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Signless Laplacian spectral radii of graphs with given chromatic number

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ABSTRACT

Let G be a simple graph with vertices $v_1, v_2, \ldots, v_n$, of degrees $\Delta = d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n = \delta$, respectively. Let A be the $(0, 1)$-adjacency matrix of G and D be the diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$. $Q(G) = D + A$ is called the signless Laplacian of G. The largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius or Q-spectral radius of G. Denote by $\chi(G)$ the chromatic number for a graph G. In this paper, for graphs with order n, the extremal graphs with both the given chromatic number and the maximal Q-spectral radius are characterized, the extremal graphs with both the given chromatic number $\chi \neq 4, 5, 6, 7$ and the minimal Q-spectral radius are characterized as well.

1. Introduction

All graphs considered here are simple, connected and undirected. Denote by $V(G)$ the vertex set and $E(G)$ the edge set for a graph G. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$, of degrees $\Delta = d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n = \delta$, respectively. If vertex v_i is adjacent to v_j, we denote by $v_i \sim v_j$. We denote by $N_G(v)$ or $N(v)$ the neighbor set of vertex v in graph G. The degree of vertex v in graph G, denoted by $d_G(v)$ or $d(v)$, is equal to $|N_G(v)|$. We denote by K_n, P_n, C_n for a complete graph, a path and a cycle with order n, respectively, in this paper.

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Denote by $|M|$ the determinant for a square matrix M. Let $A = (a_{ij})_{n\times n}$ be the $(0, 1)$-adjacency matrix of G, and let D be the diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$. The matrix $L(G) = D - A$ is the Laplacian of G, while $Q(G) = D + A$ is called the signless Laplacian of G.

The matrix $Q(G)$ is symmetric and nonnegative, and when G is connected, it is irreducible. If M is the $n \times m$ vertex-edge incidence matrix of the (n, m)-graph G, then $Q(G) = MM^T$. Thus $Q(G)$ is positive semi-definite, and its eigenvalues can be arranged as:

$$q = q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n \geqslant 0.$$

q is called the signless Laplacian spectral radius or Q-spectral radius of G. The Q-characteristic polynomial of a graph G, denoted by $P_Q(\lambda)$ or $P_{Q(G)}(\lambda)$, is the characteristic polynomial of $Q(G)$. Denoted by $\widetilde{G}$ the complement of graph G, and denoted by $P_{\widetilde{Q}}(\lambda)$ or $P_{Q(\widetilde{G})}(\lambda)$ the Q-characteristic polynomial of $\widetilde{G}$.

Computer investigations of graphs with up to 11 vertices [4] suggest that the spectrum of $D + A$ performs better than the spectrum of A or $D - A$ in distinguishing non-isomorphic graphs, study of the spectrum of $D + A$ is of interests in the literature (see [2,6], for example) recently.

In this paper, we consider the signless Laplacian spectral radii of graphs with order n and given chromatic number χ. For graphs with order n, the extremal graphs with both the given chromatic number and the maximal Q-spectral radius are characterized, the extremal graphs with both the given chromatic number $\chi \neq 4, 5, 6, 7$ and the minimal Q-spectral radius are characterized as well. This paper is organized as follows: Section 1 introduces the basic ideas and their supports; Section 2 characterizes the extremal graphs with the maximal Q-spectral radius; Section 3 characterizes the extremal graphs with the minimal Q-spectral radius.

2. Maximal Q-spectral radius

Definition 2.1 *[3].* A semi-edge walk (of length k) in an (undirected) graph G is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that for any $i = 1, 2, \ldots, k$, the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i.

Lemma 2.2 [3]. *Let Q be the signless Laplacian matrix of a graph G. The (i, j)-entry of the matrix Q^k, denoted by $q^{(k)}_{(i,j)}$, is equal to the number of semi-edge walks of length k starting at vertex i and terminating at vertex j.*

Let G be a graph with n vertices and m edges, N_k $(k \geqslant 0)$ denote the number of all the semi-edge walks with length k in G, and let $N_0 = 1$. Clearly, $N_1 = 2\sum_{i=1}^{n} d_i = 4m$. Let $H_Q(t) = \sum_{k=0}^{\infty} N_k t^k$ be the generating function of N_k $(k \geqslant 0)$. Then we have the following lemma.

Theorem 2.3. *Let G be a simple connected graph with n vertices. Then*

$$H_Q(t) = \frac{1}{t}\left(\frac{(-1)^n P_{\widetilde{Q}}\left(\frac{tn-2t-1}{t}\right)}{P_Q\left(\frac{1}{t}\right)} - 1\right).$$

Proof. Suppose M is a nonsingular $n \times n$ square matrix and J is a $n \times n$ square matrix in which all the entries are 1. Let $\|M\|_1 = \sum_{i,j} M_{i,j}$. Then the adjugate $adjM = |M|M^{-1}$ and $|M+xJ| = |M|+x\|adjM\|_1$. Let I denote the identity matrix. Note that

$$\sum_{k=0}^{\infty} Q^k t^k = (I - tQ)^{-1} = |I - tQ|^{-1} adj(I - tQ) \quad \left(t \leqslant \frac{1}{q}\right),$$

$$\sum_{k=0}^{\infty} \|Q^k\|_1 t^k = \sum_{k=0}^{\infty} N_k t^k = |I - tQ|^{-1}\|adj(I - tQ)\|_1.$$

Hence

$$H_Q(t) = \frac{\|adj(I - tQ)\|_1}{|(I - tQ)|}.$$

Let $M = I - tQ$. Then

$$|I - tQ + tJ| = |I - tQ| + t\|adj(I - tQ)\|_1. \tag{1}$$

Note that

$$I - tQ + tJ = I - tQ + tJ + (n-2)tI - (n-2)tI = (2t - tn + 1)I + t\widetilde{Q}.$$

From (1), we know that $\|adj(I - tQ)\|_1 = \frac{1}{t}(|I - tQ + tJ| - |I - tQ|)$. Hence

$$H_Q(t) = \frac{1}{t}\left(\frac{|(2t - tn + 1)I + t\widetilde{Q}|}{|I - tQ|} - 1\right) = \frac{1}{t}\left(\frac{(-1)^n P_{\widetilde{Q}}\left(\frac{tn-2t-1}{t}\right)}{P_Q\left(\frac{1}{t}\right)} - 1\right). \quad \square$$

Corollary 2.4. *Let $G = K_{n_1,n_2,\ldots,n_s}$ be a complete s-partite graph with $\sum_{i=1}^{s} n_i = n$. Then*

$$P_Q(\lambda) = (-1)^n \left(\sum_{i=1}^{s} \frac{n_i}{n - 2n_i - \lambda} + 1\right) \prod_{i=1}^{s}(n - 2n_i - \lambda)(n - n_i - \lambda)^{n_i - 1}. \tag{2}$$

Proof. Let $H_{\widetilde{Q}}(t)$ denote the semi-edge walk number generating function of $\widetilde{G}$. Let B_i denote a complete graph with n_i vertices and $N_k^{(i)}$ denote the number of semi-edge walks with length k in B_i. By Theorem 2.3, then

$$H_{\widetilde{Q}}(t) = \frac{1}{t}\left(\frac{(-1)^n P_Q\left(\frac{tn-2t-1}{t}\right)}{P_{\widetilde{Q}}\left(\frac{1}{t}\right)} - 1\right) = \sum_{k=0}^{\infty}\sum_{i=1}^{s} N_k^{(i)} t^k$$
$$= \sum_{k=0}^{\infty}\sum_{i=1}^{s} n_i(2(n_i - 1))^k t^k = \sum_{i=1}^{s} \frac{n_i}{1 - 2(n_i - 1)t}.$$

Hence

$$(-1)^n P_Q\left(\frac{tn - 2t - 1}{t}\right) = \left(t\sum_{i=1}^{s} \frac{n_i}{1 - 2(n_i - 1)t} + 1\right) P_{\widetilde{Q}}\left(\frac{1}{t}\right).$$

Let $\lambda = \frac{tn-2t-1}{t}$. Then $t = \frac{1}{n-2-\lambda}$, and then (2) follows immediately. $\square$

Lemma 2.5 [8]. *Let $\mathscr{M}_n = \{M \mid M$ is a $n \times n$ square matrix$\}$. Suppose $A, B \in \mathscr{M}_n$ $(n \geqslant 2)$, A is nonnegative irreducible and $|B| \leqslant A$ (namely $|B_{i,j}| \leqslant A_{i,j}$ for each pair of i, j). Denote by $\rho(A)$ the largest eigenvalue of A. For any eigenvalue λ of B, we have $|\lambda| \leqslant \rho(A)$, and equality holds if and only if $B = e^{i\theta}DAD^{-1}$ where $\rho(A)e^{i\theta} = \lambda$ and D is a diagonal U-matrix.*

Definition 2.6. The Turán graph $T_{(n,r)}$ is an n-vertex graph formed by partitioning the set of vertices into r parts of equal or nearly-equal size, and connecting two vertices by an edge whenever they belong to two different parts. In fact, $T_{(n,r)}$ is an n-vertex complete r-partite graph with each part of equal or nearly-equal size.

Theorem 2.7. *Suppose complete s-partite graph $G = K_{n_1,n_2,\ldots,n_s}$ with $\sum_{i=1}^{s} n_i = n$. Then $q(G) \leqslant q(T_{n,s})$ with equality if and only if $G \cong T_{n,s}$.*

Proof. Denote by $\mu(G)$ the Laplacian spectral radius for a graph G. From spectral graph theory, we know that $\mu(G) = n$ if $\widetilde{G}$ is not connected. By Lemma 2.5, we get $q(G) \geqslant \mu(G) \geqslant n$. By Corollary 2.4, we know that $q(G)$ is the largest zero of $\sum_{i=1}^{s} \frac{n_i}{n-2n_i-\lambda} + 1 = 0$.

Suppose $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s$. If $n_1 - n_s \geqslant 2$, let

$$f(\delta, \lambda) = \frac{n_1 - \delta}{n - 2(n_1 - \delta) - \lambda} + \sum_{i=2}^{s-1} \frac{n_i}{n - 2n_i - \lambda} + \frac{n_s + \delta}{n - 2(n_s + \delta) - \lambda} + 1$$

where $0 \leqslant \delta \leqslant \frac{n_1 - n_s}{2}$. So $f(0, q(G)) = 0$. Taking the derivative with respect to δ, for $\lambda \geqslant q(G)$, we have

$$\frac{df(\delta, \lambda)}{d\delta} = \frac{\lambda - n}{(2(n_1 - \delta) + \lambda - n)^2} - \frac{\lambda - n}{(2(n_s + \delta) + \lambda - n)^2} \leqslant 0.$$

Hence $f(\delta, \lambda)$ is decreasing with respect to δ for $\lambda \geqslant q(G)$, and $f(\delta, \lambda)$ is strictly decreasing with respect to δ if $0 < \delta < \frac{n_1 - n_s}{2}$. Thus, for $\lambda \geqslant q(G)$, $f(\delta, \lambda) \leqslant 0$ if $\delta \leqslant \frac{n_1 - n_s}{2}$ and $f(\delta, \lambda) < 0$ if $0 < \delta < \frac{n_1 - n_s}{2}$. This means that if we increase n_s by δ and decrease n_1 by δ in G, then $q(G)$ will increase. □

Corollary 2.8. *Let G be a simple connected graph with n vertices and chromatic number χ. Then $q(G) \leqslant q(T_{n,\chi})$ with equality if and only if $G \cong T_{n,\chi}$.*

Proof. It is well known that $q(G + e) > q(G)$ if $e \notin E(G)$. Hence the Q-spectral radius of G is less than or equal to the Q-spectral radius of a complete χ-partite graph. Then the Corollary follows from Theorem 2.7. □

3. Minimal Q-spectral radius

An internal path in some graph is a path $v_0v_1 \cdots v_{k+1}$ for which $d(v_0), d(v_{k+1}) \geqslant 3$ and $d(v_i) = 2$ for $i = 1, \ldots, k$ (here $k \geqslant 0$, or $k \geqslant 2$ whenever $v_0 = v_{k+1}$).

Lemma 3.1 [2]. *Let G_{uv} be the graph obtained from a connected graph G by subdividing its edge uv. Then the following holds:*

(i) *if uv belongs to an internal path then $q(G_{uv}) < q(G)$;*
(ii) *if $G \neq C_n$ for some $n \geqslant 3$, and if uv is not on any internal path of G, then $q(G_{uv}) > q(G)$. Otherwise, if $G = C_n$ then $q(G_{uv}) = q(G) = 4$.*

Lemma 3.2 [2]. *Let $G(k, l)$ $(k, l \geqslant 0)$ be the graph obtained from a non-trivial connected graph G by attaching pendant paths of lengths k and l at some vertex v. If $k \geqslant l \geqslant 1$ then $q(G(k, l)) > q(G(k + 1, l - 1))$.*

Lemma 3.3 [7]. *Let A be an $n \times n$ real symmetric irreducible nonnegative matrix and $X \in R^n$ be an unit vector. If $\rho(A) = X^TAX$, then $AX = \rho(A)X$.*

Definition 3.4. We say that a graph G is (color) k-critical if $\chi(G) = k$ and $\chi(H) < \chi(G)$ for every proper subgraph H of G.

Lemma 3.5 [5]. *Suppose the chromatic number $\chi(G) = k \geqslant 4$. Let G be a k-critical graph on more than k vertices (so $G \neq K_k$). Then*

$$|E(G)| \geqslant \left(\frac{k-1}{2} + \frac{k-3}{2(k^2 - 2k - 1)}\right)|V(G)|.$$

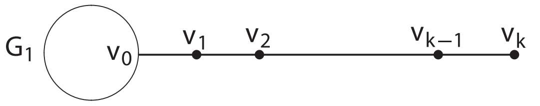

Fig. 3.1. $G_1v_0v_1P_{k+1}$.

Lemma 3.6 [9]. *Let $P(n, \lambda)$ denote the (adjacency) characteristic polynomial of path P_n and $r \geqslant 2$ be a fixed real number. If $\lambda \geqslant r$, then for any $n \geqslant 0$, $P(n+1, \lambda) > \frac{r+\sqrt{r^2-4}}{2}P(n, \lambda) > 0$, where $P(0, \lambda) = 1$.*

Let G, H be two disjoint connected graphs, and $GuvH$ denotes the graph obtained from the union of graphs G and H by adding edge uv ($u \in V(G)$), $v \in V(H)$. Let $G + v$ be obtained from G by adding a pendant edge uv and let $H + u$ be obtained from H by adding a pendant edge vu.

Lemma 3.7 [2]. *Let G, H be two connected graphs. Then*

$$P_{Q(GuvH)}(\lambda) = \frac{1}{\lambda}\left(P_{Q(G+v)}(\lambda)P_{Q(H)}(\lambda) + P_{Q(H+u)}(\lambda)P_{Q(G)}(\lambda) - (\lambda-2)P_{Q(G)}(\lambda)P_{Q(H)}(\lambda)\right).$$

Let $G = G_1v_0v_1P_k$ denote the graph obtained from graph G_1 and path P_k by adding an edge v_0v_1 between the vertex v_0 of G_1 and a pedant vertex v_1 of P_k (in G, $v_0v_1P_k$ is also called the pedant path of G_1, see Fig. 3.1). If G_1 is a complete graph K_s, $G_1v_0v_1P_k$ can be denoted by $K_s^{(k)}$ ($K_s^{(k)}$ is also known as path complete graph which is denoted by $PC_{n,1,k}$, see [1]).

Lemma 3.8. *Suppose $d_{G_1}(v_0) \geqslant 2$, $P_k = v_1v_2\cdots v_k$. Let connected graph $G = G_1v_0v_1P_k$ (see Fig. 3.1) with order n, q_i $(1 \leqslant i \leqslant n)$ be the eigenvalues of $Q(G)$. Suppose $X_i = (x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,k}, x_{i,k+1}, \ldots, x_{i,n-1})^T$ is an eigenvector corresponding to eigenvalue q_i and $x_{i,s}$ $(0 \leqslant s \leqslant n-1)$ corresponds to vertex v_s. Let $f_1 = q_i - 1$ and $f_{j+1} = q_i - 2 - \frac{1}{f_j}$. Then $x_{i,k-j} = f_jx_{i,k-j+1}$ for $1 \leqslant j \leqslant k$, and we have*

(i) $\frac{q_i-2}{2} \leqslant f_j \leqslant q_i - 2$, *if* $q_i \geqslant 4$, $j \geqslant 2$;
(ii) $f_j < f_{j-1}$ *if* $q_i \geqslant 4$, $2 \leqslant j \leqslant k$.

Proof. Note that $x_{i,k-1} = (q_i-1)x_{i,k} = f_1x_{i,k}$ and $x_{i,k-2} + x_{i,k} = (q_i-2)x_{i,k-1}$, we get

$$x_{i,k-2} = \left(q_i - 2 - \frac{1}{q_i-1}\right)x_{i,k-1} = \left(q_i - 2 - \frac{1}{f_1}\right)x_{i,k-1} = f_2x_{i,k-1}.$$

So, we can get $f_{j+1} = q_i - 2 - \frac{1}{f_j}$ and $x_{i,k-j} = f_jx_{i,k-j+1}$ for $1 \leqslant j \leqslant k$ by induction.

(i) It is easy to check that $\frac{q_i-2}{2} \leqslant f_2 \leqslant q_i - 2$ if $q_i \geqslant 4$. Suppose $\frac{q_i-2}{2} \leqslant f_j \leqslant q_i - 2$ for $2 \leqslant j < N$, then

$$-\frac{2}{q_i-2} \leqslant -\frac{1}{f_{N-1}} \leqslant -\frac{1}{q_i-2}, \quad q_i - 2 - \frac{2}{q_i-2} \leqslant f_N \leqslant q_i - 2 - \frac{1}{q_i-2}$$

because $f_N = q_i - 2 - \frac{1}{f_{N-1}}$. Note that $q_i - 2 - \frac{2}{q_i-2} \geqslant \frac{q_i-2}{2}$ if $q_i \geqslant 4$, so $\frac{q_i-2}{2} \leqslant f_N \leqslant q_i - 2$. By induction, then (i) follows.

(ii) By (i), $f_2 < f_1$ clearly. Suppose $f_j \leqslant f_{j-1}$ for $2 \leqslant j < N$, then

$$q_i - 2 - \frac{1}{f_{N-1}} \leqslant_i -2 - \frac{1}{f_{N-2}},$$

namely $f_N \leqslant f_{N-1}$. By induction, then (ii) follows. $\square$

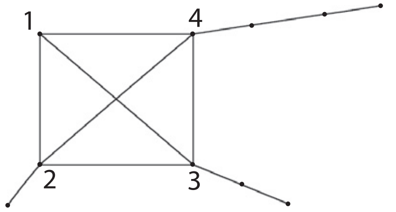

Fig. 3.2. $K_4^{(1,0;2,1;3,2;4,3)}$.

Corollary 3.9. *Suppose $d_{G_1}(v_0) \geqslant 2$, $P_k = v_1v_2 \cdots v_k$. Let connected graph $G = G_1v_0v_1P_k$ (see Fig. 3.1) with order n. Suppose $X = (x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{n-1})^T$ is Perron vector of $Q(G)$ in which x_s $(0 \leqslant s \leqslant n-1)$ corresponds to vertex v_s. If $|E(G)| \geqslant n$, then*

$$x_0 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k.$$

Proof. If $|E(G)| \geqslant n$, then G contains cycle. Hence $q(G) \geqslant 4$. Thus the corollary follows from Lemma 3.8. □

Corollary 3.10. *Suppose $P_k = v_1v_2 \cdots v_k$. Let connected graph $G = G_1v_0v_1P_k = K_{n-k}^{(k)}$ (see Fig. 3.1). Suppose $X = (x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_{n-1})^T$ is Perron vector of $Q(G)$ in which x_s $(0 \leqslant s \leqslant n-1)$ corresponds to vertex v_s. If $n-k \geqslant 3$, then*

$$x_{k+1} = x_{k+2} = \cdots = x_{n-1} \geqslant x_j$$

for $j = 1, 2, \ldots, k$.

Proof. By symmetry, we have $x_{k+1} = x_{k+2} = \cdots = x_{n-1}$. Note that

$$q(G)x_{k+1} = (2n-2k-3)x_{k+1} + x_0, \quad q(G)x_1 = 2x_1 + x_0 + x_2,$$

then

$$x_0 = (q(G) - (2n-2k-3))x_{k+1}, \quad x_0 = (q(G)-2)x_1 - x_2 \geqslant (q(G)-3)x_1,$$

and $x_{k+1} \geqslant x_1$. Then the corollary follows from Corollary 3.9. □

Let $V(K_t) = \{v_1, v_2, \ldots, v_t\}$. $K_t^{(1,s_1;2,s_2;\ldots;t,s_t)}$ $(t \geqslant 3, s_i \geqslant 0, i = 1, 2, \ldots, t)$ is obtained by adding an edge between v_i $(1 \leqslant i \leqslant t)$ and a pendant vertex of path P_{s_i} (see Fig. 3.2, for example). In particular, $s_i = 0$ means that no path joining to v_i. Then we have the following lemma.

Lemma 3.11. *If there are at least two in $\{s_i | 1 \leqslant i \leqslant t\}$ which are all at least 1 in $K_t^{(1,s_1;2,s_2;\ldots;t,s_t)}$ $(t \geqslant 3, t + \sum_{i=1}^t s_i = n)$, then $q(K_t^{(1,s_1;2,s_2;\ldots;t,s_t)}) > q(K_t^{n-t})$.*

Proof. In K_t^{n-t}, let $V(K_t) = \{v_1, v_2, \ldots, v_t\}$, and let the pedant path be $\mathcal{P} = v_1v_{t+1}v_{t+2}\ldots v_n$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of K_t^{n-t} in which x_i corresponds v_i $(1 \leqslant i \leqslant n)$. From Corollary 3.10, we know that $x_2 = x_3 = \cdots = x_t \geqslant x_j$ $(t+1 \leqslant j \leqslant n)$. Among $s_1, s_2, \ldots, s_t$, suppose $s_{i_1} \geqslant 1, s_{i_2} \geqslant 1, \ldots, s_{i_\theta} \geqslant 1$ $(1 \leqslant \theta \leqslant t)$. Let

$$\begin{aligned} G^* = K_t^{n-t} - &\left(v_{n-s_{i_2}+1}v_{n-s_{i_2}} + v_{n-s_{i_2}-s_{i_3}+1}v_{n-s_{i_2}-s_{i_3}} + \cdots + v_{n-\sum_{l=2}^{l=\theta} s_{i_l}+1}v_{n-\sum_{l=2}^{l=\theta} s_{i_l}}\right) \\ &+ v_2v_{n-s_{i_2}+1} + v_2v_{n-s_{i_2}-s_{i_3}+1} + \cdots + v_\theta v_{n-\sum_{l=2}^{l=\theta} s_{i_l}+1}. \end{aligned}$$

Then

$$\begin{aligned} X^T(Q(G^*)-Q(K_t^{n-t}))X &= 2(x_2+2x_{n-s_{i_2}+1}+x_{n-s_{i_2}})(x_2-x_{n-s_{i_2}}) \\ &\quad +(x_3+2x_{n-s_{i_2}-s_{i_3}+1}+x_{n-s_{i_2}-s_{i_3}})(x_3-x_{n-s_{i_2}-s_{i_3}}) \\ &\quad +\cdots+\left(x_\theta+2x_{n-\sum_{l=2}^{l=\theta}s_{i_l}+1}+x_{n-\sum_{l=2}^{l=\theta}s_{i_l}}\right)\left(x_\theta-x_{n-\sum_{l=2}^{l=\theta}s_{i_l}}\right) \\ &\geqslant 0. \end{aligned}$$

This means that $q(G^*) \geqslant q(K_t^{n-t})$. Suppose that $q(G^*) = q(K_t^{n-t})$. Then $X^T(Q(G^*)-Q(K_t^{n-t}))X = 0$ and $X^TQ(G^*)X = q(K_t^{n-t})$. By Lemma 3.3, we know that X is also the Perron vector of G^*. But in G^*,

$$Q_2(G^*)X = (2t-3)x_2 + x_1 + x_{n-s_{i_2}+1} > q(K_t^{n-t})x_2,$$

where $Q_2(G^*)$ denotes the row corresponding to vertex v_2. So, $q(G^*) > q(K_t^{n-t})$. Note that $G^* \cong K_t^{(1,s_1;2,s_2;\ldots;t,s_t)}$, hence $q(K_t^{(1,s_1;2,s_2;\ldots;t,s_t)}) > q(K_t^{n-t})$. □

Lemma 3.12. *Let G be a connected graph with chromatic number $\chi \geqslant 4$ and order $\chi+1$. Then G contains K_χ as subgraph, and $q(G) \geqslant q(K_\chi^1)$ with equality if and only if $G \cong K_\chi^1$.*

Proof. Suppose $V(G) = \{v_1, v_2, \ldots, v_{\chi+1}\}$. In a χ-coloring of G, there must be two vertices colored the same color. For convenience, suppose the two vertices are v_1, v_2. Then vertices $v_3, v_4, \ldots, v_{\chi+1}$ induce a complete graph in G. Let $S = \{v_3, v_4, \ldots, v_{\chi+1}\}$. There must be $(S\backslash N_G(v_1)) \bigcap (S\backslash N_G(v_2)) = \phi$, and no case $|S\backslash N_G(v_1)| \geqslant 1$, $|S\backslash N_G(v_2)| \geqslant 1$. Otherwise, G is $\chi-1$ colorable, contradicting that G is χ colorable. Hence there must be at least one of v_1, v_2 whose degree is $\chi-1$, and then G contains K_χ as subgraph. Note that for a connected graph H, if $e \notin E(H)$, then $q(H+e) > q(H)$, so $q(G) \geqslant q(K_\chi^1)$, and equality holds if and only if $G \cong K_\chi^1$. □

Lemma 3.13. *If $k \geqslant 8$, $l \geqslant 2$, then*

$$q(K_k^l) < 2(k-1) + \frac{2(k-3)}{k^2-2k-1}.$$

Proof. Note that

$$P_{Q(K_k^1)}(\lambda) = \begin{vmatrix} \lambda-(k-1) & -1 & \cdots & -1 & -1 & 0 \\ -1 & \lambda-(k-1) & \cdots & -1 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-(k-1) & -1 & 0 \\ -1 & -1 & \cdots & -1 & \lambda-k & -1 \\ 0 & 0 & \cdots & 0 & -1 & \lambda-1 \end{vmatrix}_{(k+1)\times(k+1)}$$

$$= \begin{vmatrix} \lambda-(k-1) & -1 & \cdots & -1 & -1 & 0 \\ -1 & \lambda-(k-1) & \cdots & -1 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-(k-1) & -1 & 0 \\ -1 & -1 & \cdots & -1 & \lambda-(k-1) & -1 \\ 0 & 0 & \cdots & 0 & -\lambda & \lambda-1 \end{vmatrix}_{(k+1)\times(k+1)}$$

$$= -\lambda \begin{vmatrix} \lambda-(k-1) & -1 & \cdots & -1 \\ -1 & \lambda-(k-1) & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda-(k-1) \end{vmatrix}_{(k-1)\times(k-1)}$$

$$+ (\lambda-1) \begin{vmatrix} \lambda-(k-1) & -1 & \cdots & -1 & -1 \\ -1 & \lambda-(k-1) & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-(k-1) & -1 \\ -1 & -1 & \cdots & -1 & \lambda-(k-1) \end{vmatrix}_{k\times k}$$

$$= -\lambda(\lambda-(2k-3))(\lambda-(k-2))^{k-2} + (\lambda-1)(\lambda-(2k-2))(\lambda-(k-2))^{k-1}$$
$$= (\lambda-(k-2))^{k-2}(\lambda^3-(3k-2)\lambda^2+(2k^2-k-3)\lambda-2(k-1)(k-2)).$$

By Lemma 3.7, we have

$$\begin{aligned} P_{Q(K_k^l)}(\lambda) &= \frac{1}{\lambda}\left\{P_{Q(K_k^1)}(\lambda)P_{Q(P_l)}(\lambda) + P_{Q(K_k)}(\lambda)(P_{Q(P_{l+1})}(\lambda) - (\lambda-2)P_{Q(P_l)}(\lambda))\right\} \\ &= \frac{1}{\lambda}(\lambda-k+2)^{k-2}\{(\lambda^3-(3k-2)\lambda^2+(2k^2-k-3)\lambda \\ &\quad -2(k-1)(k-2))P_{Q(P_l)}(\lambda) + (\lambda-2(k-1))(\lambda-k+2) \\ &\quad \times(P_{Q(P_{l+1})}(\lambda) - (\lambda-2)P_{Q(P_l)}(\lambda))\} \\ &= \frac{1}{\lambda}(\lambda-k+2)^{k-2}\{((1-k)\lambda+2(k-1)(k-2))P_{Q(P_l)}(\lambda) \\ &\quad +(\lambda^2-(3k-4)\lambda+2(k-1)(k-2))P_{Q(P_{l+1})}(\lambda)\}. \end{aligned} \tag{3}$$

Notice that for a graph G with incidence matrix M, we have

$$MM^T = D + A, \quad M^TM = 2I_l + A_l,$$

where A_l is the adjacency matrix of the line graph of G. So

$$P_{Q(P_l)}(\lambda) = \lambda P(l-1, \lambda-2), \quad P_{Q(P_{l+1})}(\lambda) = \lambda P(l, \lambda-2).$$

By Lemma 3.6, when $\lambda \geqslant 4$, then

$$\begin{aligned} (3) > (\lambda-k+2)^{k-2}P(l-1,\lambda-2)\{(1-k)\lambda+2(k-1)(k-2) \\ + \frac{\lambda-2+\sqrt{(\lambda-2)^2-4}}{2}(\lambda^2-(3k-4)\lambda+2(k-1)(k-2))\}. \end{aligned}$$

Let

$$\begin{aligned} g(\lambda) = (1-k)\lambda+2(k-1)(k-2) \\ + \frac{\lambda-2+\sqrt{(\lambda-2)^2-4}}{2}(\lambda^2-(3k-4)\lambda+2(k-1)(k-2))\ (\lambda \geqslant 4). \end{aligned} \tag{4}$$

Notice that, when $\lambda \geqslant 4$,

$$(4) = (1-k)\lambda + 2(k-1)(k-2) + \frac{\lambda - 2 + \sqrt{\lambda(\lambda-4)}}{2}(\lambda^2 - (3k-4)\lambda + 2(k-1)(k-2)) \geqslant (1-k)\lambda + 2(k-1)(k-2) + (\lambda-3)(\lambda^2 - (3k-4)\lambda + 2(k-1)(k-2)).$$

Let

$$f(\lambda) = (1-k)\lambda + 2(k-1)(k-2) + (\lambda-3)(\lambda^2 - (3k-4)\lambda + 2(k-1)(k-2)).$$

Then

$$f(2(k-1) + \frac{2(k-3)}{k^2-2k}) = \left(2k - 5 + \frac{2(k-3)}{k^2-2k}\right)\frac{2(k-3)}{k^2-2k}\left(\frac{2(k-3)}{k^2-2k} + k\right) - \left(2 + \frac{2(k-3)}{k^2-2k}\right)(k-1) > \frac{2k^2 - 20k + 36}{k-2} > 0 \quad (k \geqslant 8). \tag{5}$$

For $g(\lambda)$, taking the derivative with respect to λ, we get

$$g^{'}(\lambda) = 1 - k + \left(\frac{1}{2} + \frac{\lambda - 2}{2\sqrt{(\lambda-2)^2 - 4}}\right)(\lambda^2 - (3k-4)\lambda + 2(k-1)(k-2)) + \frac{\lambda - 2 + \sqrt{(\lambda-2)^2 - 4}}{2}(2\lambda - 3k + 4) > 1 - k + \lambda^2 - (3k-4)\lambda + 2(k-1)(k-2) + (\lambda-3)(2\lambda - 3k + 4).$$

Hence, when $k \geqslant 4$, $\lambda \geqslant 2k - 1$, then

$$g'(\lambda) \geqslant g'(2k-1) > 2k^2 - 6 > 0,$$

and then $g(\lambda)$ is increasing with respect to λ. From (5) we know that, when $\lambda \geqslant 2(k-1) + \frac{2(k-3)}{k^2-2k}$, then $g(\lambda) > 0$. So

$$q(K_k^l) < 2(k-1) + \frac{2(k-3)}{k^2-2k} < 2(k-1) + \frac{2(k-3)}{k^2-2k-1}. \quad \square$$

Corollary 3.14. *Let G be a connected graph with chromatic number* $\chi \geqslant 8$, *and with order n. If G does not contain* K_χ *as subgraph, then* $q(G) \geqslant q(K_\chi^{n-\chi})$ *with equality if and only if* $G \cong K_\chi^{n-\chi}$.

Proof. By Lemma 3.12, we know that $n \geqslant \chi + 2$. We assume that G contains a χ-critical subgraph H. Then $q(G) \geqslant q(H)$. By Lemma 3.5, we have

$$q(G) \geqslant q(H) \geqslant \frac{4|E(H)|}{|V(H)|} \geqslant 2(k-1) + \frac{2(k-3)}{k^2-2k-1}.$$

Then the Corollary follows from Lemma 3.13. $\square$

Theorem 3.15. *Let G be a connected graph with chromatic number* χ $(\chi \neq 4, 5, 6, 7)$ *and n vertices. Then*

(1) *If* $\chi = 2$, *then* $q(G) \geqslant q(P_n)$ *with equality if and only if* $G \cong P_n$;
(2.1) *If* $\chi = 3$ *and* n *is odd, then* $q(G) \geqslant q(C_n)$ *with equality if and only if* $G \cong C_n$;
(2.2) *If* $\chi = 3$ *and* n *is even, then* $q(G) \geqslant q(C_{n-1}^1)$ *with equality if and only if* $G \cong C_{n-1}^1$, *where* C_{n-1}^1 *is obtained from the cycle* C_{n-1} *by adding one pendent edge;*
(3) *If* $\chi \geqslant 8$, *then* $q(G) \geqslant q(K_\chi^{(l)})$ *with equality if and only if* $G \cong K_\chi^{(l)}$.

Proof. Fact 1. For a connected graph H, $q(H + e) > q(H)$ if $e \notin E(H)$.

Fact 2. For a connected graph H, $q(H - v) < q(H)$ if $v \in V(H)$.

Using Lemma 3.2 and Fact 1 repeatedly, (1) follows.

Using Facts 1, 2 and Lemma 3.1 repeatedly, (2.1), (2.2) follows.

We prove (3) next.

Case 1. G does not contain K_χ as subgraph. By Lemma 3.12, then $n \geqslant \chi + 2$, and then (3) follows from Lemma 3.13 and Corollary 3.14.

Case 2. G contains K_χ as subgraph.

If $n = \chi + 1$, then (3) follows from Lemma 3.12.

If $n \geqslant \chi + 2$, using Fact 1, Lemma 3.2 repeatedly, then (3) follows from Lemma 3.11. □

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