u''(z_0)e^{2i\alpha}] < 0$ holds. In this last case, estimating $\mathcal{I}_n$ reduces to computing the integral $\mathcal{I} = \int_{-\infty}^{\infty} e^{-at^2}\,dt$, where a is not positive, as in Laplace's method, but the condition $\mathrm{Re}\; a > 0$ holds. This integral equals $\mathcal{I} = \sqrt{\pi/a}$, and its value had been computed as early as the beginning of the 19th century.

In the literature [1–4] the method of steepest descent we have just discussed is usually ascribed to Peter Debye, who applied it in 1909 (cf. [5]) to obtain an asymptotic estimate of Bessel functions whose order tends to infinity simultaneously with the argument. Debye himself notes at the beginning of the paper that he had borrowed the idea of the method from Riemann [6]. In an unpublished paper of 1863, Riemann applied the method of steepest descent to estimate the hypergeometric function. Twelve years later, Hermann Amandus Schwarz published this paper along with some supplementary annotations and computations.

In this study, we wish to supplement considerably the history of the origin of the method of steepest descent by showing that certain elements of the method had appeared earlier in the work of Cauchy and that 25 years before the work of Debye the Russian mathematician P. A. Nekrasov had given a very general and detailed exposition of the method. The latter fact has been observed by Eugene Seneta [7].

2. APPROACHES TO THE METHOD OF STEEPEST DESCENT IN THE WORK OF CAUCHY

In his paper "Sur divers points d'analyse" [8], published in 1827, Cauchy estimated integrals of the form (1). Here, he essentially repeated Laplace's method, but in the correct form. Breaking the integral (1) in two, one over a neighborhood of a maximum point and the other over the exterior of this neighborhood, Cauchy took a neighborhood of the form $(c - a/\sqrt{n}, c + a/\sqrt{n})$ and stated explicitly that a tends to infinity, though more slowly than $\sqrt{n}$. To be sure, Cauchy did not show that the remainder term in Taylor's series in the integral over the neighborhood does not affect the estimate, he simply discarded it. As a result he obtained the estimate (2).

For us, the second part of this paper is of more interest. In that part Cauchy finds an estimate for the integral (1) for the case when the function $f(x)$, and consequently $u(x)$, is complex-valued. In so doing, Cauchy assumes that $|f(x)| = e^{\mathrm{Re}\,u(x)}$ attains its largest value at a unique interior point c of the interval (a, b), while at this point

$$u'(c) = 0, \qquad \mathrm{Re}\,u''(c) < 0. \tag{4}$$

Under these conditions, the estimate is carried out just as in the real case, but the last part of the estimate leads us to compute the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} e^{n(u''(c)/2)t^2}\,dt,$$

where $u''(c)$ is a complex quantity. As noted above, this integral equals $\mathcal{I} = \sqrt{2\pi/-nu''(c)}$. Here, the value of the square root in the right half-plane is taken.

In conditions (4), which Cauchy imposes on the function $u(x)$, the reason for the condition $u'(c) = 0$ is at first incomprehensible. Indeed, if we want to find the largest value of Re $u(x)$, we only need Re $u'(c) = 0$ at the maximum point c. However, it is not difficult to show that when this last condition holds, the integral over the neighborhood of the point c no longer gives the principal part of the integral (1) due to strong interference.

The meaning of conditions (4) is revealed in the second section of Cauchy's paper. There, he applies this method to find the radius of convergence of the Lagrange series. As is known, if the function $\varphi(z)$ is analytic at the point a, $\varphi(a) \neq 0$, and $F(z)$ is analytic at the point $z = 0$, then the equation $z = w \cdot \varphi(a + z)$ has a unique solution $z = z(w)$ in a neighborhood of the point $w = 0$, and the following expansion is valid:

$$F[z(w)] = F(0) + \sum_{n=1}^{\infty} a_n w^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{F'(z)\varphi^n(a + z)}{nz^n}\, dz. \tag{5}$$

To find the radius of convergence of this series, it is necessary to find an asymptotic estimate for its coefficients a_n as $n \to \infty$.

For that reason, Cauchy considered and estimated the integral

$$S_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(a + re^{is}) \left[\frac{\varphi(a + re^{is})}{re^{is}} \right]^n ds, \tag{6}$$

which differs only trivially from the integral a_n. (Actually Cauchy considered a slightly more general integral, but this does not affect the present discussion.) Cauchy found the largest absolute value of the function $f(re^{is}) = \varphi(a + re^{is})/re^{is}$ with respect to s. If, as above, $u(re^{is}) = \ln f(re^{is}) = \ln \varphi(a + re^{is}) - \ln r - is$, then at the point $s = s(r)$, where Re $u(re^{is})$ attains a maximal value, the following condition must hold:

$$\operatorname{Re} \frac{d}{ds} u(re^{is}) \Big|_{s=s(r)} = 0.$$

Cauchy then asserted that if the radius r is changed, there exists a point $r = r_0$ at which

$$\operatorname{Im} \frac{d}{ds} u(re^{is}) \Big|_{r=r_0}^{s=s_0} = 0, \qquad s_0 = s\,(r_0).$$

This means that $u'(z_0) = 0$, where $z_0 = r_0 e^{is_0}$. In other words, Cauchy did the following: first, he found the largest value of Re $u(re^{is})$ on a circle of radius r; then he chose $r = r_0$ for which the quantity Re $u(re^{is(r)})$ attains a minimum. We remark that when this is done, the condition

$$\frac{d^2}{ds^2} \operatorname{Re} u(re^{is}) \Big|_{s=s_0}^{r=r_0} < 0 \tag{7}$$

holds at the point $z_0 = r_0 e^{is_0}$ since a maximum over the angle s is attained at this point. Therefore, with such a choice of the radius r, the integral (5) can be estimated in accordance with the method of Cauchy discussed above.

If we compare Cauchy's reasoning with the saddle-point method considered above, we can say that Cauchy found a saddle point z_0, where $u'(z_0) = 0$, and drew a contour (a circle) through this point which, by condition (7), traverses the required sector (recall Section 1 above), yielding the required estimate. However, in so doing, Cauchy, without realizing it, made a crude error. Let us consider this question in more detail. Since the classical mathematicians rarely made fundamental mistakes, it is both important and instructive from an historical point of view to analyze such errors. The fact that Cauchy's method cannot in general yield an asymptotic estimate of the integral (5) can be seen from the following simple considerations: the function $f(z)$ can always be taken so as to have two saddle points z_1 and z_2 lying at different distances from the origin and to have the same absolute value $|f(z_1)| = |f(z_2)|$ at these points. Then, to estimate the integral (5) by the saddle-point method, one must draw a contour through both of these points, yet no circle with center at the origin can pass through both points.

Cauchy's reasoning lapsed relative to the case where the radius r of the circle varies; there, the imaginary part Im $u'(re^{is(r)})$ does not necessarily vanish at any r. At first glance, it seems that this imaginary part must vanish when the circle passes through a saddle point. This is not the case, however, since at the time when the circle passes through the saddle point $z_1 = r_1 e^{is_1}$, the maximal value of Re $u(r_1 e^{is})$ may be attained not at $s = s_1$, but at some other value $s = s_1(r_1)$. A specific example shows this most clearly. Consider the integral

$$\int_{|z|=r} \left(\frac{e^{\alpha z}}{z^2 + 2z} \right)^n dz,$$

where $\frac{1}{2} < \alpha < 1$. It can be shown that, for any $r > 0$, the largest value of $|f(z)| = e^{\mathrm{Re}[\alpha z - \ln(z^2+2z)]}$ on the circle $|z| = r$ is attained either at $z = r$ or at $z = -r$. Let $u(z) = \ln f(z) = \alpha z - \ln(z^2 + 2z)$. The equation $u'(z) = 0$ has exactly two roots

$$z_1 = \frac{1 - \alpha + \sqrt{1 + \alpha^2}}{\alpha} \qquad \text{and} \qquad z_2 = \frac{1 - \alpha - \sqrt{1 + \alpha^2}}{\alpha},$$

and $z_1 > 0$ and $-2 < z_2 < 0$. If $\alpha = \alpha_0$ is taken as a root of the equation

$$e^{\sqrt{1+\alpha^2}} = \frac{1 + \sqrt{1 + \alpha^2}}{\alpha},$$

it is easy to show that this root satisfies the inequality $\frac{1}{2} < \alpha_0 < 1$ and $|f(z_1)| = |f(z_2)|$. Figure 1 shows the graph of the function $|f(x)|$ for $\alpha = \alpha_0$.

Let us see what happens to the angle $s = s(r)$ as r varies from 0 to ∞. For small r, we have $s(r) = 0$. When r reaches the value r_1, the angle $s(r)$ makes a jump and becomes $s(r) = \pi$. Therefore, the derivative $f'(re^{is(r)})$ never vanishes, and Cauchy's method does not apply in this case.

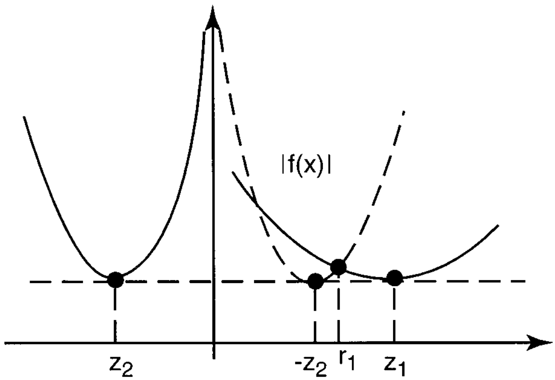

FIGURE 1

It follows from what has been said above that Cauchy had only approaches to the saddle-point method; the problem of choosing the contour did not arise with him, and he always took the contour to be a circle, thus depriving his method of the necessary generality.[2]

3. THE SADDLE-POINT METHOD IN THE WORK OF RIEMANN

In the second part of his paper [6], written in Italian in 1863, Riemann discussed the saddle-point method. As we have already mentioned, this paper was published by Schwarz with necessary annotations to the fragments left behind by Riemann at his death.

In this work, Riemann found an asymptotic estimate of the integral

$$\mathscr{I}_n = \int_0^1 s^{a+n}(1-s)^{b+n}(1-xs)^{c-n}\, ds \tag{8}$$

as $n \to \infty$. When $x < 1$ is real, the estimate of this integral is easily found by Laplace's method. However, Riemann estimated it for an arbitrary complex value of x, as he particularly emphasized. To obtain the desired estimate, Riemann wrote the integral (8) in the form

$$\mathscr{I}_n = \int_0^1 \left[\frac{s(1-s)}{1-xs}\right]^n s^a(1-s)^b(1-xs)^c\, ds$$

and studied the behavior of the level lines of the absolute value of the function $r = |s(1-s)/(1-xs)|$. If r is small, the level lines approximate small circles with centers at the points $s = 0$ and $s = 1$. As r increases, these contours expand and,

[2] The possibility that Cauchy actually "had" the "saddle-point method" was pointed out by M. V. Chirikov [9]. Our analysis qualifies this significantly.

at a certain time, meet each other for the first time at some point σ. Then we obviously have

$$\frac{d}{ds}\left[\frac{s(1-s)}{1-xs}\right]\bigg|_{s=\sigma} = 0.$$

Solving this equation, Riemann found two roots

$$\sigma_1 = \frac{1}{1+\sqrt{1-x}} \qquad \text{and} \qquad \sigma_2 = \frac{1}{1-\sqrt{1-x}},$$

at which he took the branch of the square root for which Re $\sqrt{1-x} > 0$. Therefore, $|\sigma_1| < |\sigma_2|$, and since

$$\left|\frac{\sigma_i(1-\sigma_i)}{1-x\sigma_i}\right| = |\sigma_i^2|, \qquad \text{for} \qquad i = 1, 2,$$

the level lines are tangent precisely at the point σ_1. In the integral $\mathscr{I}_n$, Riemann deformed the contour joining the points $s = 0$ and $s = 1$ so that it would pass through the point σ_1 and bisect the angle between the branches of the level lines (which intersect in a right angle at the point σ_1). In other words, he drew the contour through the saddle point σ_1 in the direction of steepest descent. He then estimated the integral $\mathscr{I}_n$ approximately as we indicated above; as a result, he obtained the desired estimate

$$\mathscr{I}_n \sim \sqrt{\frac{\pi}{n}}\frac{1}{(1+\sqrt{1-x})^{2n+a+b+1}}(1-x)^{(b+c)/2+1/4}.$$

We note that the entire text and the majority of the computations in the second half of Riemann's paper were added by Schwarz. For that reason, we cannot be completely sure that Riemann reasoned exactly as we have described above. One thing, however, is incontrovertible. Riemann understood that in estimating integrals of the form (3) one must draw the contour through a saddle point along the line of steepest descent. Confirmation that Riemann had the technique of the saddle-point method is also provided by an estimate of the function $Z(t)$ using this method found by Carl Ludwig Siegel among Riemann's papers in 1932. The function $Z(t)$ is connected with the zeta-function (the Riemann–Siegel formula, cf. [10]).

4. P. A. NEKRASOV AND THE SADDLE-POINT METHOD

Pavel Alexeevich Nekrasov was born on February 1, 1853, the son of a priest. He studied in a seminary and then in the Department of Mathematics and Physics at Moscow University, where one of his teachers, Nicolai V. Bugaev, developed in him a taste for philosophy. In 1878, he was retained at the University for 2 years to prepare to become a professor. However, in 1879, by a directive of the superintendant of the Moscow school district, he was appointed to the private Voskresenskiĭ Vocational Institute in Moscow as a teacher of mathematics.

The topic of his master's thesis (1883) was "A Study of Equations of the Form $u^m - pu^n - q = 0$," and the title of his doctoral dissertation (1886) was "The Lagrange Series." For 20 years beginning in 1885, he taught at the University of Moscow, becoming a full professor in 1886. There, he lectured on integral calculus, the theory of functions of a complex variable, and probability. From 1893 to 1897, he was rector of the University of Moscow. In 1905, he moved to St. Petersburg to work in the Ministry of Public Education. He undertook the reform of elementary education, considering it necessary to introduce the elements of probability theory into the school mathematics curriculum.

Nekrasov was an active member of the Moscow Mathematical Society, acting as its vice-president after 1891, and, from the death of Bugaev in 1903 until his own move to St. Petersburg, as its president. In addition to numerous papers, he presented several interesting reports on the scientific activity of S. V. Kovalevskaya, A. Yu. Davidov, N. V. Bugaev, V. G. Imshenetskiĭ, and others at meetings of the Society. His publications number about 50 titles, reflecting his versatile scientific and social interests. He died December 20, 1924.

The main areas of Nekrasov's research were analysis and probability, and he called himself an analyst. His doctoral dissertation, "The Lagrange Series and Approximate Expressions for Functions of Very Large Numbers," which was published as a separate monograph, was devoted to a detailed study of the convergence of the Lagrange series [11].

In order to determine the radius of convergence of this series and to study its convergence on the boundary of the disk of convergence, it is necessary to find an asymptotic estimate of the coefficients a_n of the series, expressed by the integral (5). It is this kind of problem that Nekrasov stated and solved. In the first part of his dissertation, he gives a very detailed survey of papers devoted to the problem of convergence of the Lagrange series. He devotes an especially large amount of attention to the paper of Cauchy discussed above, which, in Nekrasov's words, subsequent authors had forgotten. Nekrasov probably did not know of Riemann's work. It is not associated with the Lagrange series and, naturally, it is not found among the papers referred to in the dissertation. However, Nekrasov did not mention Riemann's paper even later, in particular in 1900, when he published a monograph devoted to methods of finding asymptotic estimates and gave a rather complete list of papers of authors who had studied this topic [12].

In the third chapter of his dissertation, which was published separately in Vol. 12 (1885) of the *Matematicheskiĭ Sbornik*, Nekrasov estimated the integral (5). If we compare this integral with the general integral (3), to the estimate of which the saddle-point method is applied, we can see that the small restriction amounts to the fact that in (5) the contour is closed, while the function $f(z) = \varphi(a + z)/z$ has a simple pole at 0. Obviously, this is unimportant for the development of the saddle-point method. Nekrasov assumed in addition that the function φ is meromorphic, i.e., its only singularities are poles, but at the end of the paper he considered also the cases of essential singularities and branch points.

Nekrasov began his analysis by choosing the contour in the integral (5). He showed that there always exists a contour enclosing the origin but no singularities of the function $\varphi(a + z)$ and passing along the directions of steepest descent through the principal saddle points. These are the points at which $|f(z)| = |\varphi(a + z)/z|$ has the same maximal value on this contour. To prove this, he used a very original geometric argument, which we summarize as follows:

Consider a horizontal complex plane over which the graph of the function $|f(z)|$ is constructed. This surface will have no vertices, but at points where $f'(z) = 0$, it will have saddle points. At the poles, it will have infinitely high peaks. If a liquid is poured onto this surface to a very high level, only small round regions corresponding to poles will lie above this level. In particular one region, which we paint red, will correspond to the pole at the origin. The other regions, at which the surface of the absolute value is above this level, we paint black. Let us now lower the level of the liquid. As we do this, the regions will expand monotonically, and at a certain moment, one or several black regions will become tangent to the red region. With this kind of reasoning Nekrasov showed that there can only be a finite number of points of tangency.

He then chose the contour C so that it enclosed only the red region and passed through all the points of tangency, which are obviously saddle points. By construction, the value of $|f(z)|$ is the same at all these points, and on the remainder of the contour C, this value will be smaller. For the sake of generality, Nekrasov assumed that at each such saddle point $z = z_0$ the expansion of the function has the form

$$f(z) = f(z_0) + \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + o[(z - z_0)^m], \qquad m \geq 2.$$

He showed that the contour C can always be drawn through all the principal saddle points in such a way that, in a neighborhood of each point, the contour passes along a line of steepest descent. Having chosen the contour in this way, Nekrasov proceeded to estimate the integral (5). He did this by breaking the integral into a sum of integrals taken over parts of the contour in a neighborhood of each saddle point, and the integral over the part of the contour outside these neighborhoods. By a rough estimate, he showed that the last integral is infinitely small in comparison with the integrals over the neighborhoods of the saddle points. To estimate the integral over each neighborhood, Nekrasov, unlike Cauchy, Riemann, and many later authors, did not reduce this estimate to the computation of an integral of exponentials. Rather, he applied his own original technique, breaking the integral over a neighborhood into two integrals, one over the part ascending to the saddle point and one over the part descending from it. Each such integral is estimated by the quantity

$$e^{i\alpha}F'(a + z_0)\frac{f^n(a_0)}{2\pi in}\int_0^\varepsilon (1 - At^m)^n \, dt, \tag{9}$$

where z_0 is the saddle point, α is the angle defining the direction of steepest descent, and $A = |f^{(m)}(z_0)/m!f(z_0)|$. The integral (9) can be estimated by the gamma function as $n \to \infty$,

$$\int_0^\varepsilon (1 - At^m)^n \, dt \sim \frac{n!\Gamma\left(1 + \dfrac{1}{m}\right)}{A^{1/m}\Gamma\left(n + 1 + \dfrac{1}{m}\right)},$$

and this last fraction is easily estimated using Stirling's formula. This, in brief, is a description of the saddle-point method in the work of Nekrasov.

5. THE WORK OF DEBYE

In 1909, 25 years after Nekrasov, the famous physicist Peter Debye published his paper [5], in which he applied the saddle-point method to find an asymptotic estimate of the Bessel functions, which are solutions of the equation

$$u'' + \frac{1}{x}u' + \left(1 - \frac{a^2}{x^2}\right) u = 0$$

for the case when the argument x tends to infinity, while the ratio $a/x = \xi$ remains constant. (We remark that an estimate of the Bessel functions as $x \to \infty$ with fixed a can be obtained without the saddle-point method.)

At the beginning of his paper, Debye notes that the idea of the saddle-point method came to him from reading the note of Riemann discussed above. A large part of Debye's paper is devoted to estimating the Hankel functions, which are represented by contour integrals of the form

$$H_1^a(x) = -\frac{1}{\pi}\int_{C_1} e^{-ix\sin z} \cdot e^{iaz} \, dz, \qquad H_2^a(x) = -\frac{1}{\pi}\int_{C_2} e^{-ix\sin z} \cdot e^{iaz} \, dz.$$

The contours C_1 and C_2 have the following form (cf. Fig. 2).

Debye took the function $H_2^a(x)$ (everything is analogous for $H_1^a(x)$) and wrote it in the form

$$H_2^a(x) = -\frac{1}{\pi}\int_{C_2} e^{-xu(z)} \, dz,$$

where $u(z) = i(\sin z - \xi z)$ and $\xi = a/x$. He found a saddle point z_0 at which $u'(z_0) = 0$ and drew a contour through this point so that the imaginary part Im $u(z)$ is constant on the contour and the real part Re $u(z)$ has a maximum at the point z_0. In other words, he also drew the contour C_2 through the saddle point along the line of steepest descent. He then divided the contour C_2 into two parts, the part up to the saddle point and the part after the saddle point. In each integral,

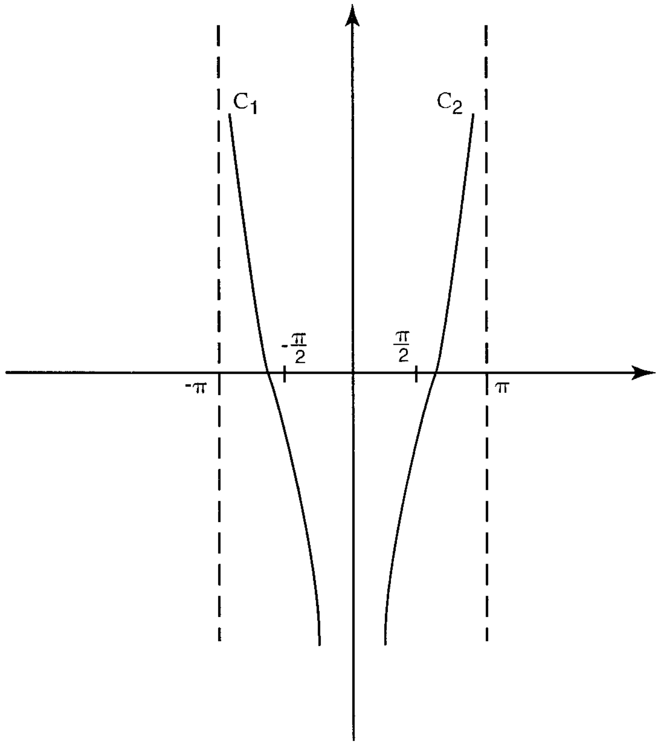

FIGURE 2

he made the change of variable $R(z) - R(z_0) = t$. Thus he brought the Hankel function into the form

$$H_2^a(x) = \frac{e^{-xu(z_0)}}{\pi}\left[\int_0^\infty e^{-xt}\varphi(t)\,dt - \int_0^\infty e^{-xt}\varphi_1(t)\,dt\right].$$

The functions $\varphi(t)$ and $\varphi_1(t)$ were not found explicitly, since the equation $R(z) - R(z_0) = t$ cannot be solved explicitly for z; however, Debye had no need to do so. By methods known since the time of Newton, he expanded the functions φ and φ_1 at zero into series in fractional powers of the variable t. Integrating these series termwise, he obtained an expansion of the function $H_2^a(x)$ into an asymptotic series (in particular, the first term of this series gives the desired asymptotic relation).

Debye thus used, without proof, the following proposition that later came to be known as Watson's lemma [4, ch. 17]:

If $\mathscr{I}(x) = \int_0^\infty e^{-xt}\varphi(t)\,dt$ converges absolutely for some x, and the function $\varphi(t)$ can be expanded in an asymptotic series

$$\varphi(t) \sim \sum_{n=0}^{\infty} a_n t^{np+q}, \qquad p > 0, q > -1,$$

on a right-hand neighborhood of $t = 0$, then as $x \to \infty$ the integral $\mathcal{I}(x)$ can be expanded into an asymptotic series obtained by termwise integration:

$$\mathcal{I}(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(np + q + 1)}{x^{np+q+1}}.$$

Next, proceeding in the same manner, Debye obtained asymptotic expansions for the other Bessel functions.

The saddle-point method in the works of Nekrasov and Debye may now be compared in this way:

1. Nekrasov developed the saddle-point method for the general integral (3), while Debye estimated a specific and rather simple integral by the saddle-point method.
2. Nekrasov considered the most general case when there are several saddle points and each has an arbitrary multiplicity. Debye considered only the case of a single saddle point at which $u''(z_0) \neq 0$. For this reason, we can say with complete justification that Nekrasov established the general saddle-point method. (However, Debye obtained not only an asymptotic estimate but also the complete asymptotic expansion, something Nekrasov did not have.)
3. Nekrasov proved the existence of a closed contour passing through the saddle points along the directions of steepest descent in the general case, whereas Debye did this for a specific and simple integral, which is, of course, conceptually much easier.

ACKNOWLEDGMENT

The authors are very grateful to Professor Roger Cooke for his help in editing the present article and for his very useful remarks.

REFERENCES

1. Edward T. Copson, *Asymptotic Expansions*, Cambridge, UK: Cambridge Univ. Press, 1965.
2. Marat Andreevich Evgrafov, *Asymptotic Estimates and Entire Functions*, trans. Allen L. Shields, New York: Gordon & Breach, 1961.
3. Nicolaas Govert De Bruijn, *Asymptotic Methods in Analysis*, Amsterdam: North-Holland, 1958.
4. Harold Jeffreys and Bertha Swirles, *Methods of Mathematical Physics*, 3rd ed., Cambridge, UK: Cambridge Univ. Press, 1956.
5. Peter Debye, Näherungsformeln für die Zylinderfunktionen für große Werte des Arguments und unbeschränkt veränderliche Werte des Index, *Mathematische Annalen* **67** (1909), 535–558.
6. Bernhard Riemann, Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, in *Gesammelte mathematische Werke*, ed. H. Weber, Berlin/New York: Springer-Verlag, 1990, pp. 456–462.
7. Eugene Seneta, The Central Limit Problem and Linear Squares in Pre-revolutionary Russia: The Background, *Mathematical Scientist* **9** (1984), 37–77.
8. Augustin Louis Cauchy, Mémoire sur divers points d'analyse, *Mémoires de l'Académie de France* **8** (1829), 130–138; reprinted in *Oeuvres complètes d'Augustin Cauchy*, I[re] série, vol. 2, Paris: Gauthier–Villars, 1908, pp. 52–58.

9. M. V. Chirikov, On the History of the Origin of Asymptotic Methods of Mathematical Analysis in the Nineteenth Century, in *History and Methodology of the Natural Sciences*, no. 9, Moscow: Moscow Univ. Press, 1970, pp. 239–247. [In Russian]

10. Harold Edwards, *Riemann's Zeta Function*, New York: Academic Press, 1974.

11. Pavel Alexeevich Nekrasov, *The Lagrange Series and Approximate Expressions for Functions of Very Large Numbers*, Moscow: Moscow Univ. Printers, 1886; *Matematicheskiĭ Sbornik* **12** (1886), 481–579.

12. Pavel Alexeevich Nekrasov, *Computation of Approximate Expressions for Functions of Very Large Numbers*, Moscow: Moscow Univ. Printers, 1900; *Matematicheskiĭ Sbornik* **21** (1901), 564–762.