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1-HOMOGENEOUS GRAPHS

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1. Introduction

Roughly speaking an n-homogeneous graph G (defined formally below) may be thought of as having the following property: No matter what pair A, B of induced subgraphs are chosen, if A, B cannot be distinguished by a search of their environments n points deep, then there is an automorphism of G which takes A to B. The properties of n-homogeneous structures have been studied for many years in logic, particularly by Clark and Krauss [4]. It appears that Sheehan first considered finite 0-homogeneous graphs, and these were later classified by Gardiner [5], who called them ultrahomogeneous. More recently Lachlan and Woodrow [7] studied the countably infinite 0-homogeneous graphs. The notion of 1-homogeneity generalizes 0-homogeneity, and turns out to be closely related to distance-transitivity. The 'type' of a pair x, y will generalize the notion of distance. We will classify the trivalent 1-homogeneous graphs, and present a complete list which contains all combinations of types which might be realized in some regular, 1-homogeneous graph of diameter 2.

2. Basic notions

2.1. *1-Homogeneity*

Let G be a graph with vertex set VG and X, Y induced subgraphs. A bijection $f:X \to Y$ is a 0-isomorphism if f is an isomorphism in the usual sense. Inductively, we say f is an $(n+1)$-isomorphism if, for any choice of $g_x \in \text{VG}$, there is a choice $g_y \in \text{VG}$ such that the extension of f with $f(g_x) = g_y$ is an n-isomorphism, and vice-versa. Thus, $f:X \to Y$ is a 1-isomorphism if, for each choice of g_x (g_y) there is a g_y (g_x) such that $f:X+g_x \to Y+g_y$ is an isomorphism. If f is a 1-isomorphism, then the restriction of f to any induced subgraph of X is a 1-isomorphism as well.

A graph G is said to be n-homogeneous if every n-isomorphism (between induced subgraphs X, Y) can be extended to an automorphism of G. Thus G is

0-homogeneous if it is ultrahomogeneous in the sense used by Gardiner. If G is n-homogeneous, then G is k-homogeneous for every $k \geq n$, and G^c is also n-homogeneous.

Proposition 2.1. *Let G be a 1-homogeneous. Then*

(1) *The automorphism group of G is transitive on each of the following sets:*

(a) *points of zero valency,*

(b) *points of full valency, that is, adjacent to every other point,*

(c) *points whose valency is neither zero nor full;*

(2) *Nontrivial components of G are isomorphic;*

(3) *If G is connected, then G has at most two valencies. Moreover, if G has two valencies, one of them is full;*

(4) *If G is 1-homogeneous connected, and regular, then G has diameter ≤ 3.*

This proposition tells us that each nontrival component of a 1-homogeneous graph consists of points of full valency plus points of some other valency k. We define the *regular core* K of a connected, 1-homogeneous graph to be the regular subgraph of G induced by the set of all points of valency k.

Proposition 2.2. *The regular core is 1-homogeneous.*

Proof. Let X, Y be induced subgraphs of K, and $f: X \to Y$ a 1-isomorphism in K. We claim that f is also a 1-isomorphism in G. For, let $x \in VG$ be chosen. We must exhibit $y \in VG$ such that the extension of f with $f(x) = y$ is an isomorphism. If $x \in K$, we can choose $y \in K$ since f is a 1-isomorphism in K. Otherwise x has full valency in G, and we choose y to be any point of full valency. Now use the 1-homogeneity of G to find an automorphism α of G extending f. The restriction of α to K is the desired automorphism of K. □

If the regular core is not connected then its components must be identical. So the structure of 1-homogeneous graphs actually reduces to the study of connected, regular, 1-homogeneous graphs, which will be the emphasis of the remainder of this paper.

We shall use the following notation:

$L(G)$ denotes the line graph of G,
K_t denotes the complete graph on t points,
tG denotes t copies of G,
$K_{t,t}$ denotes the complete bipartite graph with t points in each block,
$K_{q;t}$ denotes the complete q-partite graph with t points in each block,
G^c denotes the complement of G,
k denotes the valency of G,
n denotes the number of points in G,

$G_1[G_2]$ the composition of G_1, G_2; obtained by replacing each points of G_1 by a copy of G_2, and for x, y in distinct copies of G_2, x is adjacent to y if the points replaced in G_1 were adjacent,

x adj y is an abbreviation for 'x is adjacent to y',

x nadj y is an abbreviation for 'x is not adjacent to y',

$d(x, y)$ denotes the distance from x to y,

$d(G)$ is the diameter of G,

$(x_1, \ldots, x_m) \sim_n (y_1, \ldots, y_m)$ signifies that the mapping $x_i \to y_i$, $1 \leqslant i \leqslant m$, is an n-isomorphism.

We will also need the following results:

Theorem 1 (Gardiner [5]). *The* 0*-homogeneous graphs are*

(1) C_5,

(2) qK_t, $q, t \geqslant 1$,

(3) $K_{q;t}$ *(the complement of* qK_t*)*, $q, t \geqslant 1$,

(4) $L(K_{3,3})$.

Theorem 2 (Myers [8]). $L(K_t)$ *is* 1*-homogeneous.*

2.2. *Types*

Let G be a finite, regular, 1-homogeneous graph. For each pair of points $x_1, x_2 \in VG$, we wish to define the type $t(x_1, x_2)$. The general idea of the notion is that types in a 1-homogeneous graph will play the role of distances in a distance-transitive graph. In particular we will require that, for x_1, x_2 and y_1, y_2, pairs of points in the same regular, 1-homogeneous graph, $t(x_1, x_2) = t(y_1, y_2)$ iff the mapping $x_i \to y_i$, $i = 1, 2$ is a 1-isomorphism. So, we define

$t(x_1, x_2) = 0$ if $x_1 = x_2$,

$t(x_1, x_2) = 1$ if x_1 adj x_2, no point is adjacent to both, some other point is adjacent to x_1 or x_2, and there is some other point adjacent to neither,

$t(x_1, x_2) = \alpha$ iff x_1 adj x_2, no other point is adjacent to either x_1 or x_2, and there is some point adjacent to neither,

$t(x_1, x_2) = 2$ iff x_1 adj x_2, some point is adjacent to both, there is a point adjacent to $x_1(x_2)$ but not $x_2(x_1)$ and some other point is adjacent to neither,

$t(x_1, x_2) = 3$ iff x_1 adj x_2, x_1, x_2 are otherwise adjacent to exactly the same nonempty set of points, and some other point is adjacent to neither,

$t(x_1, x_2) = 4$ iff x_1 nadj x_2, x_1, x_2 are adjacent to exactly the same nonempty set of points, and some other points is adjacent to neither,

$t(x_1, x_2) = 5$ iff x_1 nadj x_2, some point is adj to both, some other point is adjacent to $x_1(x_2)$ but not to $x_2(x_1)$, and some other point is adjacent to neither,

$t(x_1, x_2) = 6$ iff x_1 nadj x_2, some other point is adjacent to x_1, or x_2, no point is adjacent to both, and some other point is adjacent to neither,

$t(x_1, x_2) = \beta$ iff x_1 nadj x_2, no point is adjacent to either x_1 or x_2, and some other point is adjacent to neither.

In addition to these we define the types $1', 2', \ldots, 6', \alpha', \beta'$ by replacing in each definition the phrase "some other point is adj to neither" by "every other point is adj to x_1 or x_2". It is not difficult to see that, in a finite, regular, 1-homogeneous graph, each pair of points falls into exactly one of the 17 types, and that $t(x_1, x_2) = t(x_2, x_1)$. The types $\alpha, \alpha', \beta, \beta'$ have been distinguished from the others, because they can be realized only in unique regular graphs. Note also that if $t(x, y) = \alpha(\beta)$ in G, then $t(x, y) = 4'(3')$ in G^c. We have:

Proposition 2.3. *The only* 1*-homogeneous, regular graphs realizing types* $\alpha, \alpha', \beta, \beta', 3', 4'$ *are*

(1) *Type* α (α') *is realized only in* nK_2, $n > 1$ $(n = 1)$;
(2) *Type* β (β') *is realized only in* nK_1, $n > 2$ $(n = 2)$;
(3) *Type* $3'$ *is realized only in* K_n, $n > 2$;
(4) *Type* $4'$ *is realized only in* $K_{t;2}$, $t > 1$.

Note that for G connected, $t(x, y) = 6$ or $6'$ iff $d(x, y) = 3$. For G not connected, and x, y in distinct nontrivial components, we also have $t(x, y) = 6$ or $6'$. Therefore the components of a nonconnected, regular, 1-homogeneous graph must have diameter $\leqslant 2$.

It is very helpful to think of 'type' a generalization of distance, for the role of types in a 1-homogeneous graph is similar to that of distance in a distance-transitive graph. In fact, the algebraic methods for distance-transitive graphs described by Biggs [1, pp. 82–108] adapt readily to the 1-homogeneous case. Later, when we discuss the notion of feasibility, the analogy will be developed further. For now, we will just introduce some notation which will be useful in the following section, which classifies the trivalent 1-homogeneous graphs.

Let (x, y) be a pair of type j in a regular 1-homogeneous graph. Since (x, y) can be mapped under an automorphism to any other pair of type j, the number s_{fij} of points z such that $t(x, z) = f$ and $t(y, z) = i$ is independent of the particular x, y chosen. These 'intersection numbers' also have the property

(I1) $\quad s_{fij} = s_{ifj},$

which follows since $t(x, y) = j = t(y, x)$.

We use k_i to denote the number of points z such that $t(x, z) = i$. Suppose that $i1, i2, \ldots, ir$ is a list of all types occurring in G, except for type 0, which is always realized, and n be the number of points in G. Then we have, naturally,

(I2) $\quad n = 1 + k_{i1} + k_{i2} + \cdots + k_{ir}.$

A useful relationship between the k_i and the s_{fij} can be derived by counting:

(I3) $\quad k_j \cdot s_{fij} = k_i \cdot s_{fji}.$

To see this, we count the ordered triples (x, y, z) such that $t(x, y) = j$, $t(y, z) = i$, and $t(x, z) = f$, in two ways. So, for the first count, we consider each point x in turn (n in all). We look to each y such that $t(x, y) = j$ (there are k_j of these), and observe that for each of these y's there are exactly s_{fij} points z for which (x, y, z) is one of the desired triples. Thus our first count of all such triples is $n \cdot k_j \cdot s_{fij}$. The second count considers each y (n in all). We then look to each z for which $t(y, z) = i$ (there are k_i of these), and for each such z there are exactly s_{fji} points x for which (x, y, z) is a desired triple. This time the count is $n \cdot k_i \cdot s_{fji}$, from which (I3) follows.

3. Trivalent 1-homogeneous graphs

In this section we classify the connected trivalent 1-homogeneous graphs. The main fact is that $d(G) \leqslant 3$, which requires $n \leqslant 22$.

Lemma 3.1. *Let G be trivalent, connected, and 1-homogeneous.*

(1) *Type* 3 *is not realized, and if type* 3′ *is realized, then* $G = K_4$.

(2) (a) *If type* 1′ *occurs, then* $n = 6$;

(b) *Type* 2′ *cannot occur at all*;

(c) *If type* 6; *occurs, then* $n = 8$.

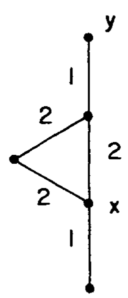
(3) *If type* 2 *occurs, then* $n = 6$.

(4) *If* x, y *are adjacent*, $G \neq K_4$, *and* $n \neq 6$, *then* $t(x, y) = 1$.

(5) *If type* 4 (*or* 4′) *occurs, then* $n = 6$.

(6) *If type* 5′ *occurs, then* $n \leqslant 6$.

Proof. (3) Suppose type 2 occurs, then either type 1 or type 1′ must also occur. If type 1′ occurs, then $n = 6$. If type 1 occurs, then clearly $k_1 = 1$, $k_2 = 2$, and $k_{5'} = 0$ ($k_{5'} \neq 0$ would again imply $n = 6$). Since every point is then in exactly one 3-cycle, we know that n is a multiple of 3.



$t(x, y) = 5$

We have $s_{125} = 1 = s_{152}$, and by property (I3) of Section 2, $k_5 s_{125} = k_2 s_{152}$, which implies $k_5 = 2$.

Therefore, if our graph has diameter 2, then $n = 1 + k_1 + k_2 + k_5 = 6$, as desired. Otherwise type 6 must occur, so suppose this is the case. Thus if $t(x, y) = 5$, there must be at least one point z adjacent to y such that $t(x, z) = 6$. Moreover, there can be only one, since $s_{125} = s_{215} = 1$. Hence if $t(x, z_1) = 6$ and $t(x, z_2) = 6$, then z_1, z_2 are adjacent to distinct y_1, y_2 which are each of type 5 with x. Therefore $k_6 \leq k_5$, and $n \leq 8$. But n must be a multiple of 3, so type 6 cannot occur. □

Proposition 3.2. *Let G be connected, trivalent, and* 1*-homogeneous.*

(1) *If* $d(G) = 1$, *then* $G = K_4$.

(2) *If* $d(G) = 2$, *then G is one of the following:*

(a) *the complement of* C_6 (C_6 *is the cycle on* 6 *points*),

(b) *the complete bipartite graph* $K_{3,3}$,

(c) *Petersen's graph.*

(3) *If* $d(G) = 3$, *then G is one of the following:*

(a) *the cube,*

(b) *Heawood's graph* (see Fig. 1).

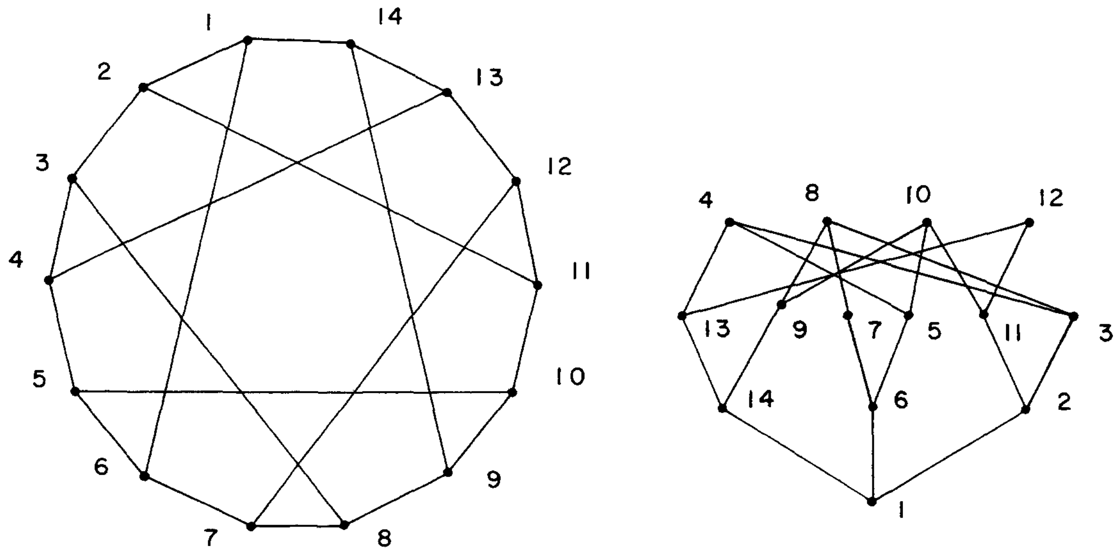


Fig. 1. Heawood's graph.

Proof. For $n = 6$, the only possibilities are $K_{3,3}$ and C_6^c. Now suppose $n > 6$. In 1–6 of Lemma 3.1 it is shown that the only types which can occur are 1, 5, 6, and 6′. Moreover, type 6′ occurs only if $n = 8$, and the only exemplar is the cube. In all other cases, G must be distance-transitive (for each distance there is only one type which can be realized). Trivalent distance-transitive graphs were classified by Biggs and Smith [3], and include for $d(G) \leq 3$ all the graphs we have listed except C_6^c.

It remains only to show that each graph listed is in fact 1-homogeneous. The cases which are not rather obvious are Petersen's graph and Heawood's graph. Petersen's graph is the complement of $L(K_5)$, which is 1-homogeneous by Theorem 2.

We must now establish that the Heawood graph is indeed 1-homogeneous. Let A, B be induced subgraphs of the Heawood graph, and $f: A \to B$ a 1-isomorphism. We must show that there is an automorphism α which agrees with f

on A. Since the Heawood graph is distance-transitive, we assume that A contains ≥ 3 points.

We will use heavily the fact that the Heawood graph is (fortunately) known to be 4-unitransitive [6, pp. 173–175], that is, any isomorphism between two paths of length 4 can be extended uniquely to an automorphism. In certain cases we will use the following sort of argument:

Argument A. We locate certain points $a, b, \ldots, \varepsilon A$, a path PA of length 4 containing $a, b, \ldots$, a path PB of length 4 continuing $f(a) = a', f(b) = b', \ldots$ and an isomorphism π : PA$\rightarrow$PB which agrees with f on $a, b, \ldots$. Then there is a unique automorphism α extending π. We then show that for each x in the Heawood graph there is a property P such that

(i) x is the unique point for which $P(x, a, b, \ldots)$ holds,

(ii) P is preserved under any 1-isomorphism, that is

$$P(x, a, b, \ldots) \Rightarrow P(f(x), a', b', \ldots) \quad \text{for } x \in \operatorname{dom} f.$$

Since α is an automorphism extending π, there is a unique point $y = \alpha(x)$ for which $P(y, a', b', \ldots)$ holds. Now suppose $x \in \operatorname{dom} f$. By (ii), $P(f(x), a', b', \ldots)$ holds, which shows that $\alpha(x) = f(x)$. Thus f agrees with α on dom f, and so f can be extended to an automorphism (namely α).

Let $a, b \in A$ and a', b' be their images under f. Since f is a 1-isomorphism, $d(a, b) = d(a', b')$. So for example, if $1 \in A$ and there are j members of A at distance i, then B has exactly j members at distance i from $1'$. We will use this to identify several cases which will be treated separately.

Case 1. A contains two points, a, b, at distance 2 from each other.

Subcase A. Among the remaining members of A, there is a point, c, at distance 2 from both a and b.

Subsubcase (i). There is no point mutually adjacent to a, b, c. For example, let $a = 1$, $b = 3$, $c = 5$. We use Argument A. Let PA have points 1, 2, 3, 4, 5. By the 1-homogeneity of f there is a path PB with points a', p_2, $3'$, p_4, $5'$. Let π be the obvious bijection from PA to PB and α the unique automorphism extending π. For each x we will find a P satisfying (i), (ii) above. First of all $P(x, a, b, c)$ will be understood always to include the following property $Q(a, b, c)$: $d(a, b) = 2$, $d(a, c) = 2$, $d(b, c) = 2$, and there is no point adjacent to each of a, b, c. Now for each x in the Heawood graph there is a property (preserved under 1-isomorphism) which uniquely identifies x in relation to any such a, b, c, see Table 1.

So for each x we form $P(x, a, b, c)$ by adding to $Q(a, b, c)$ the appropriate property above. It is readily verified that, if $x \in \operatorname{dom} f$, then (i), (ii) are satisfied, and hence f can be extended by Argument A.

Subsubcase (ii). The points a, b, c are mutually adjacent to another point in the graph. For example, let $a = 2$, $b = 6$, $c = 14$, with point 1 adjacent to all three. The

Table 1

Property	x identified for 1, 3, 5
x adj a and x adj b	2
x adj b and x adj c	4
x adj a and x adj c	6
x nadj a, b, c, and there is a z such that z adj a, c, x	7
x adj b and x nadj a, c	8
x nadj a, b, c, and there is no z such that z adj a, b, x, and there is a z such that z adj a, x	9
x adj c and x nadj a, b	10
x nadj a, b, c and there is a z such that z adj a, b, x	11
x nadj a, b, c and there is no z such that z adj x and z adj a or b or c	12
x nadj a, b, c and there is a z such that z adj b, c, x	13
x adj a and x nadj b or c	14

1-isomorphism of f guarantees that $f(2)$, $f(6)$, $f(14)$ are mutually adjacent to some point, say $1'$.

If $A = \text{dom}\, f$ contains only 2, 6, 14, consider any automorphism α extending the mapping from $\{2, 1, 6\}$ to $\{f(2), 1', f(6)\}$ (α exists by distance-transitivity). Clearly $\alpha(14) = f(14)$, since 14 adj 1. Thus α extends f.

If A contains any point from Level 3, we have an instance of Subsubcase (i); for any such point p, there is a pair p_1, p_2 from $\{2, 6, 14\}$ such that p_1, p_2, p are at distance 2 from each other, and no point is adjacent to all 3.

So suppose A contains a point from Level 2, and no points from Level 3. If there is only one such point, say 3, we just consider the mapping π from the path with points $\{3, 2, 1, 6\}$ to the path with points $\{f(3), f(2), 1', f(6)\}$. Any automorphism α extending π will extend f. Now suppose there is at least one other point from Level 2, (which is not adjacent to point 2), say 5. We can extend π by defining $\pi(5) = f(5)$, and then consider the unique automorphism extending π. If $7 \in A$, then clearly $f(7) = \alpha(7)$, since 7 adj 6.

But what if $9 \in A$ or $13 \in A$? Note that 13 is the unique point with the property, P, that "there is a point (4) to which 3, 5, and 13 are all adjacent". This property must be preserved by any 1-isomorphism, and, naturally, by α. Therefore, if $13 \in A$, then $f(13) = \alpha(13)$. If $9 \in A$, similar reasoning applies, modifying P to read "there is no point . . .".

Subcase B. Among the remaining members of A, there is no point at distance two from both a and b. For example, let $a = 1$, $b = 3$. Then the additional members of A are a subset of $\{2, 6, 14, 4, 8, 10, 12\}$.

Subsubcase (i). The additional members of A are a subset of $\{2, 6, 14\}$. In this case no matter which of the seven nonempty subsets is considered the required α is easily found using 4-unitransitivity.

Subsubcase (ii). The additional members of A are a subset of $\{4, 8, 10, 12\}$.

First suppose $4 \in A$ (or $8 \in A$; the arguments are similar). Consider the path PA whose points are 1, 2, 3, 4, and PB whose points are $1'$, p_2, $3'$, $4'$. The paths are of length 3 and we will not use Argument A, but argue directly. For any automorphism α extending the map $\pi: \text{PA} \to \text{PB}$ we have $\alpha(8) = f(8)$ if $8 \in A$, but not necessarily $\alpha(10) = f(10)$. Indeed, if $10 \in A$, $f(10)$ could be either of two points, namely the two points at distance 3 from $1'$, $3'$, since, if x denotes either of these points, we have $(1, 3, 4, 10) \sim_1 (1', 3', 4', x)$. However, whichever of the two possible points $f(10)$ is, there is an automorphism α extending π with $\alpha(10) = f(10)$, as we can see by extending PA to include 5, and the path PB to include the unique point to which $4'$ and $f(10)$ are mutually adjacent. The automorphism extending this mapping must agree with f.

Now suppose A includes neither 4 nor 8, but includes, say, 10 (for 12 the argument is similar). The 1-isomorphism f preserves distances, so $10' = f(10)$ is one of the two points at distance 3 from $1'$, $3'$. Let PA be the path with points 1, 2, 3, 4, 5, and PB the path with points $1'$, p_2, $3'$, p_4, p_5, where p_4 is either of the points which is adjacent to $3'$ and at distance 3 from $1'$, and p_5 is the unique point adjacent to both p_4 and $10'$. Again, the automorphism extending the mapping $\pi: \text{PA} \to \text{PB}$ must agree with f.

Subsubcase (iii). The additional members of A include at least one point from Level 1 and one from Level 3. Here we examine all nonempty subsets X, Y of additional points with x from Level 1 and y from Level 3. In all cases either Argument A applies or we can argue similar to Subsubcase (ii) above.

Case 2. A contains no pair of points at distance 2 from each other. This is impossible, as can be seen by considering two subcases.

Subcase A. A contains a pair of adjacent points, say 1, 2. The remaining members of A would have to be at distance 3 from both, but no such points exist, so this case is impossible.

Subcase B. A contains no pair of adjacent points. Then every pair of points in A would be at distance 3 from each other. Three such points do not exist. □

4. Feasibilty of types in regular, 1-homogeneous graphs of diameter 2

Our overall goal will be the presentation (in Section 4.3) of a list containing all combinations of types which are potentially realizable in regular, 1-homogeneous graphs of diameter 2. Toward this end, Section 4.1 gives methods for constructing exemplars for certain combinations, while Section 4.2 introduces the notion of a feasible combination of types. Feasibility is an algebraic property which is necessary for a combination of types to be realized. The list in Section 4.3 contains all combinations of types which either have an exemplar, or are feasible and are not ruled out by other considerations.

4.1. *Expansion and collapsing theorems*

In this section we give some results which allow us to obtain new 1-homogeneous graphs from old. These results will help to justify the table in Section 4.3.

Definitions. Let G be a regular, 1-homogeneous graph which realizes one of the types 3, 3′, 4, 4′ (G can in fact realize at most one of these types). Then

$[x]$ denotes the collection of all points y such that, for each $z \neq x, y$, z adj x iff z adj y. In other words, letting δ denote whichever of 3, 3′, 4, 4′ appears in G, $[x]=\{y \mid t(x, y)=\delta\} \cup \{x\}$. The collection $[\mathrm{VG}]=\{[x] \mid x \in \mathrm{VG}\}$ is a partition of VG.

G^{δ} denotes the collapse of G under type δ, that is, the graph with vertex set $[\mathrm{VG}]$, and for distinct $[x],[y] \in [\mathrm{VG}]$, $[x] \operatorname{adj} [y]$ in G^{δ} iff x adj y. Note that the statement "x adj y iff $[x] \operatorname{adj} [y]$" may be false if $\delta=3$ or 3′, and $y \in [x]$. Also, for distinct $[x]$, $[y]$, $[x] \operatorname{adj} [y]$ iff each $x_1 \in [x]$ is adj to each $y_1 \in [y]$.

Proposition 4.1. *Let G be a regular, 1-homogeneous graph realizing $\delta \in \{3, 3', 4, 4'\}$. Then G^{δ} is 1-homogeneous, and does not realize δ.*

Proof. Let $([x_1], \ldots, [x_n]) \sim_1 ([y_1], \ldots, [y_n])$ in G^{δ}. We claim that $(x_1, \ldots, x_n) \sim_1 (y_1, \ldots, y_n)$ in G. For, let $x \in G$ be chosen. If $x \in [x_i]$ for some i, $1 \leqslant i \leqslant n$, then choose $y \in [y_i]$, and we have $(x_1, \ldots, x_n, x) \sim_0 (y_1, \ldots, y_n, y)$. If $[x]$ is distinct from each $[x_i]$, then by 1-isomorphism we can choose $[y]$ such that $([x_1], \ldots, [x_n], [x]) \sim_0 ([y_1], \ldots, [y_n], [y])$, and hence $(x_1, \ldots, x_n, x) \sim_0 (y_1, \ldots, y_n, y)$. □

Next a few minor observations will prove useful later.

Lemma 4.2. *Let G be regular, 1-homogeneous, and realize type δ.*

(1) *Let $\delta=4$ and suppose that, if $t(x, y)=1$ (1′), then there is a point x_1 which is adjacent to x with $t(x_1, y) \neq 4$. Then $t(x, y)=1$ (1′) in G iff $t([x],[y])=1$ (1′) in G^{δ}.*

(2) *Let $\delta=4$. Then type 5′ cannot occur.*

(3) *Let $\delta=4$, and suppose that, if $t(x, y)=5$, then there is a point z which is not adjacent to x or y, and such that neither $t(x, z)=4$ nor $t(y, z)=4$. Then $t(x, y)=5$ in G iff $t([x],[y])=5$ in G^{δ}.*

(4) *Let $\delta=4'$. Then G is $K_{t;2}$, $t \geqslant 2$, so G^{δ} is complete.*

(5) *Let $\delta=3'$. Then G is complete (and G^{δ} is trivial).*

(6) *Let $\delta=3$. Then x nadjj y in G implies $[x]$ nadj $[y]$ in G^{δ}.*

Proof (1). Suppose $t(x, y)=1$ (1′). Then $[x]$, $[y]$ are adjacent in G^{δ}, and there are no points $[t]$ adjacent to both $[x]$ and $[y]$. The point x_1 given in the hypotheses is adjacent to x but not in $[y]$, so $[x_1]$ is a distinct point adjacent to $[x]$. This guarantees that type 1 (1′) cannot 'collapse' to type α (α') in G^{δ}. Finally, if z is nonadjacent to both x and y, then $[z]$ is distinct from $[x]$, $[y]$, and nonadjacent to both, and conversely. Therefore $t([x],[y]) = 1$ (or $1'$, if no such z exists). The converse is similar.

(2). Let $t(x, z) = 4$, and suppose $t(x, y) = 5'$. Then z must be adjacent to either x or y, but this is impossible, since $t(x, z) = 4$, and if y were adjacent to z, then y would also be adjacent to x. □

In view of Proposition 4.1, it is natural to consider the process inverse to collapse, namely expansion. Given a graph G not realizing, say, type 4, we can consider the graph $G[tK_1]$, which does realize type 4 (or $4'$). The question is: When is $G[tK_1]$ 1-homogeneous? The following provides a partial answer.

Proposition 4.3. *Let G be 0-homogeneous and realize none of the types 3, $3'$, 4, $4'$. Then each of the graphs $G[tK_1]$, $G[K_t]$ is 1-homogeneous.*

Proof. Let G^* denote either of $G[tK_1]$, $G[K_t]$, $[x]$ denote the copy of K_t or tK_1 in which x is found, and suppose $(x_1, \ldots, x_n) \sim_1 (y_1, \ldots, y_n)$ in G^*. We will show that the mapping $f(x_i) = y_i$ can be extended to an automorphism of G^*. For some $\delta \in \{3, 3', 4, 4'\}$, we have $t(x_1, x_2) = \delta$ for each $x_1, x_2 \in [x]$, and clearly $G \sim_0 (G^*)^\delta$. Let F be the bijection defined by: $F([x_i]) = [y_i]$ if there is $x \in [x_i]$, $y \in [y_i]$, $[x_i] \neq [y_i]$ and $f(x) = y$. We claim that, since f must respect types, F is a 0-isomorphism in G. First, $[x_i] = [x_j]$ iff $t(x_i, x_j) = \delta$ iff $t(y_i, y_j) = \delta$ iff $[y_i] = [y_j]$. If $[x_i] \neq [x_j]$, then $[x_i]$ adj $[x_j]$ in G iff x_i adj x_j, and since $t(x_i, x_j) = t(y_i, y_j)$, the preceding statements are true iff y_i adj y_j and $[y_i]$ adj $[y_j]$. Now since F is a 0-isomorphism, use the 0-homogeneity of G to extend F to an automorphism of G. Thus for each $[x] \in G$, $F([x]) = [y]$ is defined. Extend the mapping f by mapping each collection $[x]$ to its image $[y]$ in any way which respects the mapping $x_i \to y_i$, $1 \leq i \leq n$. □

Proposition 4.4. *Let G be 1-homogeneous, regular, of diameter 2. Then*
(1) *tG is 1-homogeneous;*
(2) *If G does not realize any of $1'$, α', $2'$, $3'$, then $K_t[G]$ is 1-homogeneous.*

Proof (1). Since G does not realize type 6, $x, y \in tG$ realize type 6 iff x, y lie in distinct copies of G. Any 1-isomorphism $f: A \to B$ in tG must therefore respect components of tG, and the component-wise pieces of f are each 1-isomorphisms of G. Extend these pieces by 1-homogeneity of G to component-wise automorphisms of G. □

(2). Consider $(K_t[G])^c = tG^c$. Now G^c can realize none of the types 6, $6'$, β, β', or else G would realize type $2'$, $1'$, $3'$, or α' respectively. Therefore G^c has diameter ≤ 2, and so by (1), tG^c is 1-homogeneous. □

4.2. *Feasible combination of types*

In this section we will present some algebraic properties enjoyed by types in a 1-homogeneous graph, and define the meaning of feasibility for a combination of types. This will be a straightforward adaptation of the methods described by Biggs [1, pp. 82–108]. Feasibility is a necessary, but not sufficent property which a combination of types must have to be realized in some graph. In Section 4.3, we will use the notion to show that certain combinations of types cannot be realized.

So, throughout this section, let G be a finite, regular, 1-homogeneous graph. In Section 2 we introduced the notations s_{fij} and k_i, and a few facts concerning them which proved useful in Section 3.

Let $t1, t2, \ldots, tr$ be a list of all types occurring in G, except for type 0, which is always realized. We will call G a $(t1, t2, \ldots, tr)$-graph.

Let $X = VG = \{1, 2, \ldots, n\}$ and consider the action of the automorphism group A on $X \times X$. If G is a $(t1, t2, \ldots, tr)$-graph, then A has exactly $r+1$ orbits, one for each type realized in G. Let f be one of the types $t1, t2, \ldots, tr$, and define A_f by

$$(A_f)_{ij} = \begin{cases} 1 & \text{if } t(i,j) = f, \\ 0 & \text{otherwise.} \end{cases}$$

A_f is a real, symmetric, $n \times n$ matrix, wth real eigenvalues.

For each $g \in A$, the permutation matrix $P(g)$ is the $n \times n$ matrix with $(P(g))_{ij} = 1$ if $g(i) = j$, and 0 otherwise. It turns out that $P(g)$ commutes with A_f. The commuting algebra of G is the centralizer of the group of matrices $P(G) = \{P(g) \mid g \in G\}$. It can be shown that the matrices $I, A_{t1}, \ldots, A_{tr}$ form a basis for the commuting algebra, and that the commuting algebra has a faithful representation as an algebra of $(r+1) \times (r+1)$ matrices, in which the representative $\tilde{A}_f$ of A_f is given by

$$(\tilde{A}_f)_{ij} = s_{fij}$$

where i, j each denote one of the types $0, t1, t2, \ldots, tr$.

For example, suppose G is a 1-homogeneous graph which realizes exactly types 0, 1, 2, and 5. The matrices $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_5$ are given by

$$\tilde{A}_1 = \begin{array}{c} \\ \text{Type 0} \\ \text{Type 1} \\ \text{Type 2} \\ \text{Type 5} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Type 0} & \text{Type 1} & \text{Type 2} & \text{Type 5} \end{array} \\ \begin{bmatrix} s_{100} & s_{101} & s_{102} & s_{105} \\ s_{110} & s_{111} & s_{112} & s_{115} \\ s_{120} & s_{121} & s_{122} & s_{125} \\ s_{150} & s_{151} & s_{152} & s_{155} \end{bmatrix} \end{array},$$

$$\tilde{A}_2 = \begin{array}{c} \\ \text{Type 0} \\ \text{Type 1} \\ \text{Type 2} \\ \text{Type 5} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Type 0} & \text{Type 1} & \text{Type 2} & \text{Type 5} \end{array} \\ \begin{bmatrix} s_{200} & s_{201} & s_{202} & s_{205} \\ s_{210} & s_{211} & s_{212} & s_{215} \\ s_{220} & s_{221} & s_{222} & s_{225} \\ s_{250} & s_{251} & s_{252} & s_{255} \end{bmatrix} \end{array},$$

$$\tilde{A}_5 = \begin{array}{c} \\ \text{Type 0} \\ \text{Type 1} \\ \text{Type 2} \\ \text{Type 5} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Type 0} & \text{Type 1} & \text{Type 2} & \text{Type 5} \end{array} \\ \begin{bmatrix} s_{500} & s_{501} & s_{502} & s_{505} \\ s_{510} & s_{511} & s_{512} & s_{515} \\ s_{520} & s_{521} & s_{522} & s_{525} \\ s_{550} & s_{551} & s_{552} & s_{555} \end{bmatrix} \end{array}.$$

From Section 2 we already have two relationships which must hold among the matrix entries, namely

(M1) $s_{fij} = s_{ifj}$,

(M2) $k_j \cdot s_{fij} = k_i \cdot s_{fji}$.

To these we now add

(M3) The sum of each column of $\tilde{A}_f$ equals k_f. That is, $(1, 1, \ldots, 1)$ is a left eigenvector for the eigenvalue k_f.

(M4) The vector $(1, k_{t1}, \ldots, k_{tr})^{\mathrm{T}}$ is a right eigenvector for k_f in $\tilde{A}_f$.

Proof (M3). This is just a restatement of the fact that k_f is the number of neighbors of type f of a point. For, consider the 'type j' column of $\tilde{A}_f$. If $t(x, y) = j$, then the set of neighbors of type f with the point x is partitioned according to type with y. Thus $k_f = \sum_i s_{fij}$, where i ranges over $\{0, t1, \ldots, tr\}$. □

(M4). This proof (which, again, comes straight from Biggs), combines (M2) and (M3). The assertion is that $\sum_j k_j s_{fij} = k_f k_i$, which is demonstrated as follows:

$$\begin{aligned} \sum_j k_j s_{fij} &= \sum_j k_i s_{fji} && \text{(by (M2))} \\ &= k_i \sum_j s_{fji} \\ &= k_i k_f && \text{(by (M3))} \quad \square \end{aligned}$$

The assertions (M1)–(M4) place severe restrictions on the 'intersection matrices' (the matrices $\tilde{A}_f$) that a 1-homogeneous graph may have. We will say that a combination of types $(t1, t2, \ldots, tr)$ is *feasible* if there is some choice of n, $k_{t1}, k_{t2}, \ldots, k_{tr}$, and matrices $\tilde{A}_{t1}, \tilde{A}_{t2}, \ldots, \tilde{A}_{tr}$, such that (M1)–(M4) are satisfied.

How, for instance, does one verify that the combination (1, 2, 5) is feasible? Let $n = 42$, $k_1 = 6$, $k_2 = 5$, $k_5 = 30$ and the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_5$ be

$$\tilde{A}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 6 & 4 \end{bmatrix} \end{array}, \quad \tilde{A}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 4 & 0 \\ 0 & 5 & 0 & 4 \end{bmatrix} \end{array}$$

$$\tilde{A}_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 5 \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 5 & 6 & 4 \\ 0 & 5 & 0 & 4 \\ 30 & 20 & 24 & 21 \end{bmatrix} \end{array}$$

Verification of (M1) and (M2) is tedious but straightforward; we just check for all possible choices of f, i, and j, sixty-four in all. With $f = 5$, $i = 2$, and $j = 1$ we have, for example,

(M1) $s_{521} = 5 = s_{251}$,

(M2) $k_1 \cdot s_{521} = 6 \cdot 5 = 5 \cdot 6 = k_2 s_{512}$.

For (M3) and (M4), first consider matrix $\tilde{A}_1$. It is evident that the sum of each column is $k_1 = 6$, which shows (M3). Moreover, multiplying $\tilde{A}_1$ on the right by $(1, k_1, k_2, k_5)^{\mathrm{T}} = (1, 6, 5, 30)^{\mathrm{T}}$ gives the vector $(6, 36, 30, 180)^{\mathrm{T}} = k_1(1, k_1, k_2, k_5)^{\mathrm{T}}$, which shows (M4). The verification for $\tilde{A}_2$ and $\tilde{A}_5$ is similar.

It should be mentioned that deeper properties of the matrices A_f can be derived concerning the multiplicities of eigenvalues. This has been done for distance-transitive graphs and incorporated into the definition of feasibility for distance-transitive graphs presented by Biggs. The bulk of the argument carries over to 1-homogeneous graphs, with the important exception that if λ is an eigenvalue for $\tilde{A}_f$, the multiplicity of λ need not be 1, as is true for distance-transitive graphs. However, we can get by without these deeper results in this paper.

4.3. *A list of potentially realizable combinations*

The aim of this section is to justify Table 2, which lists, for 1-homogeneous, regular graphs of diameter 2, every combination of types which is feasible and not ruled out by other considerations; that is, potentially realizable.

Table 2

Combination	Comment of exemplar
1, 5 (2, 5′)	Petersen graph
1, 5′	C_5
1, 4, 5 (2, 3, 5′)	$C_5[2K_1]$
2, 5	$L(K_6), L(K_{3,3})$
2, 4, 5 (2, 3, 5)	$L(K_{3,3})[2K_1]$
2, 5, 5′ (1, 2, 5)	Feasible, realizability unknown
1′, 4	$K_{3,3}$
1′, 4′	C_4
1′, 2, 5	Feasible only for $k_{1'} = 1$, realizability unknown
1′, 2, 5′	complement of the cube
2′, 3, 5	$K_q[mK_t]$, $m, t \geqslant 3, q \geqslant 2$
2′, 4	$K_{t;q}$, $t \geqslant 3, q \geqslant 3$
2′, 4′	$K_{t;2}$, $t \geqslant 3$
2′, 3, 5′	$K_q[2K_t]$, $t \geqslant 3, q \geqslant 2$
1, 2, 4, 5 (2, 3, 5, 5′)	Feasible, realizability unknown
1′, 2, 4, 5	Feasible, realizability unknown
2, 2′, 5	K_2[Petersen graph]
2, 2′, 5′	$K_2[C_5]$
2, 2′, 4, 5	$K_2[C_5[2K_1]]$
2, 2′, 5, 5′	Feasible, realizability unknown
2, 2′, 3, 5	The complement of $2\{L(K_{3,3})[2K_1]\}$
2, 2′, 5, 5′	Feasible, realizability unknown
2, 2′, 3, 5	The compement of $2\{L(K_{3,3})[2K_1]\}$
2, 2′, 3, 5, 5′	Feasible, realizability unknown

Throughout this section G is assumed to be regular, 1-homogeneous, and of diameter 2 unless stated otherwise. Besides type 0, there are 10 types which could conceivably appear in G: 1, 2, 3, 4, 5, 1′, 2′, 3′, 4′, 5′, leading to a great many combinations. Most of these combinations can be ruled out by the following observations.

Lemma 4.5. *Let G be 1-homogeneous, regular, of diameter 2.*

(0) *Type 3′ is impossible.*

(1) (a) *If G realizes type 4′, then G is $K_{t;2}$, $t \geqslant 2$;*

(b) *G cannot realize the pair of types $\{3, 4\}$.*

(2) *G can realize no pair of types from $\{1, 1', 2'\}$.*

(3) *G can realize no pair of types from $\{1, 1', 3\}$.*

(4) (a) *The pair of types 1, 5′ occurs only in C_5;*

(b) *The pair of types 1′, 5′ can occur only in the combination $(1', 2, 5')$.*

(5) *If G^c realizes type 2, then G^c must realize one of $\{4, 4', 5, 5'\}$.*

(6) *If G is distance-transitive and realizes 1′ or 2′, then G must realize one of $\{4, 4'\}$.*

Proof (0). As was mentioned in Proposition 2.3.3, if G realizes 3′, then G is complete.

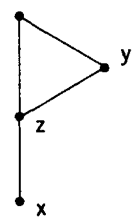
(1b). Let $t(x, y)=3$ and $t(x, z)=4$. Then, since x, z are adjacent to exactly the same set of points, z adj y. But now, since x, y are adjacent to exactly the same points, z adj x, which is a contradiction.

(2). Just observe that if type 1′ occurs, then $n=2k$; if 1 occurs, then $n>2k$; if 2′ occurs, then $n<2k$.

(4). We claim that if either of the combinations 1, 5′ or 1′, 5′ is realized in G, then $k_1 \leqslant 2$ (or $k_{1'} \leqslant 2$). For let $t(x, y)=5'$, and consider any path of length 2 from x to y with midpoint z. Suppose $t(w, z)=1$ (or 1′), where $w \neq x$ or y. Then since $t(x, y)=5'$, w adj x or y, which is impossible. Thus the only points which can be type 1 (or 1′) with z are x and y.

Moreover, if $k \geqslant 3$, let $w \neq x$, y be adjacent to z. Then again since $t(x, y)=5'$, w is adjacent to either x or y, so we cannot have both $t(x, z)=1$ and $t(y, z)=1$. We conclude that if the pair 1, 5′ (or 1′, 5′) is realized, and $k \geqslant 3$, then $k_1=1$ (or $k_1'=1$).

So, suppose $k \geqslant 3$ and the pair 1, 5′ or 1′, 5′ is realized. From the preceding paragraph we have that $k_1=1$ (or $k_1'=1$), and since, by (0), (2), and (3) above, none of the types 3′, 2′, or 3 can appear, G must realize type 2. Thus the list of types realized by G is either $(1, 2, 5')$ or $(1', 2, 5')$. But we claim that $(1, 2, 5')$ is impossible. For a contradiction, suppose that G realizes $(1, 2, 5')$, and consider the situation in the vicinity of point z, letting x be the unique point such that $t(x, z)=1$, and $y \in \{u: t(u, z)=2\}$.



Note that $t(u, z)=2 \Rightarrow t(u, x)=5'$ and so $\{u: t(u, z)=2\} \subseteq \{u: t(u, x)=5'\}$. Our next goal is to show that $k_2=k_{5'}$, whence we will have equality between these sets. It can be seen that

(i) $s_{125'}=1$ (since $t(x, y)=5'$, and z is the unique point with $t(x, z)=1$ and $t(y, z)=2$), and

(ii) $s_{15'2}=1$ (since $t(z, y)=2$, and x is the unique point with $t(z, x)=1$ and $t(y, x)=5'$).

Using (i) and (ii) in the equation $k_{5'}s_{125'}=k_2 s_{15'2}$, we obtain $k_{5'}=k_2$, which gives us equality between the set above. But any point which is not adjacent to x must be in $\{u: t(u, x)=5'\}$, so every point in G is either adjacent to x or adjacent to z. Thus $t(x, z)=1'$, which is a contradiction to the hypothesis that $(1, 2, 5')$ is realized.

For $k \geqslant 3$, then, only the pair 1′, 5′ is possible, and only in the combination $(1', 2, 5')$. For $k=2$, the cycle C_5 realizes the pair 1, 5′.

(5). This is entirely trivial. Let $t(x, y)=2$, and z be any point adjacent to x but not to y. We must have $t(y, z)$ equal to one of 4, 4′, 5, or 5′.

(6). Since G is distance-transitive, every adjacent pair of points realizes the same type, by hypothesis either 1′ or 2′. Likewise, every non-adjacent pair must realize the same type, one of 4, 4′, 5 or 5′, since G has diameter 2. Suppose, for instance, that G realized the pair 1′, 5. Then G^c would realize exactly the types 6′ and 2, which is impossible by (5). If G realized 1′, 5′, then G^c would realize exactly the types 6′ and 1, which is clearly impossible. The cases where G realizes 2′, 5 or 2′, 5′ are similar. Thus nonadjacent pairs in G must realize either 4 or 4′. □

We now enumerate the combinations of types which our previous results allow for G 1-homogeneous, regular, and connected with diameter 2. Of course only 9 types (besides type 0) are ostensibly possible: 1, 2, 3, 4, 5, 1′, 2′, 4′, 5′ (recall that 3′ is impossible by Lemma 5.4(0)). Each combination of types which is 'potentially realizable', that is feasible and not ruled out by other considerations, must include some nonempty subset of the 'adjacent' types $A = \{1, 1', 2, 2', 3\}$ and some nonempty subset of the 'nonadjacent' types $B = \{4, 4', 5, 5'\}$. Thus there are $31 \times 15 = 465$ pairs of nonempty subsets which must be considered, but our previous results substantially restrict the pairs.

First, by Lemma 4.5(1a), 4′ can be realized only with the type 1′ (in $K_{2;2}$) or with type 2′ (in $K_{t;2}$, $t > 2$). So now we need consider only 3 nonadjacent types, which leaves $31 \times 7 = 217$ combinations.

Next, using parts (2) and (3) of Lemma 4.5, one finds that 18 of the possible 31 subsets of adjacent types are elminated. This brings us to $13 \times 7 = 91$ combinations.

Now if G were to realize types 4 and 5′, then G^c would realize types 3 and 1, which is impossible by Lemma 4.5(3). This eliminates two of the seven nonadjacent subsets. Moreover, note that:

If 4 is the only nonadjacent type realized by G, then G^δ must be complete, and hence G is $K_{t;n}$, where $t \geq 2$, $n \geq 3$, which realizes 1′, 4 if $t = 2$, and 2′, 4 if $t > 2$. (∗)

We need now only consider four nonadjacent subsets, and we are down to $13 \times 4 = 52$ combinations. With parts (4a) and (4b) of Lemma 4.5, ten more combinations are eliminated, leaving 42. Part (1b) eliminates six more, leaving 36, and (6) rules out 3 more, so 33 combinations remain. We can exhibit exemplars for sixteen of the remaining combinations (see Table 1). The remaining seventeen are:

(3, 5)	(1′, 4, 5)	(2′, 4, 5)	(1, 2, 5)	(1, 2, 4, 5)	(2, 5, 5′)
(3, 5′)			(1, 2, 5′)	(1′, 2, 4, 5)	(2′, 5, 5′)
(3, 5, 5′)			(1′, 2, 5)		(2, 2′, 5, 5′)
			(1, 2, 5, 5′)		(2, 3, 5, 5′)
					(2′, 3, 5, 5′)
					(2, 2′, 3, 5, 5′)

Of these, several can be eliminated simply because if G were to realize them, G^c would realize some combination of types which has already been ruled out.

If G were to realize	Then G^c would realize
$(3, 5)$	$(2, 4)$; ruled out by $(*)$
$(3, 5')$	$(1, 4)$; ruled out by $(*)$
$(3, 5, 5)'$	$(1, 2, 4)$; ruled out by $(*)$
$(1, 2, 5')$	$(1, 5, 5')$ ruled out by Lemma 4.5(4a)

Other combinations can be deleted simply because if type 2 is realized by G^c, then (by Lemma 4.5(5)) type 4, 4′, 5 or 5′ must also be realized.

If G were to realize	Then G^c would realize
$(1', 4, 5)$	$(2, 3, 6')$
$(2', 4, 5)$	$(2, 3, 6)$
$(2', 5, 5')$	$(1, 2, 6)$

Finally consider the combination $(2', 3, 5, 5')$. If G realizes this, then G^c realizes $(1, 2, 4, 6)$. If G^c is not connected, then each component is 1-homogeneous, regular, and of diameter 2, realizing $(1, 2, 4)$, which has already been ruled out. If G^c is connected, consider the collapse of G^c under type 4, $(G^c)^\delta = H$. According to Proposition 4.1, H is 1-homogeneous and does not realize type 4, nor in fact can H realize 4′. H cannot realize type 5 or 5′, since this would require one of these be realized in G^c. According to Lemma 4.2(1), however, H realizes type 1, which in turn implies that H realizes one of 4, 4′, 5 or 5′, a contradiction.

We wish to verify that the remaining combinations are indeed feasible, though not necessarily realizable, as we have not found exemplars for them. First consider $(1, 2, 5)$. The matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_5$ which were exhibited in Section 4.2 verify feasibility for this combination. However, an exemplar is not known to the author, and the realizability of $(1, 2, 5)$ remains an open problem.

Since $(1, 2, 5)$ is feasible (as well as its complementary case $(2, 5, 5')$, it is not surprising that $(1, 2, 4, 5)$ (and $2, 3, 5, 5'$) is feasible, for if G realizes $(1, 2, 5)$, then $G[2K_1]$ should realize $(1, 2, 4, 5)$. Of course, unless G is 0-homogeneous, we are not certain that $G[2K_1]$ is 1-homogeneous, but for feasibility purposes this is not necesssary. The characteristic polynomial of $\tilde{A}_1$ is found to be

$$t^5 - ct^4 - (bk_1 + ad + k_1k_4 + k_1)t^3 + (k_1k_4c + ck_1)t^2 + (bk_1k_4 + bk_1^2)t$$

where

$$a = \frac{k_1(k_1-(k_4+1))}{k_5}, \qquad b = \frac{k_1 k_2}{k_5}, \qquad c = k_1-(a+b), \qquad d = k_1-k_4-1.$$

Using $k_1 = 12$, $k_2 = 10$, $k_4 = 1$, $k_5 = 60$, We find that $a = 2$, $b = 2$, $c = 8$, and $d = 10$, and the above polynomial does indeed have $k_1 = 12$ as a root, as the definition of feasibility requires. The matrix $\tilde{A}_1$ turns out to be:

$$\tilde{A}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 12 & 0 & 0 & 12 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 10 & 12 & 0 & 8 \end{bmatrix} \end{array}$$

Note that the sum of each column is $k_1 = 12$, and it can be verified that $(1, 12, 10, 1, 60)^T$ is a right eigenvector for $k_1 = 12$. And indeed, taking $\tilde{A}_2$, $\tilde{A}_4$, and $\tilde{A}_5$ as below, one can verify each of the conditions (M1)–(M4) in the definition of feasibility.

$$\tilde{A}_2 = \begin{array}{c} \\ 1 \\ 1 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 10 & 0 & 8 & 10 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 8 \end{bmatrix} \end{array}, \qquad \tilde{A}_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\tilde{A}_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 10 & 12 & 0 & 8 \\ 0 & 10 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 \\ 60 & 40 & 48 & 60 & 42 \end{bmatrix} \end{array}.$$

Progress toward determining the realizability of the combinations $(1', 2, 5)$ and $(1', 2, 4, 5)$ can be made by calling upon algebraic techniques. Suppose G realizes $(1', 2, 5)$. Then $1 + k_{1'} + k_2 + k_5 = n = 2k = 2(k_{1'} + k_2)$, and so $k_5 = k_{1'} + k_2 - 1$.

We consider the matrix

$$\tilde{A}_{1'}=\begin{array}{c} \\ 0 \\ 1' \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1' & 2 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_{1'} & 0 & 0 & s_{1'1'5} \\ 0 & 0 & 0 & s_{1'25} \\ 0 & k_{1'}-1 & k_{1'} & 0 \end{bmatrix} \end{array}.$$

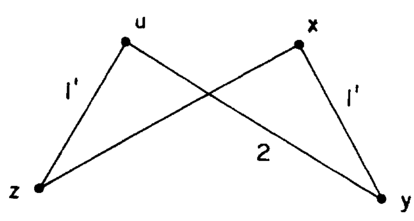
Columns 1′, 2 are obvious, using the definition of 1′. Then we have $k_5 s_{1'1'5} = k_{1'} s_{1'51'}$, so $s_{1'1'5} = k_{1'}(k_{1'}-1)/k_5$, and $k_5 s_{1'25} = k_2 s_{1'52}$, so $s_{1'25} = k_{1'}k_2/k_5$. Finally we have $k_{1'}(k_{1'}-1)/k_5 + k_{1'}k_2/k_5 = k_{1'}(k_{1'}-1+k_2)/(k_{1'}+k_2-1) = k_{1'}$, and since the columns of this matrix must add up to $k_{1'}$, $s_{1'55}=0$.

The eigenvalues of such a matrix are easy to compute, and the characteristic polynomial turns out to be:

$$t^4 - (bk_{1'} + a(k_{1'}-1) + k_{1'})t^2 + bk_{1'}^2$$

where $a = s_{1'1'5}$ and $b = s_{1'25}$. Now this is a polynomial in t^2, so for any root t, $-t$ is also a root. We known that $k_{1'}$ must be a root, so $-k_{1'}$ is also.

First we wish to show that $k_{1'} > 1$ is not possible. So, assume that $k_{1'} > 1$ (hence $s_{1'1'5} > 0$) and consider the graph $G_{1'}$ obtained from G be deleting each line between pairs of type 2; thus $G_{1'}$ has only 'type 1' lines. We claim that $G_{1'}$ is connected and in fact has diameter 3. To see this, suppose that x nadj y in $G_{1'}$. Then in G, either $t(x, y) = 5$ or $t(x, y) = 2$. If $t(x, y) = 5$, then since $s_{1'1'5} \neq 0$, there is a path of length 2 from x to y consisting of 'type 1' lines. If $t(x, y) = 2$, consider a point z such that $t(x, z) = 1'$. Clearly $t(y, z) = 5$, so there is a type 1 path of length 2 from y to z, and hence a type 1 path of length 3 from x to y. Also note that $A_{1'}$ is the adjacency matrix for $G_{1'}$. Since $A_{1'}$ and $\tilde{A}_{1'}$ have the same eigenvalues, $-k_{1'}$ is an eigenvalue of $A_{1'}$, which shows that $G_{1'}$ is bipartite (see [1, pp. 77–78]), with sets X and Y such that any line of $G_{1'}$ connects points in X and Y. Now let $t(u, y) = 2$ and $t(x, y) = 1'$, with $x \in X$, and $y \in Y$.



Since $t(u, y) = 2 \Rightarrow t(x, u) = 5$, there must be some point z with $t(u, z) = t(x, z) = 1'$, and so $u \in X$. Thus $t(q, r) = 2$ implies q, r are on 'opposite sides' of the bipartition. But now let v be any point adjacent to both u and y. We have $t(v, y) = 2$, and $t(v, u) = 2$, and so v must be opposite from both u and y, a contradiction. Therefore $k_{1'} > 1$ cannot be realized.

For $k_{1'} = 1$, however, the combination $(1', 2, 5)$ is feasible. With $k_{1'} = 1$, $k_2 = 3$,

$k_5 = 3$, $n = 8$, one can verify that (M1)–(M4) hold for the following matrices:

$$\tilde{A}_0 = \begin{array}{c} \\ 0 \\ 1' \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1' & 2 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}, \quad \tilde{A}_{1'} = \begin{array}{c} \\ 0 \\ 1' \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1' & 2 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

$$\tilde{A}_2 = \begin{array}{c} \\ 0 \\ 1' \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1' & 2 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \end{bmatrix} \end{array}, \quad \tilde{A}_5 = \begin{array}{c} \\ 0 \\ 1' \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1' & 2 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 3 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

While not required to satisfy (M1)–(M4), we note that s_{552} and s_{555} are both nonzero, as they must be in any graph actually realizing $(1', 2, 5)$. An exemplar, however, has not yet been found.

The feasibility of $(1', 2, 4, 5)$ can be verified by 'doubling' appropriate intersection numbers in the above matrices, similar to the strategy for verifying feasibility of $(1, 2, 4, 5)$.

The case $(2, 2', 5, 5')$ has $(1, 2, 5, 6)$ as its complementary case. If G realizes $(1, 2, 5)$, then $2G$ realizes $(1, 2, 5, 6)$, and so this case should be feasible. Indeed this turns out to be the case. Similar reasoning applies to $(2, 2', 3, 5, 5')$, whose complementary case is $(1, 2, 4, 5, 6)$.

Each of the exemplars in Table 2 may be seen to be 1-homogeneous using Theorems 1, 2, and Propositions 4.3, 4.4. We do not contend that the exemplars given below are the only ones. Complementary combinations appear in parentheses.

We hope that Table 2 will contribute to the eventual classification of 1-homogeneous graphs. But while the study of types of pairs provides much information, it may easily be that types of larger sets must be considered to complete the classification.

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