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# Solving the convection–diffusion equation by means of the optimal q-homotopy analysis method (Oq-HAM)

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ABSTRACT

The main aim of this paper is to propose a new and simple algorithm namely the optimal q-homotopy analysis method (Oq-HAM), to obtain approximate analytical solutions of the convection diffusion (CD) equation. Comparison of Oq-HAM with the homotopy analysis method (HAM) and the homotopy perturbation method (HPM) is made. The results reveal that the Oq-HAM has more accuracy than the others. Finally, numerical example is given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions shows that the proposed method is very efficient and computationally attractive. A new efficient approach is proposed to obtain the optimal value of convergence controller parameter $\hbar$ to guarantee the convergence of the obtained series solution.



## 1. Introduction

Nonlinear partial differential equations are known to describe a wide variety of phenomena not only in physics, where applications extend over magneto fluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasma, just to name a few, but also in biology and chemistry, and several other fields.

Several methods have been suggested to solve nonlinear equations. These methods include the homotopy perturbation method (HPM) [1], Lyapunov's artificial small parameter method [2], Adomian decomposition method [3,4], variation iterative method [5,6] and so on. Homotopy analysis method (HAM), first proposed by Liao in his Ph.D dissertation [7], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [8–14]. The HAM contains a certain auxiliary parameter $h$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution [15]. In 2005 Liao [16] has pointed out that the HPM is only a special case of the HAM (The case of $h = -1$). El-Tawil and Huseen [17] proposed a method namely q-homotopy analysis method (q-HAM) which is a more general method of homotopy analysis method (HAM). The q-HAM contains an auxiliary parameter $n$ as well as $h$ such that the cases of (q-HAM; $n = 1$) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to solve many types of nonlinear problems [17–24]. In addition to the auxiliary parameter $h$ present in HAM, q-HAM contains a fraction factor that gives it much flexibility to adjust and control the convergence region and rate of convergence of the series of solution. Hamed et al. [25–29], used the improved and generalized $(G'/G)$-expansion method to construct explicit traveling wave solutions involving parameters of the fractional generalized Kolmogrove–Petrovskii Piskunov equation. Also some partial differential equations such as: $(3+1)$-dimensional potential-YTSF equation, combined KdV and mKdV equations, FitzHugh–Nagumo equation, fifth-order KdV equation and combined KdV and mKdV equations. The exact solutions for the generalized fractional Kolmogrove–Petrovskii Piskunov equation by using the generalized $(G'/G)$)-expansion method are obtained by Hamed et al. [30]. Hamed et al. [31], applied a complex transformation using a modified chain rule to convert fractional generalized coupled MKDV and KDV of an ordinary differential equation and then we obtain a new exact solution. Hamed et al. [32], applied a fractional complex transform with optimal homotopy analysis method (OHAM) to obtain numerical and analytical solutions for the nonlinear time–space fractional Fornberg–Whitham. The results reveal that the method is very effective, simple and controls the convergence region and rate of convergence of the series solution. Mohamed S. Mohamed and Hamed [33], used the optimal q-homotopy analysis method

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(Oq-HAM) to solve the fractional Fornberg–Whitham equation. Shaheed N. Huseen [34], illustrated the application of a newly developed efficient method namely, optimal q-homotopy analysis method (Oq-HAM) for solving second order initial and boundary value problems. The results reveal that the Oq-HAM has high accuracy to determine the convergence-control parameter; hence the results match well with the exact solutions and this proves the effectiveness of the method.

Convection diffusion (CD) equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection. This equation is a parabolic partial differential equation which describes physical phenomena. Some numerical and analytical methods for solving it have been presented so far, which are sorted by explicit and implicit methods in general [35].

In [35] the CD equation was solved with HAM, the Adomian decomposition method was used to solve CD equation and in the homotopy perturbation method was applied to find the solution of CD equation. In this work, we consider the following linear CD equation and the Oq-HAM is applied to solve it:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2},\quad \mathbf{0}\leqslant\mathbf{x}\leqslant\mathbf{l},\ t\geqslant 0, \tag{1}$$

subject to the initial condition

$$u(x,0)=f(x),\quad \mathbf{0}\leqslant\mathbf{x}\leqslant\mathbf{l}, \tag{2}$$

where $c$ and $\gamma$ are arbitrary constants.

The rest of this paper is as follows. In Section 2, we introduce the basic definitions of optimal q-homotopy analysis method. In Section 3, examples solved to show the importance of the proposed method and in the last section the conclusion is stated.

## 2. Basic idea of the optimal q-homotopy analysis method (Oq-HAM)

To describe the basic ideas of the optimal Oq-HAM for nonlinear partial differential equations. Let us consider the following nonlinear partial differential equation:

$$N[u(x,t)]=0, \tag{3}$$

where $N$ is linear and nonlinear operator for this problem, $x$ and $t$ denote the independent variables, and $u(x,t)$ is an unknown function. We first construct the zero-order deformation equation as follows:

$$(1-nq)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)N[\phi(x,t;q)], \tag{4}$$

where $n>1, q\in[0,\frac{1}{n}]$ is the embedding parameter, $h\neq 0$ is an auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator and $u_0(x,t)$ is an initial guess. Clearly, when $q=0$ and $q=\frac{1}{n}$, Eq. (4) becomes:

$$\phi(x,t;0)=u_0(x,t),\quad \phi\left(x,t;\frac{1}{n}\right)=u(x,t), \tag{5}$$

respectively. so, as $q$ increases from 0 to $\frac{1}{n}$ the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. If $u_0(x,t)$, $L,h,H(x,t)$ are chosen appropriately, solution of Eq. (5) exists for $q\in[0,\frac{1}{n}]$.

Taylor series expression of $\phi(x,t,q)$ with respect to $q$ in the form

$$\varphi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{6}$$

where

$$\varphi_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m\varphi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{7}$$

We assume that the auxiliary linear operator, the initial guess, the auxiliary parameter $h$ and the auxiliary function $H(x,t)$ is selected such that the series (7) is convergent when $q\to\frac{1}{n}$, then the approximate solution (6) takes the form:

$$u(x,t)=\varphi\left(x,t;\frac{1}{n}\right)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{8}$$

Let us define the vector

$u_n^{\rightarrow}(t)=\{u_0(x,t),u_1(x,t),u_2(x,t),\ldots,u_n(x,t)\}$.

Differentiating (4) $m$ times with respect to $q$, then setting $q=0$ and dividing then by $m!$, we have the $m$th-order deformation equation [7,8] as

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hH(x,t)\mathbf{R}_m(u_{m-1}^{\rightarrow}(x,t)), \tag{9}$$

with initial conditions

$$u_m^{(k)}(x,t)=0,\quad k=0,1,2,3,\ldots,m-1$$

where

$$\mathbf{R}_m(u_{m-1}^{\rightarrow}(x,t))=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathbf{N}[\varphi(x,t;q)]}{\partial q^{m-1}}\right|_{q=0}, \tag{10}$$

and

$$\chi_m=\begin{cases}0 & m\leqslant 1,\\ n & m>1.\end{cases} \tag{11}$$

It should be emphasized that $u_m(x,t)$ for $m\geqslant 1$ is governed by the linear equation (9) with linear boundary conditions that come from the original problem. Due to the existence of the factor $(\frac{1}{n})^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $n=1$ in Eq. (4), standard HAM can be reached.

The $h$-curves cannot tell us the best convergence-control parameter, which corresponds to the fastest convergent series. In 2007, Yabushita et al. [36] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called optimization method to find out the two optimal convergence-control parameters by means of the minimum of the squared residual error of governing equations. In 2008, Akyildiz and Vajravelu [37] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

Basiri et al. [38] and Mohamed S. Mohamed et al. [13] have discussed the optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error.

Let $\Delta(h)$ denote the square residual error of the governing equation (3) and expressed as:

$$\Delta(h)=\int_{\Omega}(N[u_n(t)])^2d\Omega, \tag{12}$$

where

$$u_m(t)=u_0(t)+\sum_{k=1}^{m}u_k(t). \tag{13}$$

The optimal value of the auxiliary parameter $h$ is given by solving the following nonlinear algebraic equation

$$\frac{d\Delta(h)}{dh}=0. \tag{14}$$

## 3. Numerical results

To demonstrate the effectiveness of the Oq-HAM algorithm discussed above, example of variation problems will be studied in this section. In this section, example is solved according to the mentioned algorithm in previous section. The results have been provided by Mathematica.

Consider the CD equation:

$$u_t - \mathbf{0.02}u_{xx} + 0.1u_x = 0 \quad 0 \leqslant x \leqslant 1, \tag{15}$$

with the initial condition

$$\mathbf{u}(\mathbf{x},\mathbf{0}) = e^{1.1771243x},$$

with the exact solution

$$\mathbf{u}(\mathbf{x},\mathbf{t}) = e^{1.1771243x-t}. \tag{16}$$

This problem is solved by HAM [35]. For Oq-HAM solution we choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t},$$

with the property that

$$L[c] = 0, \quad \text{where } c \text{ is constant.}$$

We define a nonlinear operator as

$$\mathbf{N}[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} - 0.02\frac{\partial \phi^2(x,t;q)}{\partial x^2} + 0.1\frac{\partial \phi(x,t;q)}{\partial x}. \tag{17}$$

We construct the zero order deformation equation

$$(1-nq)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)].$$

For $q = 0$ and $q = 1$, we can write

$$\begin{aligned} \varphi(x,t;0) &= \ u_0(x,t) = u(x,0), \\ \varphi(x,t;1) &= \ u(x,t). \end{aligned}$$

We can take $H(x,t) = 1$, and the $m$th-order deformation equation is

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = h\mathbf{R}_m(\vec{u}_{m-1}(x,t)), \tag{18}$$

with the initial conditions for $m \geqslant 1$

$$\mathbf{u_m}(\mathbf{x},\mathbf{0}) = 0, \tag{19}$$

where $\chi_m$ as defined by (11) and

$$\mathbf{R}_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - 0.02\frac{\partial^2}{\partial x^2} + 0.1\frac{\partial}{\partial x}u_{m-1}. \tag{20}$$

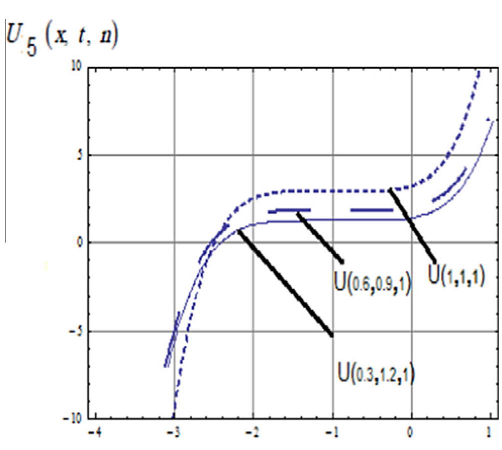


**Fig. 1.** $h$-curve for the HAM (q-HAM; $n = 1$) approximation solution $U_5$ ($x$, $t$; 1) of problem (15) at different values of $x$ and $h_{\text{optmal}} = -0.97$.

Now the solution of the $m$th-order deformation equations for $m \geqslant 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\int \mathbf{R}_m(\vec{u}_{m-1}(x,s))ds + c_1, \tag{21}$$

where the constant of integration $c_1$ is determined by the initial conditions (19). Then, the components of the solution using Oq-HAM are

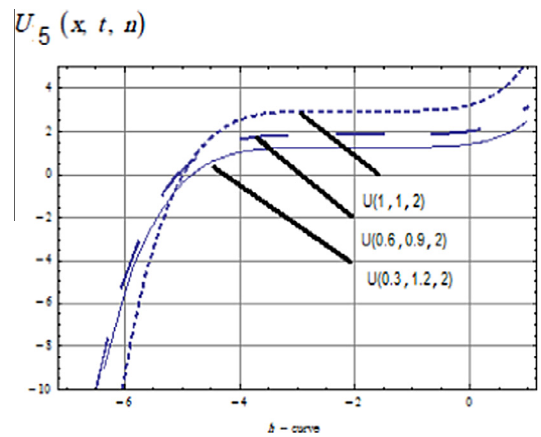


**Fig. 2.** $h$-curve for the HAM (q-HAM; $n = 2$) approximation solution $U_5$ ($x$, $t$; 2) of problem (15) at different values of $x$ and $h_{\text{optmal}} = -1.99$.

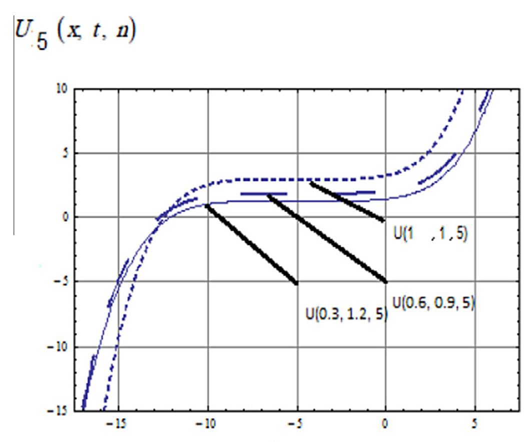


**Fig. 3.** $h$-curve for the HAM (q-HAM; $n = 5$) approximation solution $U_5$ ($x$, $t$; 5) of problem (15) at different values of $x$ and $h_{\text{optmal}} = -4.5$.

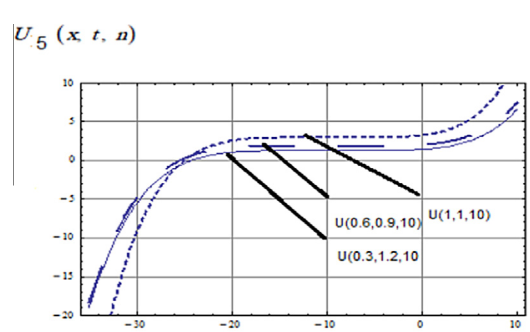


**Fig. 4.** $h$-curve for the HAM (q-HAM; $n = 10$) approximation solution $U_5$ ($x$, $t$; 10) of problem (15) at different values of $x$ and $h_{\text{optmal}} = -11.45$.

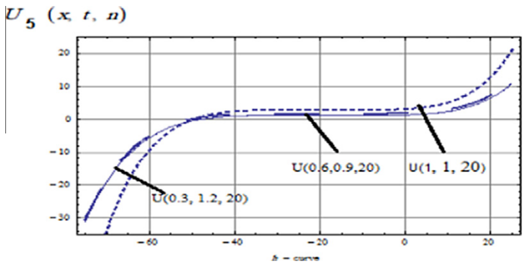


**Fig. 5.** $h$-curve for the HAM (q-HAM; $n = 20$) approximation solution $U_5\,(x, t; 20)$ of problem (15) at different values of $x$ and $th_{\text{optmal}} = -16.75$.

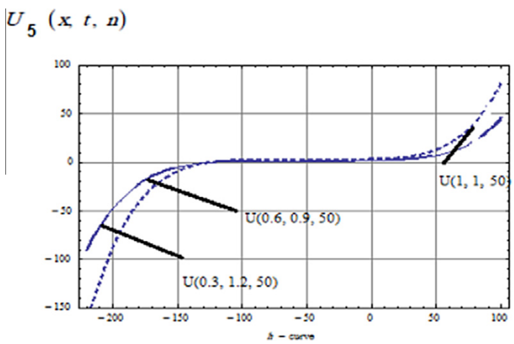


**Fig. 6.** $h$-curve for the HAM (q-HAM; $n = 50$) approximation solution $U_5\,(x, t; 50)$ of problem (15) at different values of $x$ and $h_{\text{optmal}} = -22.05$.

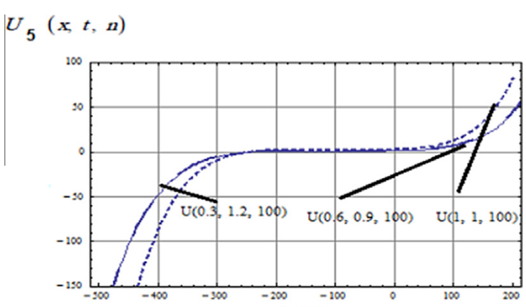


**Fig. 7.** $h$-curve for the HAM (q-HAM; $n = 100$) approximation solution $U_5\,(x, t; 100)$ of problem (15) at different values of $x$ and $h_{\text{optmal}} = -55.65$.

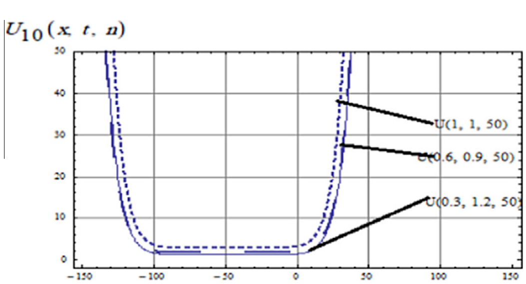


**Fig. 8.** $h$-curve for the HAM (q-HAM; $n = 50$) approximation solution $U_{10}\,(x, t; 50)$ of problem (15) at different values of $x$ and $h_{\text{optmal}} = -10.25$.

$$u(x,0) = e^{1.17712434446770x},$$
$$u_1(x,t) = 0.09e^{1.17712434446770x}ht,$$
$$u_2(x,t) = ht(0.09h + 0.09n) + 0.00405e^{1.17712434446770x}h^2t^2, \qquad (22)$$
$$\ldots$$
$$\ldots$$

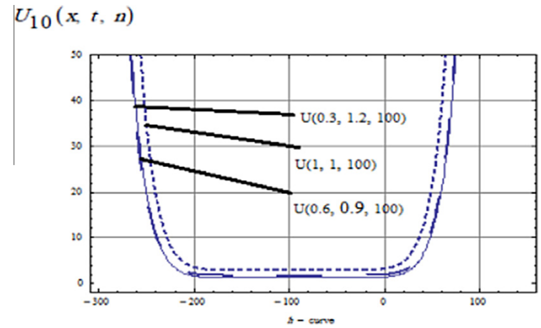


**Fig. 9.** $h$-curve for the HAM (q-HAM; $n = 100$) approximation solution $U_{10}\,(x, t; 100)$ of problem (15) at different values of $x$ and $h_{\text{optmal}} = -15.43$.

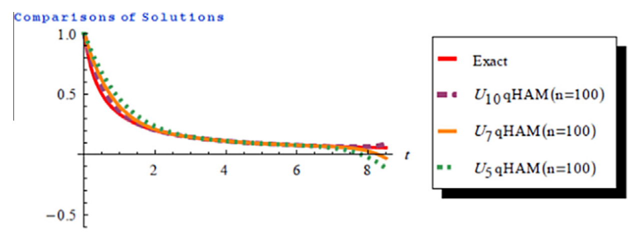


**Fig. 10.** Comparison between $U_5$, $U_7$, $U_{10}$ of (q-HAM; $nn = 100$) and exact solution of (16) at $x = 1$ with $h_{\text{optmal}} = -15.43$, $0 < t \leqslant 8$.

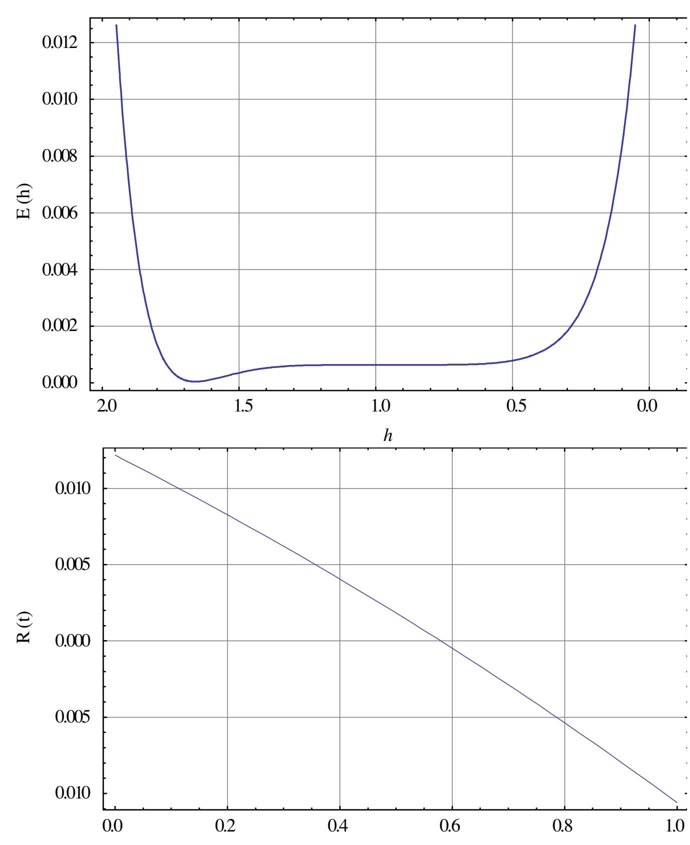


**Fig. 11.** The residual of the 5th order approximation for $N = 1$ and $h_{\text{optmal}} = -1.65954$ and $x = 0.2$.

**Table 1**
The errors of the approximate solution at the points ($x$; 0.1).

| $x$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| Error | 1.11022E−16 | 2.22045E−16 | 4.44089E−16 | 8.88178E−16 | 1.33227E−15 | 4.44089E−16 |

According to the optimal q-homotopy analysis, we can conclude that

$$u(x,t;n;h) \simeq U_m(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{23}$$

Eq. (23) is an approximate solution to the problem (15) in terms of convergence parameter $h$ and $n$. Then we have,

$$\begin{aligned} u_{app} &= u_0(x,t) + \left(\frac{1}{n}\right)u_1(x,t) + \left(\frac{1}{n}\right)^2 u_2(x,t) + \left(\frac{1}{n}\right)^3 u_3(x,t) \\ &\quad + \left(\frac{1}{n}\right)^4 u_4(x,t) + \cdots \\ &= e^{1.17712434446770x} + 0.09e^{1.17712434446770x}ht + ht(0.09h \\ &\quad + 0.09n) + 0.00405e^{1.17712434446770x}h^2t^2 + \cdots. \end{aligned} \tag{24}$$

As special case if $n = 1$ and $h = -1$, then we obtain the same result in [26].

Eq. (24) is an approximate solution to the problem (15) in terms of the convergence parameters $h$ and $n$. To find the valid region of $h$, the $h$-curves given by the 5th order q-HAM approximation at different values of $x$, $t$, and $n$ are drawn in Figs. 1–9. These figures show the interval of $h$ at which the value of $U_5(x, t; n)$ is constant at certain values of $x$, $t$ and $n$. We choose the horizontal line parallel to *x-axis* ($h$) as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution. From these figures, the valid intersection region of $h$ for the values of $x$, $t$ and $n$ in the curves becomes larger as $n$ increases.

**Remark 1.** Using the $h$-curve, it is possible to locate the valid region of $h$ which corresponds to the line segment nearly parallel to the horizontal axis.

Fig. 10 shows the comparison between $U_5$, $U_7$ and $U_{10}$ using different values of $n$ with the exact solution (16). Therefore, based on these present results, we can say that q-HAM is more effective than HAM and HPM.

Fig. 11 shows the residual error of the approximate solution of Eq. (15) at $N = 1$ and $x = 0.2$ (see Fig. 11 and Table 1).

## 4. Comparison between the optimal q-homotopy analysis method and the homotopy analysis method

A. Fallahzadeh and Shakibi. [35], used the HAM to solve a linear convection–diffusion equation. The convergence theorem was proved to ensure the validity and reliability of the proposed method. The examples showed the accuracy and efficiency of the method. Therefore, the HAM is able to evaluate a satisfactory solution for CD equation.

B. In our study, we applied optimal q-homotopy analysis method to get the approximate solutions of CD equation. It was shown that the convergence of Oq-HAM is faster than the convergence of HAM and this method is powerful and efficient in finding exact and approximate solutions for equations.

## 5. Conclusions

An approximate solution of CD equation was found by using the Oq-homotopy analysis method. The results show that the convergence region of series solutions obtained by Oq-HAM is increasing as q is decreased. The comparison of Oq-HAM with the HAM and HPM [35] was made. It was shown that the convergence of Oq-HAM is faster than the convergence of HAM and HPM. The results show that the method is powerful and efficient in finding exact and approximate solutions for equations and also, this method uses simple computation with acceptable solution. Our results show that Oq-HAM can be applied to many complicated linear and strongly nonlinear partial differential equations. The method to choose the appropriate auxiliary parameter $h$ for better convergence of the series solution is given in the $h$-curve. All the numerical analyses in this study were carried out using *Mathematica 9*.

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