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Two-dimensional gauge theory and matrix model

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ABSTRACT

We study a matrix model obtained by dimensionally reducing Chern–Simons theory on S^3 . We find that the matrix integration is decomposed into sectors classified by the representation of $SU(2)$. We show that the N -block sectors reproduce $SU(N)$ Yang–Mills theory on S^2 as the matrix size goes to infinity.

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1. Introduction

Matrix models have been proposed as non-perturbative formulation of superstring or M-theory [1–3]. Since low energy physics predicted by string theory depends on topological aspects of compactification, it is relevant to investigate how they are incorporated in matrix models. The topological field theories have been developed to efficiently describe the topological aspects of field theories. It is, therefore, worthwhile to study realization of the topological field theories in matrix models.

Hinted by the work [4], the authors of [5] found the following classical relationships among Chern–Simons (CS) theory on S^3 , two-dimensional Yang–Mills (2d YM) on S^2 and a matrix model. The latter two theories are obtained by dimensionally reducing the first theory. The theory around each multiple monopole background of 2d YM is obtained by expanding the matrix model around a certain multiple fuzzy sphere background in the continuum limit (see also [6]). CS theory is obtained by applying an extension of compactification in matrix models developed in [4,6,7] to the theory around a multiple monopole background of 2d YM. Eventually, CS theory is obtained by expanding the matrix model around a certain multiple fuzzy sphere background and imposing the orbifolding condition. 2d YM is also viewed as BF theory with a mass term on S^2 . The matrix model takes the form of the superpotential for $\mathcal{N} = 1^*$ theory. The classical relationships between CS on S^3 and 2d YM on S^2 are generalized to those between CS theory on a $U(1)$ bundle over a Riemann surface Σ_g of genus g and 2d YM on Σ_g .

In this Letter, we show that 2d YM on S^2 is obtained from the matrix model also at quantum level. We find that the matrix integration is decomposed into sectors classified by the representation of $SU(2)$. We show that the N -block sectors reproduce the partition function of 2d $SU(N)$ YM on S^2 .

It has been already shown in [8,9] that different types of matrix models give 2d YM on S^2 . Moreover, the authors of [9] have shown that the localization works also for the matrix model in the same way as it works for the continuum 2d YM. We hope to elucidate the relation of our work with [8,9] in the future.

This Letter is organized as follows. In Section 2, we briefly review part of the results in [5], which are associated with the present work. In Section 3, we reduce the path-integral in the matrix model to the integral over the eigenvalues of a single matrix, which is decomposed into the sectors classified by the representation of $SU(2)$. In Section 4, we show that part of the above sectors reproduce 2d YM on S^2 . Section 5 is devoted to conclusion and outlook. In appendix, we summarize some useful properties of S^3 and S^2 .

2. Classical relationships among CS theory, 2d YM and a matrix model

In this section, we briefly review only part of the results in [5] which are concerned with the present Letter. We start with CS theory on S^3 with the gauge group $U(M)$:

$$S_{\text{CS}} = \frac{k}{4\pi} \int_{S^3} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.1)$$

We expand the gauge field in terms of the right-invariant 1-form defined in (A.3) as

$$A = iX_i E^i. \quad (2.2)$$

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Then, we rewrite (2.1) as

$$S_{CS} = -\frac{k}{4\pi} \int \frac{d\Omega_3}{(\mu/2)^3} \text{Tr} \left(i\mu \epsilon^{ijk} X_i \mathcal{L}_j X_k + \mu X_i^2 + \frac{2i}{3} \epsilon^{ijk} X_i X_j X_k \right), \tag{2.3}$$

where \mathcal{L}_i is the Killing vector dual to E^i and defined in (A.7).

By dropping the derivative of the fiber direction y , we obtain a gauge theory on S^2 :

$$S_{BF} = -\frac{1}{g_{BF}^2 \mu} \int \frac{d\Omega_2}{\mu^2} \text{Tr} \left(i\mu \epsilon^{ijk} X_i L_j^{(0)} X_k + \mu X_i^2 + \frac{2i}{3} \epsilon^{ijk} X_i X_j X_k \right), \tag{2.4}$$

where $g_{BF}^2 = 1/k^1$ and $L_i^{(0)}$ are the angular momentum operators on S^2 given in (A.13) with $q = 0$. In order to see that (2.4) is BF theory with a mass term, we define $L_i^{(0)\mu}$ ($\mu = \theta, \varphi$) by $L_i^{(0)} = L_i^{(0)\mu} \partial_\mu$ and introduce N_i ($i = 1, 2, 3$) given by

$$N_1 = \sin\theta \cos\varphi, \quad N_2 = \sin\theta \sin\varphi, \quad N_3 = \cos\theta. \tag{2.5}$$

Then, it is easy to see that $L_i^{(0)\mu}$ and N_i satisfy the following relations:

$$\begin{aligned} L_i^{(0)\mu} L_i^{(0)\nu} &= -g^{\mu\nu}, & N_i N_i &= 1, & L_i^{(0)\mu} N_i &= 0, \\ L_i^{(0)\mu} \partial_\mu L_j^{(0)\nu} - L_j^{(0)\mu} \partial_\mu L_i^{(0)\nu} &= i\epsilon_{ijk} L_k^{(0)\nu}, \\ L_i^{(0)\mu} \partial_\mu N_j - L_j^{(0)\mu} \partial_\mu N_i &= 2i\epsilon_{ijk} N_k, \\ \epsilon_{ijk} N_i L_j^{(0)\mu} L_k^{(0)\nu} &= -\epsilon^{\mu\nu}, \end{aligned} \tag{2.6}$$

where $g^{\mu\nu}$ and $\epsilon^{\mu\nu}$ can be read off from (A.12). We expand X_i as [6,10]

$$X_i = \mu (iL_i^{(0)\mu} a_\mu + N_i \chi). \tag{2.7}$$

a_μ and χ turn out to be the gauge field and a scalar field on S^2 , respectively. By using the relations (2.6), we can show that (2.4) is equivalent to

$$S_{BF} = -\frac{\mu^2}{g_{BF}^2} \int \frac{d\Omega_2}{\mu^2} \text{Tr} (\chi \epsilon^{\mu\nu} f_{\mu\nu} - \chi^2), \tag{2.8}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + i[a_\mu, a_\nu]$ is the field strength. Indeed, the first term is the BF term and the second term is a mass term.

By integrating χ out in (2.8), we obtain 2d YM on S^2 :

$$S_{2dYM} = \frac{\mu^4}{g_{YM}^2} \int \frac{d\Omega_2}{\mu^2} \text{Tr} \left(\frac{1}{4} f^{\mu\nu} f_{\mu\nu} \right), \tag{2.9}$$

where $1/g_{YM}^2 = -2/(g_{BF}^2 \mu^2)$.

By dropping all the derivatives in (2.4) and rescale X_i as $X_i \rightarrow \mu X_i$, we obtain $\mathcal{N} = 1^*$ matrix model:

$$S = -\frac{1}{g^2} \text{Tr} \left(X_i^2 + \frac{i}{3} \epsilon^{ijk} X_i [X_j, X_k] \right), \tag{2.10}$$

where $1/g^2 = 4\pi/g_{BF}^2$. In the sense of the Dijkgraaf–Vafa theory [11], this matrix model is regarded as a mass deformed superpotential of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (super-YM), which gives the so-called $\mathcal{N} = 1^*$ theory. We call the matrix model (2.10) the $\mathcal{N} = 1^*$ matrix model in this Letter.

Inversely, we can obtain the BF theory with the mass term from the matrix model as follows. The matrix model (2.10) possesses the following classical solution,

$$\hat{X}_i = L_i = \begin{pmatrix} L_i^{[j_1]} & & & \\ & L_i^{[j_2]} & & \\ & & \ddots & \\ & & & L_i^{[j_N]} \end{pmatrix}, \tag{2.11}$$

where $L_i^{[j_s]}$ ($s = 1, \dots, N$) are the spin j_s representation of the $SU(2)$ generators obeying $[L_i^{[j_s]}, L_j^{[j_s]}] = i\epsilon_{ijk} L_k^{[j_s]}$, and the relation $\sum_{s=1}^N (2j_s + 1) = M$ is satisfied. We label the blocks by s . We put $2j_s + 1 = N_0 + n_s$ with N_0 and n_s integers and take the limit in which

$$N_0 \rightarrow \infty \quad \text{with} \quad \frac{N_0}{g^2} = \frac{4\pi}{g_{BF}^2} = -\frac{8\pi^2}{g_{YM}^2 A} = \text{fixed}, \tag{2.12}$$

where $A = 4\pi/\mu^2$ is the area of S^2 . Then, we can show classically [5] that the theory around (2.11) is equivalent to the theory around the following classical solution of (2.4),

$$\mu L_i^{(0)} + \hat{X}_i = \mu \text{diag}(L_i^{(q_1)}, L_i^{(q_2)}, \dots, L_i^{(q_N)}), \tag{2.13}$$

where $q_s = n_s/2$, and $L_i^{(q_s)}$ are the angular momentum operators in the presence of a monopole with the monopole charge q_s , which are given in (A.13). This theory can also be viewed as the theory around the following classical solution of (2.8),

$$\begin{aligned} \hat{\chi} &= -\text{diag}(q_1, q_2, \dots, q_N), \\ \hat{a}_\theta &= 0, \\ \hat{a}_\varphi &= (\cos\theta \mp 1) \hat{\chi}, \end{aligned} \tag{2.14}$$

where the upper sign is taken in the region $0 \leq \theta < \pi$ and the lower sign in the region $0 < \theta \leq \pi$, and \hat{a}_θ and \hat{a}_φ represent the monopole configuration.

3. Exact integration of the partition function

In this section, we evaluate the partition function of (2.10). We reduce the path-integral in the matrix model to the integral over the eigenvalues of a single matrix. In (2.10), we redefine the matrices as

$$Z = X_1 + iX_2, \quad Z^\dagger = X_1 - iX_2, \quad \Phi = X_3. \tag{3.1}$$

Z is an $M \times M$ complex matrix while Φ is an $M \times M$ Hermitian matrix. Using (2.10) and (3.1), we define the partition function of $\mathcal{N} = 1^*$ matrix model (2.10) by

$$\mathcal{Z} = \lim_{\epsilon \rightarrow 0} \int d\Phi dZ dZ^\dagger e^{-\frac{i}{g^2} \text{Tr}(Z[\Phi, Z^\dagger] + (1-i\epsilon)ZZ^\dagger + \Phi^2)}, \tag{3.2}$$

where we introduce the ‘ $-i\epsilon$ ’ term in the action to make the integral converge. Integral over Z and Z^\dagger leads to a one matrix model with respect to Φ [11–13]

$$\mathcal{Z} = \lim_{\epsilon \rightarrow 0} \int d\Phi \frac{1}{\det([\Phi, \cdot] + 1 - i\epsilon)} e^{-\frac{i}{g^2} \Phi^2},$$

where $[\Phi, \cdot]$ represents an adjoint action. Furthermore, if we diagonalize Φ as $\Phi = \text{diag}(\phi_1, \phi_2, \dots, \phi_M)$, the matrix integral reduces integrals over the eigenvalues ϕ_i

$$\mathcal{Z} = \frac{1}{M!} \lim_{\epsilon \rightarrow 0} \int \prod_i d\phi_i \prod_{i \neq j} \frac{\phi_i - \phi_j}{\phi_i - \phi_j + 1 - i\epsilon} e^{-\frac{i}{g^2} \sum_i \phi_i^2}, \tag{3.3}$$

where $\prod_{i \neq j} (\phi_i - \phi_j)$ in the numerator of the integrand comes from the Vandermonde determinant owing to the diagonalization of Φ .

¹ While k in (2.1) must be integer, such a restriction is not imposed on k in (2.4).

As a simple example, we consider the $M = 2$ case. In this case, (3.3) is explicitly written as

$$\mathcal{Z} = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int d\phi_1 d\phi_2 \frac{(\phi_1 - \phi_2)(\phi_2 - \phi_1)}{(\phi_1 - \phi_2 + 1 - i\epsilon)(\phi_2 - \phi_1 + 1 - i\epsilon)} \times e^{-\frac{i}{g^2}(\phi_1^2 + \phi_2^2)}. \quad (3.4)$$

In what follows, we frequently use the identity

$$\lim_{\epsilon \rightarrow 0} \frac{1}{x - i\epsilon} = \text{P.V.} \frac{1}{x} + i\pi \delta(x), \quad (3.5)$$

where P.V. stands for Cauchy's principal value of an integral. Applying (3.5) to (3.4) leads to

$$\begin{aligned} \mathcal{Z} &= \frac{1}{2} \text{P.V.} \int d\phi_1 d\phi_2 \frac{(\phi_1 - \phi_2)(\phi_2 - \phi_1)}{(\phi_1 - \phi_2 + 1)(\phi_2 - \phi_1 + 1)} e^{-\frac{i}{g^2}(\phi_1^2 + \phi_2^2)} \\ &\quad - \frac{i\pi}{4} \int d\phi_1 \left(e^{-\frac{i}{g^2}(\phi_1^2 + (\phi_1 + 1)^2)} + e^{-\frac{i}{g^2}(\phi_1^2 + (\phi_1 - 1)^2)} \right) \\ &= \frac{1}{2} \text{P.V.} \int d\phi_1 d\phi_2 \frac{(\phi_1 - \phi_2)(\phi_2 - \phi_1)}{(\phi_1 - \phi_2 + 1)(\phi_2 - \phi_1 + 1)} e^{-\frac{i}{g^2}(\phi_1^2 + \phi_2^2)} \\ &\quad - \frac{i\pi}{2} e^{-\frac{i}{2g^2}} \int d\phi e^{-\frac{2i}{g^2}\phi^2}. \end{aligned} \quad (3.6)$$

We generalize the above calculation to the case of arbitrary M . We apply (3.5) to the factor in the integrand of (3.3),

$$\prod_{i \neq j} \frac{\phi_i - \phi_j}{\phi_i - \phi_j + 1 - i\epsilon}, \quad (3.7)$$

and obtain the sum of the terms, each of which includes some delta functions. It is easily seen that any term giving non-vanishing contribution must be proportional to

$$\begin{aligned} &\left(\frac{-i\pi}{2} \right)^{\sum_{s=1}^N 2j_s} \delta(\phi_1^{(1)} - \phi_2^{(1)} - 1) \delta(\phi_2^{(1)} - \phi_3^{(1)} - 1) \cdots \\ &\quad \times \delta(\phi_{2j_1}^{(1)} - \phi_{2j_1+1}^{(1)} - 1) \\ &\quad \times \delta(\phi_1^{(2)} - \phi_2^{(2)} - 1) \delta(\phi_2^{(2)} - \phi_3^{(2)} - 1) \cdots \delta(\phi_{2j_2}^{(2)} - \phi_{2j_2+1}^{(2)} - 1) \\ &\quad \times \cdots \\ &\quad \times \delta(\phi_1^{(N)} - \phi_2^{(N)} - 1) \delta(\phi_2^{(N)} - \phi_3^{(N)} - 1) \cdots \\ &\quad \times \delta(\phi_{2j_N}^{(N)} - \phi_{2j_N+1}^{(N)} - 1), \end{aligned} \quad (3.8)$$

where we have reordered and relabeled the eigenvalues of Φ , ϕ_i ($i = 1, \dots, M$), as

$$\Phi = \text{diag}(\phi_1^{(1)}, \dots, \phi_{2j_1+1}^{(1)}, \phi_1^{(2)}, \dots, \phi_{2j_2+1}^{(2)}, \dots, \phi_1^{(N)}, \dots, \phi_{2j_N+1}^{(N)}), \quad (3.9)$$

with $\sum_{s=1}^N (2j_s + 1) = M$, such that the form of (3.8) is obtained. $\phi_i^{(s)}$ ($i = 1, \dots, 2j_s + 1$) represents the i th component of the s th block. (3.8) and (3.9) specify an M -dimensional irreducible representation of $SU(2)$ consisting of N blocks as seen in (2.11), with a $U(1)$ degree of freedom in each block. We label the irreducible representation by r and denote the $U(1)$ part in the s th block by a_s , putting $a_s \equiv \phi_{2j_s+1}^{(s)} + j_s$. Then, we find that the contribution of (3.8) to (3.2) is

$$\begin{aligned} \mathcal{N}_r (-i\pi)^{M-N} \prod_{s=1}^N \frac{1}{2j_s + 1} \\ \times \text{P.V.} \int \prod_{s=1}^N da_s \prod_{s \neq t} \prod_{m_s = -j_s}^{j_s} \prod_{m_t = -j_t}^{j_t} \frac{a_s + m_s - a_t - m_t}{a_s + m_s - a_t - m_t + 1} \\ \times e^{-\frac{i}{g^2} \sum_{s=1}^N \sum_{m_s = -j_s}^{j_s} (a_s + m_s)^2}, \end{aligned} \quad (3.10)$$

where

$$\mathcal{N}_r = \prod \frac{1}{(\# \text{ of blocks with the same length})!} \quad (3.11)$$

and the other factor in (3.10) is obtained from the following calculation:

$$\prod_{s=1}^N \left(\frac{-i\pi}{2} \right)^{2j_s} \sum_{s=1}^N \prod_{k=2}^{2j_s} \left(\frac{k^2}{k^2 - 1} \right)^{2j_s - k + 1} = (-i\pi)^{M-N} \prod_{s=1}^N \frac{1}{2j_s + 1}. \quad (3.12)$$

We further do some algebra for the exponent in (3.10):

$$\sum_{s=1}^N \sum_{m_s = -j_s}^{j_s} (a_s + m_s)^2 = \sum_{s=1}^N \left((2j_s + 1)a_s^2 + \frac{1}{3} j_s(j_s + 1)(2j_s + 1) \right). \quad (3.13)$$

By composing the angular momenta, we also evaluate the product appearing in (3.10):

$$\begin{aligned} &\prod_{s \neq t} \prod_{m_s = -j_s}^{j_s} \prod_{m_t = -j_t}^{j_t} \frac{a_s + m_s - a_t - m_t}{a_s + m_s - a_t - m_t + 1} \\ &= \prod_{s \neq t} \prod_{J = |j_s - j_t|}^{j_s + j_t} \prod_{m = -J}^J \frac{m + a_s - a_t}{1 + m + a_s - a_t} \\ &= \prod_{s \neq t} \prod_{J = |j_s - j_t|}^{j_s + j_t} \frac{J + a_s - a_t}{-J - 1 + a_s - a_t} \\ &= \prod_{s < t} \frac{(j_s - j_t)^2 - (a_s - a_t)^2}{(j_s + j_t + 1)^2 - (a_s - a_t)^2}. \end{aligned} \quad (3.14)$$

Gathering all the above results, we eventually find that (3.3) results in

$$\begin{aligned} \mathcal{Z} &= \sum_r \mathcal{N}_r (-i\pi)^{M-N} \prod_{s=1}^N \frac{1}{2j_s + 1} e^{-\frac{i}{3g^2} \sum_{s=1}^N \text{tr}(L_i^{[j_s]})^2} \\ &\quad \times \text{P.V.} \int \prod_{s=1}^N da_s \prod_{s < t} \frac{(j_s - j_t)^2 - (a_s - a_t)^2}{(j_s + j_t + 1)^2 - (a_s - a_t)^2} \\ &\quad \times e^{-\frac{i}{g^2} \sum_{s=1}^N (2j_s + 1)a_s^2}, \end{aligned} \quad (3.15)$$

where $L_i^{[j_s]}$ is the spin j_s representation of the $SU(2)$ generators seen in (2.11). Thus the partition function of the $\mathcal{N} = 1^*$ matrix model is decomposed into the sectors classified by the irreducible representation of $SU(2)$. Indeed, it is ensured by P.V. that the whole integral region of a_s are decomposed into these sectors without overlap, which means that the full matrix integral over X_1, X_2, X_3 is decomposed into these sectors without overlap.

4. Relation to continuum field theory

In this section, we reproduce 2d YM on S^2 from the $\mathcal{N} = 1^*$ matrix model in the large matrix size limit. As we will see, the number of the matrix blocks in the irreducible representation of $SU(2)$, N , corresponds to the rank of the gauge group of 2d YM. Since there is no overlap between the decomposed sectors in the matrix model partition function (3.15), we can extract the sectors with a fixed N . But one question arises: What type of the partition of blocks is dominated in the large matrix size limit with fixed N ?

To see this, let us investigate the ‘‘potential’’ in the N -block sectors in the partition function (3.15)

$$V(\vec{d}, \vec{\lambda}) = \sum_{s=1}^N \left(d_s a_s^2 + \frac{1}{12} d_s (d_s^2 - 1) \right) + \lambda \left(\sum_{s=1}^N d_s - M \right),$$

where we put $d_s = 2j_s + 1$ and λ is a Lagrange multiplier for the constraint $\sum_{s=1}^N d_s = M$. This potential is minimized at $a_s = 0$ and $d_s = M/N$ for $\forall s$, that is, a configuration of almost equal size blocks is dominated.

Thus we now consider the fluctuation around the dominated configuration

$$d_s \equiv N_0 + n_s,$$

where $M = NN_0$ and $\sum_{s=1}^N n_s = 0$. In the large matrix size limit, we take the limit (2.12) with fixed N , which reduces the N -block sectors to

$$\mathcal{Z}_N = C \sum_{\sum_s n_s = 0} \int \prod_{s=1}^N da'_s \prod_{1 \leq s < t \leq N} \left\{ (a'_s - a'_t)^2 - \frac{1}{4}(n_s - n_t)^2 \right\} \times e^{\frac{8\pi^2 i}{g_{\text{YM}}^2 A} \sum_{s=1}^N (a_s'^2 + n_s^2/4)}, \tag{4.1}$$

where $a'_s = a_s - \frac{1}{N}a$, $a = \sum_s a_s$ and the integral over a has been performed. Irrelevant constants and divergences are absorbed into a renormalized constant C . In this limit, the poles in the integral measure have disappeared, then we have taken integral domains as whole space of integral variables a'_s . By rescaling a'_s by $y_s \equiv 2a'_s$ and making an analytical continuation $g_{\text{YM}}^2 \rightarrow -ig_{\text{YM}}^2$, we finally obtain

$$\mathcal{Z}_N = C' \sum_{\sum_s n_s = 0} \int \prod_{s=1}^N dy_s \prod_{1 \leq s < t \leq N} \{ (y_s - y_t)^2 - (n_s - n_t)^2 \} \times e^{-\frac{2\pi^2}{g_{\text{YM}}^2 A} \sum_{s=1}^N (y_s^2 + n_s^2)}, \tag{4.2}$$

where irrelevant constants are again absorbed into a constant C' . \mathcal{Z}_N exactly agrees with the partition function of 2d $SU(N)$ YM on S^2 [14–18].²

The physical meaning of the integers n_s can be understood from the following argument. The localization theorem in the continuum $SU(N)$ YM on S^2 [15,18] says that the path integral of the partition function is localized at the solutions of the classical equation of motion

$$D_\mu f^{\mu\nu} = 0, \tag{4.3}$$

which are given by (2.14). Substituting the solution (2.14) into the YM action (2.9) which gives the equation of motion (4.3) yields

$$S_{\text{2dYM}} = \frac{\mu^4}{g_{\text{YM}}^2} \int_{S^2} \frac{d\Omega_2}{\mu^2} \text{Tr} \left(\frac{1}{4} f^{\mu\nu} f_{\mu\nu} \right) = \frac{2\pi^2}{g_{\text{YM}}^2 A} \sum_{s=1}^N n_s^2. \tag{4.4}$$

This coincides with the exponent appearing in (4.2). Thus we can identify the fluctuations of the size of blocks n_s with the monopole charges of the classical solution, which is consistent with the classical equivalence reviewed in Section 2 and suggests that the localization works for the matrix model in a manner analogous to the case of the continuum field theory.

5. Conclusion and discussion

In this Letter, we study the $\mathcal{N} = 1^*$ matrix model which is obtained by dimensionally reducing CS theory on S^3 . We decompose the matrix integral into the sectors classified by the representation of $SU(2)$. We show that the N -blocks sectors reproduce 2d $SU(N)$ YM on S^2 in the large matrix size limit.

² Note that (3.7) in [16] represents the partition function of $U(N)$ YM on S^2 . By applying the procedure in [16] to the partition function of $SU(N)$ YM in [18], it is easy to see that the corresponding expression of the partition function of $SU(N)$ YM takes the form (4.2).

We reproduced the partition function of 2d YM on S^2 from the $\mathcal{N} = 1^*$ matrix model. It is relevant to investigate whether the correlation functions of the physical observables in 2d YM on S^2 can be reproduced from the matrix model. For instance, the vev of $\text{Tr}_R e^\Phi$, where the trace is taken over a representation R of the matrix Φ , is easily calculated in the matrix model. This kind of the observables should be interpreted as a Wilson loop-like operator in 2d YM.

Our result suggests that the localization also works for the $\mathcal{N} = 1^*$ matrix model as for 2d YM theory. It has been discussed in [9] by using a different matrix model. We need further investigation on the localization mechanism of $\mathcal{N} = 1^*$ matrix model and relationship of our work to [9].

We expect that CS theories on S^3 and the lens space S^3/Z_q are obtained from the $\mathcal{N} = 1^*$ matrix model also at quantum level as $\mathcal{N} = 4$ super-YM on $R \times S^3$ is obtained from the plane wave matrix model [19–21]. In this case, the operator $\text{Tr}_R e^\Phi$ in the matrix model should correspond to the Wilson loop operator in CS theory [5], and hopefully the knot invariant is derived from the matrix model.

The $\mathcal{N} = 1^*$ matrix model is also interesting from the point of view of 4d super-YM, since the large N limit of the matrix model describes the effective superpotential of $\mathcal{N} = 1^*$ theory which is mass deformed theory from $\mathcal{N} = 4$ theory [11,13]. The different sectors of the $SU(2)$ representations that we have investigated should be related to the different Higgs branches of the $\mathcal{N} = 1^*$ theory. The effective superpotential in the different Higgs branches can be investigated by using the direct integration of the matrix model partition function.

While we have extracted the N -block sectors ‘by hand’ in the present Letter, we may expect that the large N limit with the large N_0 limit realizes 2d large N YM on S^2 naturally, as the planar limit of $\mathcal{N} = 4$ super-YM is realized in [19]. The large N limit seems relevant for the following reason. It has been already pointed out that the $1/N$ expansion of 2d YM describes the genus expansion of (non-critical) string theory [22]. One can deduce a world-sheet description of string theory from the partition function of 2d YM. On the other hand, in this Letter, we have derived the partition function of 2d YM from the $\mathcal{N} = 1^*$ matrix model in the large matrix size limit. Matrix models are often regarded as regularization of (non-critical) string theory, giving world-sheet description in the large matrix size limit. Our investigation strongly suggests the relationship between the matrix models in the large matrix size limit, gauge theory in the large N limit and string theory and the $\mathcal{N} = 1^*$ matrix model is a good example to understand the relationship (see also [23,24]). Further investigation of the $\mathcal{N} = 1^*$ matrix model may shed lights on nonperturbative definition of string theory.

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Appendix A. S^3 and S^2

In this appendix, we summarize some useful facts about S^3 and S^2 (see also [6,19]). S^3 is viewed as the $SU(2)$ group manifold. We parameterize an element of $SU(2)$ in terms of the Euler angles as

$$g = e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{-i\psi\sigma_3/2}, \quad (\text{A.1})$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$, $0 \leq \psi < 4\pi$. The periodicity with respect to these angle variables is expressed as

$$(\theta, \varphi, \psi) \sim (\theta, \varphi + 2\pi, \psi + 2\pi) \sim (\theta, \varphi, \psi + 4\pi). \quad (\text{A.2})$$

The isometry of S^3 is $SO(4) = SU(2) \times SU(2)$, and these two $SU(2)$'s act on g from left and right, respectively. We construct the right-invariant 1-forms,

$$dg g^{-1} = -i\mu E^i \sigma_i / 2, \quad (\text{A.3})$$

where the radius of S^3 is given by $2/\mu$. They are explicitly given by

$$\begin{aligned} E^1 &= \frac{1}{\mu} (-\sin\varphi d\theta + \sin\theta \cos\varphi d\psi), \\ E^2 &= \frac{1}{\mu} (\cos\varphi d\theta + \sin\theta \sin\varphi d\psi), \\ E^3 &= \frac{1}{\mu} (d\varphi + \cos\theta d\psi), \end{aligned} \quad (\text{A.4})$$

and satisfy the Maurer–Cartan equation

$$dE^i - \frac{\mu}{2} \epsilon_{ijk} E^j \wedge E^k = 0. \quad (\text{A.5})$$

The metric is constructed from E^i as

$$ds^2 = E^i E^i = \frac{1}{\mu^2} (d\theta^2 + \sin^2\theta d\varphi^2 + (d\psi + \cos\theta d\varphi)^2). \quad (\text{A.6})$$

The Killing vector dual to E^i is given by

$$\mathcal{L}_i = -\frac{i}{\mu} E_i^M \partial_M, \quad (\text{A.7})$$

where $M = \theta, \varphi, \psi$ and E_i^M are inverse of E^i_M . The explicit form of the Killing vector is

$$\begin{aligned} \mathcal{L}_1 &= -i \left(-\sin\varphi \partial_\theta - \cot\theta \cos\varphi \partial_\varphi + \frac{\cos\varphi}{\sin\theta} \partial_\psi \right), \\ \mathcal{L}_2 &= -i \left(\cos\varphi \partial_\theta - \cot\theta \sin\varphi \partial_\varphi + \frac{\sin\varphi}{\sin\theta} \partial_\psi \right), \\ \mathcal{L}_3 &= -i \partial_\varphi. \end{aligned} \quad (\text{A.8})$$

Because of the Maurer–Cartan equation (A.5), the Killing vector satisfies the $SU(2)$ algebra $[\mathcal{L}_i, \mathcal{L}_j] = i\epsilon_{ijk} \mathcal{L}_k$.

One can also regard S^3 as a $U(1)$ bundle over $S^2 = SU(2)/U(1)$. S^2 is parametrized by θ and φ and covered with two local patches: the patch I defined by $0 \leq \theta < \pi$ and the patch II defined by $0 < \theta \leq \pi$. In the following expressions, the upper sign is taken in the patch I while the lower sign in the patch II. The element of $SU(2)$ in (A.1) is decomposed as

$$g = L \cdot h \quad (\text{A.9})$$

with

$$\begin{aligned} L &= e^{-i\varphi\sigma_3/2} e^{-i\theta\sigma_2/2} e^{\pm i\psi\sigma_3/2}, \\ h &= e^{-i(\psi \pm \varphi)\sigma_3/2}. \end{aligned} \quad (\text{A.10})$$

L represents an element of S^2 , while h represents the fiber $U(1)$. The fiber direction is parametrized by $y = \psi \pm \varphi$. Note that L has no φ -dependence for $\theta = 0, \pi$. The zweibein of S^2 is given by the $i = 1, 2$ components of the left-invariant 1-form, $-iL^{-1} dL = \mu e^i \sigma_i / 2$. It takes the form

$$\begin{aligned} e^1 &= \frac{1}{\mu} (\pm \sin\varphi d\theta + \sin\theta \cos\varphi d\varphi), \\ e^2 &= \frac{1}{\mu} (-\cos\varphi d\theta \pm \sin\theta \sin\varphi d\varphi). \end{aligned} \quad (\text{A.11})$$

This zweibein gives the standard metric of S^2 with the radius $1/\mu$:

$$ds^2 = e^i e^i = \frac{1}{\mu^2} (d\theta^2 + \sin^2\theta d\varphi^2). \quad (\text{A.12})$$

Making a replacement $\partial_y \rightarrow -iq$ in (A.8) leads to the angular momentum operator in the presence of a monopole with magnetic charge q at the origin [25]:

$$\begin{aligned} L_1^{(q)} &= i(\sin\varphi \partial_\theta + \cot\theta \cos\varphi \partial_\varphi) - q \frac{1 \mp \cos\theta}{\sin\theta} \cos\varphi, \\ L_2^{(q)} &= i(-\cos\varphi \partial_\theta + \cot\theta \sin\varphi \partial_\varphi) - q \frac{1 \mp \cos\theta}{\sin\theta} \sin\varphi, \\ L_3^{(q)} &= -i\partial_\varphi \mp q, \end{aligned} \quad (\text{A.13})$$

where q is quantized as $q = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$, because y is a periodic variable with the period 4π . These operators act on the local sections on S^2 and satisfy the $SU(2)$ algebra $[L_i^{(q)}, L_j^{(q)}] = i\epsilon_{ijk} L_k^{(q)}$. Note that when $q = 0$, these operators are reduced to the ordinary angular momentum operators (A.13) on S^2 (or R^3), which generate the isometry group of S^2 , $SU(2)$. The $SU(2)$ acting on g from left survives as the isometry of S^2 .

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