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A nonlinear Korn inequality on a surface

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Abstract

Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{R}^3$ be a smooth immersion. The main purpose of this paper is to establish a "nonlinear Korn inequality on the surface $\boldsymbol{\theta}(\overline{\omega})$", asserting that, under ad hoc assumptions, the $\boldsymbol{H}^1(\omega)$-distance between the surface $\boldsymbol{\theta}(\overline{\omega})$ and a deformed surface is "controlled" by the $\boldsymbol{L}^1(\omega)$-distance between their fundamental forms. Naturally, the $\boldsymbol{H}^1(\omega)$-distance between the two surfaces is only measured up to proper isometries of $\mathbb{R}^3$.

This inequality implies in particular the following interesting per se sequential continuity property for a sequence of surfaces. Let $\boldsymbol{\theta}^k:\omega\to\mathbb{R}^3$, $k\geqslant 1$, be mappings with the following properties: They belong to the space $\boldsymbol{H}^1(\omega)$; the vector fields normal to the surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, are well defined a.e. in ω and they also belong to the space $\boldsymbol{H}^1(\omega)$; the principal radii of curvature of the surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, stay uniformly away from zero; and finally, the fundamental forms of the surfaces $\boldsymbol{\theta}^k(\omega)$ converge in $\boldsymbol{L}^1(\omega)$ toward the fundamental forms of the surface $\boldsymbol{\theta}(\overline{\omega})$ as $k\to\infty$. Then, up to proper isometries of $\mathbb{R}^3$, the surfaces $\boldsymbol{\theta}^k(\omega)$ converge in $\boldsymbol{H}^1(\omega)$ toward the surface $\boldsymbol{\theta}(\overline{\omega})$ as $k\to\infty$.

Such results have potential applications to nonlinear shell theory, the surface $\boldsymbol{\theta}(\overline{\omega})$ being then the middle surface of the reference configuration of a nonlinearly elastic shell.

Résumé

Soit ω un domaine de $\mathbb{R}^2$ et soit $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{R}^3$ une immersion régulière. L'objet principal de cet article est d'établir une "inégalité de Korn non linéaire sur la surface $\boldsymbol{\theta}(\overline{\omega})$", affirmant que, moyennant des hypothèses convenables, la distance dans $\boldsymbol{H}^1(\omega)$ entre la surface $\boldsymbol{\theta}(\overline{\omega})$ et une surface déformée est "controlée" par la distance dans $\boldsymbol{L}^1(\omega)$ entre leurs formes fondamentales. Naturellement, la distance dans $\boldsymbol{H}^1(\omega)$ entre les deux surfaces est mesurée seulement modulo les isométries propres de $\mathbb{R}^3$.

Cette inégalité implique en particulier la propriété de continuité séquentielle suivante, intéressante par elle-même. Soit $\boldsymbol{\theta}^k:\omega\to\mathbb{R}^3$, $k\geqslant 1$, des applications ayant les propriétés suivantes : Elles appartiennent à l'espace $\boldsymbol{H}^1(\omega)$; les champs de vecteurs normaux aux surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, sont définis presque partout dans ω et appartiennent aussi à l'espace $\boldsymbol{H}^1(\omega)$; les modules des rayons de courbure principaux des surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, sont uniformément minorés par une constante strictement positive ; finalement, les formes fondamentales des surfaces $\boldsymbol{\theta}^k(\omega)$ convergent dans $\boldsymbol{L}^1(\omega)$ vers les formes fondamentales de la surface $\boldsymbol{\theta}(\overline{\omega})$ lorsque $k\to\infty$. Alors, à des isométries propres de $\mathbb{R}^3$ près, les surfaces $\boldsymbol{\theta}^k(\omega)$ convergent dans $\boldsymbol{H}^1(\omega)$ vers la surface $\boldsymbol{\theta}(\overline{\omega})$ lorsque $k\to\infty$.

Ce type de résultat a des applications potentielles à la théorie non linéaire des coques, la surface $\boldsymbol{\theta}(\overline{\omega})$ étant alors la surface moyenne de la configuration de référence d'une coque non linéairement élastique.

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1. Introduction

Let ω be a bounded and connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary, let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be a smooth enough immersion, and let $\boldsymbol{\theta}(\overline{\omega})$ be the middle surface of the reference configuration of a *nonlinearly elastic shell*. Let $\mathbb{S}^2$ denote the space of all symmetric matrices of order two.

Let $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ denote the first and second fundamental forms of the "undeformed" middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and let $(\tilde{a}_{\alpha\beta})$ and $(\tilde{b}_{\alpha\beta})$ denote the first and second fundamental forms of a "deformed" surface $\tilde{\boldsymbol{\theta}}(\overline{\omega})$ associated with a smooth enough mapping $\tilde{\boldsymbol{\theta}}$, whose normal vector field is well defined a.e. in ω (so as to insure that the second fundamental form $(\tilde{b}_{\alpha\beta})$ is well defined). Then the *change of metric tensor field* $(\tilde{a}_{\alpha\beta} - a_{\alpha\beta}) : \overline{\omega} \to \mathbb{S}^2$ and the *change of curvature tensor field* $(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}) : \overline{\omega} \to \mathbb{S}^2$ associated with such a *deformation* $\tilde{\boldsymbol{\theta}}$ play a major rôle in *two-dimensional nonlinear shell theories*.

For instance, the well-known stored energy function w_K proposed by Koiter [22, Eqs. (4.2), (8.1), and (8.3)] for modeling shells made with a homogeneous and isotropic elastic material takes the form

$$w_K = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau}(\tilde{a}_{\sigma\tau} - a_{\sigma\tau})(\tilde{a}_{\alpha\beta} - a_{\alpha\beta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau}\big(\tilde{b}_{\sigma\tau} - b_{\sigma\tau}\big)\big(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}\big),$$

where 2ε is the (constant) thickness of the shell and

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu\big(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}\big),$$

where $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$ and $\lambda > 0$ and $\mu > 0$ denote the Lamé constants of the elastic material.

The stored energy functions of a *nonlinearly elastic membrane shell* and of a *nonlinearly elastic flexural shell* have been identified and fully justified by means of Γ-convergence theory in two key contributions, respectively by Le Dret and Raoult [25] and Friesecke, James, Mora and Müller [20] (a nonlinearly elastic shell is a "membrane shell" if there are no nonzero admissible deformations of its middle surface S that preserve the metric of S; otherwise, the shell is a "flexural shell"). It then turns out that the stored energy function of a membrane shell is an *ad hoc* quasiconvex envelope that is only a function of the change of metric tensor field, and that the stored energy function w_F of a flexural shell is of the form

$$w_F = \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau}\big(\tilde{b}_{\sigma\tau} - b_{\sigma\tau}\big)\big(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}\big),$$

i.e., it is only a function of the change of curvature tensor field (in this case, the minimizers of the total energy are sought in a set of admissible deformations that preserve the metric of S; see again [20], or Ciarlet and Coutand [11]).

Conceivably, an alternative approach to existence theory in nonlinear shell theory could thus regard the *change of metric and change of curvature tensors*, or equivalently, the *first and second fundamental forms* $(\tilde{a}_{\alpha\beta})$ *and* $(\tilde{b}_{\alpha\beta})$ *of the unknown deformed middle surface*, as the *primary unknowns*, instead as the deformation $\tilde{\boldsymbol{\theta}}$ itself as is customary.

This observation is one of the reasons underlying the present study, the other one being *differential geometry* per se. As such, it is a continuation of the works initiated by Ciarlet [8] and continued by Ciarlet and Mardare [18] for "smooth" topologies, respectively those of the spaces $\mathcal{C}^m(\omega)$ and $\mathcal{C}^m(\overline{\omega})$.

Let us henceforth restrict ourselves to deformations $\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3)$ whose normal vector field $\tilde{\boldsymbol{a}}_3 = \frac{\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|}$, where $\tilde{\boldsymbol{a}}_\alpha = \partial_\alpha \tilde{\boldsymbol{\theta}}$, is well defined a.e. in ω and satisfies $\tilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbb{R}^3)$. The covariant components of the three fundamental forms of the deformed surface $\tilde{\boldsymbol{\theta}}(\omega)$, viz.,

$$\tilde{a}_{\alpha\beta} = \tilde{\boldsymbol{a}}_\alpha \cdot \tilde{\boldsymbol{a}}_\beta, \qquad \tilde{b}_{\alpha\beta} = -\partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \tilde{\boldsymbol{a}}_\beta, \qquad \tilde{c}_{\alpha\beta} = \partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \partial_\beta \tilde{\boldsymbol{a}}_3,$$

are then well defined as functions in $L^1(\omega)$ and clearly, the mapping

$$(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{a}}_3) \in \big[H^1\big(\omega; \mathbb{R}^3\big)\big]^2 \to \big((\tilde{a}_{\alpha\beta}), \big(\tilde{b}_{\alpha\beta}\big), (\tilde{c}_{\alpha\beta})\big) \in \big[L^1\big(\omega; \mathbb{S}^2\big)\big]^3,$$

restricted to such deformations $\tilde{\boldsymbol{\theta}}$, is *continuous*.

One of the purposes of this paper is to show that, *under appropriate assumptions, the converse also holds*, i.e., *the surfaces* $\tilde{\boldsymbol{\theta}}(\omega)$, *together with their normal vector fields* $\tilde{\boldsymbol{a}}_3$, *depend continuously on their three fundamental forms, the topologies being those of the same spaces, viz.,* $[H^1(\omega;\mathbb{R}^3)]^2$ *and* $[L^1(\omega;\mathbb{S}^2)]^3$.

This continuity result is itself a consequence of the following *"nonlinear Korn inequality on a surface"*, which constitutes the main result of this paper (see Theorem 4.1): Assume that $\boldsymbol{\theta}\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ is an immersion with a normal vector field $\boldsymbol{a}_3\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$. Then, for each $\varepsilon>0$, there exists a constant $c(\boldsymbol{\theta},\varepsilon)$ with the following property: Given any mapping $\tilde{\boldsymbol{\theta}}\in H^1(\omega;\mathbb{R}^3)$ such that the normal vector field $\tilde{\boldsymbol{a}}_3$ to the surface $\tilde{\boldsymbol{\theta}}(\omega)$ is well defined and satisfies $\tilde{\boldsymbol{a}}_3\in H^1(\omega;\mathbb{R}^3)$, and such that the principal radii of curvature $\widetilde{R}_\alpha$ of the surface $\tilde{\boldsymbol{\theta}}(\omega)$ satisfy $|\widetilde{R}_\alpha|\geqslant\varepsilon$ a.e. in ω, there exists a vector $\boldsymbol{b}:=\boldsymbol{b}(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},\varepsilon)\in\mathbb{R}^3$ and a matrix $\boldsymbol{R}=\boldsymbol{R}(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},\varepsilon)\in\mathbb{O}^3_+$ such that

$$\begin{aligned}&\big\|\big(\boldsymbol{b}+\boldsymbol{R}\tilde{\boldsymbol{\theta}}\big)-\boldsymbol{\theta}\big\|_{H^1(\omega;\mathbb{R}^3)}+\varepsilon\|\boldsymbol{R}\tilde{\boldsymbol{a}}_3-\boldsymbol{a}_3\|_{H^1(\omega;\mathbb{R}^3)}\\&\quad\leqslant c(\boldsymbol{\theta},\varepsilon)\big\{\big\|(\tilde{a}_{\alpha\beta}-a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon^{1/2}\big\|\big(\tilde{b}_{\alpha\beta}-b_{\alpha\beta}\big)\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon\big\|(\tilde{c}_{\alpha\beta}-c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}\big\},\end{aligned}$$

where $\mathbb{O}^3_+$ denotes the set of all proper orthogonal matrices of order three.

The proof of the above inequality relies in an essential way on a *nonlinear Korn inequality in an open set of* $\mathbb{R}^3$ recently established by Ciarlet and Mardare [16] (see Theorem 3.1). This inequality in turn makes an essential use of the fundamental "*geometric rigidity lemma*" of Friesecke, James, and Müller [21] and of the methodology developed in Ciarlet and Laurent [14].

That a vector $\boldsymbol{b}\in\mathbb{R}^3$ and a matrix $\boldsymbol{R}\in\mathbb{O}^3_+$ should appear in the left-hand side of this inequality is no surprise in light of the following extension, due to Ciarlet and Mardare [15], of the classical *rigidity theorem*: Let $\boldsymbol{\theta}\in\mathcal{C}^1(\omega;\mathbb{R}^3)$ be an immersion that satisfies $\boldsymbol{a}_3\in\mathcal{C}^1(\omega;\mathbb{R}^3)$ and let $\tilde{\boldsymbol{\theta}}\in H^1(\omega;\mathbb{R}^3)$ be a mapping that satisfies

$$\tilde{a}_{\alpha\beta}=a_{\alpha\beta}\quad\text{a.e. in }\omega,\qquad\tilde{\boldsymbol{a}}_3\in H^1\big(\omega;\mathbb{R}^3\big),\qquad\tilde{b}_{\alpha\beta}=b_{\alpha\beta}\quad\text{a.e. in }\omega$$

(as shown in ibid., the assumption $\tilde{a}_{\alpha\beta}=a_{\alpha\beta}$ a.e. in ω insures that the normal vector field $\boldsymbol{a}_3$ is well defined a.e. in ω). Then the two surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $\tilde{\boldsymbol{\theta}}(\omega)$ are *properly isometrically equivalent*, i.e., there exist a vector $\boldsymbol{b}\in\mathbb{R}^3$ and a matrix $\boldsymbol{R}\in\mathbb{O}^3_+$ such that

$$\tilde{\boldsymbol{\theta}}(y)=\boldsymbol{b}+\boldsymbol{R}\boldsymbol{\theta}(y)\quad\text{for almost all } y\in\omega.$$

One application of the nonlinear Korn inequality on a surface is the following *sequential continuity property* (cf. Corollaries 5.1 and 5.2; in the same spirit, the same inequality is also recast as one involving distances in Corollary 5.3). Let $\boldsymbol{\theta}^k:\omega\to\mathbb{R}^3$, $k\geqslant 1$, be mappings with the following properties: They belong to the space $\boldsymbol{H}^1(\omega)$; the vector fields normal to the surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, are well defined a.e. in ω and they also belong to the space $\boldsymbol{H}^1(\omega)$; the principal radii of curvature of the surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, stay uniformly away from zero; and finally, the three fundamental forms of the surfaces $\boldsymbol{\theta}^k(\omega)$ converge in $\boldsymbol{L}^1(\omega)$ toward the three fundamental forms of the surface $\boldsymbol{\theta}(\overline{\omega})$ as $k\to\infty$. Then, for each $k\geqslant 1$, there exists a surface $\hat{\boldsymbol{\theta}}^k(\omega)$ that is properly isometrically equivalent to the surface $\boldsymbol{\theta}^k(\omega)$ such that the surfaces $\hat{\boldsymbol{\theta}}^k(\omega)$ and their normal vector fields converge in $\boldsymbol{H}^1(\omega)$ to the surface $\boldsymbol{\theta}(\overline{\omega})$ and its normal vector field.

Should the fundamental forms of the unknown deformed surface be viewed as the primary unknowns in a shell problem (as suggested earlier), this kind of sequential continuity result could thus prove to be useful when considering *infimizing sequences* of the energy of a nonlinearly elastic shell (in particular for handling the part of the energy that takes into account the applied forces and the boundary conditions, which are both naturally expressed in terms of the deformation itself).

In this respect, it is worth mentioning that a similar program has been successfully carried out in the *linear case*. More specifically, Ciarlet and Gratie [12] have recently revisited from a similar perspective the quadratic minimization problem proposed by Koiter [23] for modeling a *linearly elastic shell*. As expected, the stored energy function then takes the form

$$w_K^{\mathrm{lin}}=\frac{\varepsilon}{2}a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\tilde{\boldsymbol{\eta}})\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})+\frac{\varepsilon^3}{6}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\tilde{\boldsymbol{\eta}})\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}),$$

where $(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})):\omega\to\mathbb{S}^2$ and $(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})):\omega\to\mathbb{S}^2$ are the *linearized change of metric*, and *linearized change of curvature, tensor fields* associated with a displacement field $\tilde{\boldsymbol{\eta}}=\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}$ of the middle surface of the shell ("linearized" means that only the linear parts with respect to $\tilde{\boldsymbol{\eta}}$ are retained in the "complete" differences $(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})$ and $(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})$). Then the novelty in [12] has consisted in *considering these linearized tensors as the new unknowns*, instead of the displacement field $\tilde{\boldsymbol{\eta}}$ as is customary in linear shell theory. A new existence theory for the resulting minimization problem has been established in [12], which interestingly also provides a new proof of the *linear Korn inequality on a surface* (in so doing, an essential use is made of a similar approach, which has been successfully applied to linearized three-dimensional elasticity by Ciarlet and Ciarlet, Jr [10]).

This linear inequality on a surface is also briefly reviewed here in Section 7, for the (different) purpose of showing that it is indeed a linearization of the nonlinear inequality established here, thus justifying the terminology "nonlinear Korn inequality on a surface" proposed in the present paper.

The results of this paper have been announced in [13].

2. Notations and definitions

The symbols $\mathbb{M}^n$, $\mathbb{S}^n$, and $\mathbb{O}^n_+$ respectively designate the sets of all real matrices of order n, of all real symmetric matrices of order n, and of all real orthogonal matrices $\boldsymbol{R}$ of order n with $\det \boldsymbol{R} = 1$. The Euclidean norm of a vector $\boldsymbol{b} \in \mathbb{R}^n$ is denoted $|\boldsymbol{b}|$ and $|\boldsymbol{A}| := \sup_{|\boldsymbol{b}|=1} |\boldsymbol{A}\boldsymbol{b}|$ denotes the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^n$.

Let U be an open subset in $\mathbb{R}^n$. Given any smooth enough mapping $\boldsymbol{\chi} : U \to \mathbb{R}^n$, we let $\nabla\boldsymbol{\chi}(x) \in \mathbb{M}^n$ denote the gradient matrix of the mapping $\boldsymbol{\chi}$ at $x \in U$ and we let $\partial_i \boldsymbol{\chi}(x)$ denote the ith column of the matrix $\nabla\boldsymbol{\chi}(x)$. Given any mapping $\boldsymbol{F} \in L^p(U; \mathbb{M}^n)$, $p \geqslant 1$, we let

$$\|\boldsymbol{F}\|_{L^p(U;\mathbb{M}^n)} := \left\{ \int_U \left|\boldsymbol{F}(x)\right|^p \mathrm{d}x \right\}^{1/p},$$

and we define $\|\boldsymbol{F}\|_{L^p(U;\mathbb{S}^n)}$ in an analogous manner if $\boldsymbol{F} \in L^p(U; \mathbb{S}^n)$. Given any mapping $\boldsymbol{\chi} \in H^1(U; \mathbb{R}^n)$, we let

$$\|\boldsymbol{\chi}\|_{H^1(U;\mathbb{R}^n)} := \left\{ \int_U \left(\left|\boldsymbol{\chi}(x)\right|^2 + \sum_{i=1}^n \left|\partial_i \boldsymbol{\chi}(x)\right|^2 \right) \mathrm{d}x \right\}^{1/2}.$$

A *domain U* in $\mathbb{R}^n$ is an open and bounded subset of $\mathbb{R}^n$ with a boundary that is Lipschitz-continuous in the sense of Adams [2] or Nečas [26], the set U being locally on the same side of its boundary. If U is a domain in $\mathbb{R}^n$, the space $\mathcal{C}^1(\overline{U}; \mathbb{R}^m)$ consists of all vector-valued mappings $\boldsymbol{\chi} \in \mathcal{C}^1(U; \mathbb{R}^m)$ that, together with all their partial derivatives of the first order, possess continuous extensions to the closure $\overline{U}$ of U. The space $\mathcal{C}^1(\overline{U}; \mathbb{R}^m)$ also consists of restrictions to $\overline{U}$ of all mappings in the space $\mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^m)$ (for a proof, see, e.g., [29] or [17]).

Latin indices and exponents henceforth range in the set $\{1, 2, 3\}$ save when they are used for indexing sequences, Greek indices and exponents range in the set $\{1, 2\}$, and the summation convention is used in conjunction with these rules.

The notations $(a_{\alpha\beta})$, $(a^{\alpha\beta})$, (b^β_α), and (g_{ij}) respectively designate matrices in $\mathbb{M}^2$ and $\mathbb{M}^3$ with components $a_{\alpha\beta}$, $a^{\alpha\beta}$, b^β_α, and g_{ij}, the index or exponent denoted here α or i designating the row index.

3. Preliminaries

The proof of our main result (Theorem 3.1) relies on several preliminaries, which are gathered in this section. The key preliminary is the following *nonlinear Korn inequality on an open subset in* $\mathbb{R}^n$ recently established by Ciarlet and Mardare [16], the proof of which is sketched below for the sake of completeness. See also Reshetnyak [28] for related results.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^n$, $n \geqslant 2$. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying* $\det \nabla\boldsymbol{\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $C(\boldsymbol{\Theta})$ with the following property*: *Given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a matrix $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\left\|\left(\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}\right) - \boldsymbol{\Theta}\right\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta}) \left\|\nabla\widetilde{\boldsymbol{\Theta}}^T \nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^T \nabla\boldsymbol{\Theta}\right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Proof. We sketch the main parts of the proof under the additional assumption that the mapping $\boldsymbol{\Theta}$ is *injective* in $\overline{\Omega}$. The proof in the general case is substantially more technical and relies on a methodology reminiscent to that proposed in Ciarlet and Laurent [14].

(i) *Let a matrix* $\boldsymbol{F} \in \mathbb{M}^n$ *be such that* $\det \boldsymbol{F} > 0$. *Then*

$$\operatorname{dist}\big(\boldsymbol{F}, \mathbb{O}^n_+\big) := \inf_{\boldsymbol{Q}\in\mathbb{O}^n_+} |\boldsymbol{F} - \boldsymbol{Q}| \leqslant \big|\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\big|^{1/2}.$$

It is known that

$$\operatorname{dist}\big(\boldsymbol{F}, \mathbb{O}^n_+\big) = \big|\big(\boldsymbol{F}^T\boldsymbol{F}\big)^{1/2} - \boldsymbol{I}\big|.$$

Let $0 < v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n$ denote the singular values of the matrix $\boldsymbol{F}$. Then

$$\begin{aligned}\big|\big(\boldsymbol{F}^T\boldsymbol{F}\big)^{1/2} - \boldsymbol{I}\big| &= \max\big\{|v_1 - 1|, |v_n - 1|\big\} \leqslant \max\big\{\big|v_1^2 - 1\big|^{1/2}, \big|v_n^2 - 1\big|^{1/2}\big\}\\ &= \big|\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\big|^{1/2}.\end{aligned}$$

(ii) *Let* Ω *be a domain in* $\mathbb{R}^n$. *Then there exists a constant* $\Lambda(\Omega)$ *with the following property*: *Given any mapping* $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ *satisfying* $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ *a.e. in* Ω, *there exists a matrix* $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}) \in \mathbb{O}^n_+$ *such that*

$$\big\|\boldsymbol{R}\nabla\widetilde{\boldsymbol{\Theta}} - \boldsymbol{I}\big\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant \Lambda(\Omega)\big\|\nabla\widetilde{\boldsymbol{\Theta}}^T\nabla\widetilde{\boldsymbol{\Theta}} - \boldsymbol{I}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

By the "*geometric rigidity lemma*" of Friesecke, James and Müller [21, Theorem 3.1], there exists a constant $\Lambda(\Omega)$ depending only on the set Ω with the following property: For each $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$, there exists a rotation $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}) \in \mathbb{O}^n_+$ such that

$$\big\|\boldsymbol{R}\nabla\widetilde{\boldsymbol{\Theta}} - \boldsymbol{I}\big\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant \Lambda(\Omega)\big\|\operatorname{dist}\big(\nabla\widetilde{\boldsymbol{\Theta}}, \mathbb{O}^n_+\big)\big\|_{L^2(\Omega)}.$$

If in addition the mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfies $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω, then part (i) implies that

$$\operatorname{dist}\big(\nabla\widetilde{\boldsymbol{\Theta}}(x), \mathbb{O}^n_+\big) \leqslant \big|\nabla\widetilde{\boldsymbol{\Theta}}(x)^T\nabla\widetilde{\boldsymbol{\Theta}}(x) - \boldsymbol{I}\big|^{1/2}$$

for almost all $x \in \Omega$. Hence

$$\big\|\operatorname{dist}\big(\nabla\widetilde{\boldsymbol{\Theta}}, \mathbb{O}^n_+\big)\big\|_{L^2(\Omega)} \leqslant \big\|\nabla\widetilde{\boldsymbol{\Theta}}^T\nabla\widetilde{\boldsymbol{\Theta}} - \boldsymbol{I}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

(iii) *Let* Ω *be a domain in* $\mathbb{R}^n$. *Given any injective mapping* $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ *satisfying* $\det \nabla\boldsymbol{\Theta} > 0$ *in* $\overline{\Omega}$, *there exists a constant* $c(\boldsymbol{\Theta})$ *with the following property*: *Given any mapping* $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ *satisfying* $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ *a.e. in* Ω, *there exists a rotation* $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ *such that*

$$\big\|\boldsymbol{R}\nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}\big\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant c(\boldsymbol{\Theta})\big\|\nabla\widetilde{\boldsymbol{\Theta}}^T\nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^T\nabla\boldsymbol{\Theta}\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Since Ω is a domain, any mapping $\boldsymbol{\Theta}$ in the space $\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ can be extended to a mapping $\boldsymbol{\Theta}^\flat$ in the space $\mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$. Moreover, since $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ and Ω is bounded, there exists a connected open subset $\Omega^\sharp$ containing $\overline{\Omega}$ such that the restriction $\boldsymbol{\Theta}^\sharp \in \mathcal{C}^1(\Omega^\sharp; \mathbb{R}^n)$ to $\Omega^\sharp$ of such an extension $\boldsymbol{\Theta}^\flat$ satisfies $\det \nabla\boldsymbol{\Theta}^\sharp > 0$ in $\Omega^\sharp$. Consequently, the set $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$ is also a domain in $\mathbb{R}^n$. Besides, the inverse mapping $\widehat{\boldsymbol{\Theta}} : \{\widehat{\Omega}\}^- \to \overline{\Omega}$ of $\boldsymbol{\Theta}$ belongs to the space $\mathcal{C}^1(\{\widehat{\Omega}\}^-; \mathbb{R}^n)$.

Given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$, the composite mapping $\widehat{\boldsymbol{\Phi}} := \widetilde{\boldsymbol{\Theta}} \circ \widehat{\boldsymbol{\Theta}}$ belongs to the space $H^1(\widehat{\Omega}; \mathbb{R}^n)$ since the bijection $\boldsymbol{\Theta} : \overline{\Omega} \to \{\widehat{\Omega}\}^-$ is bi-Lipschitzian. Moreover,

$$\widehat{\nabla}\widehat{\boldsymbol{\Phi}}(\hat{x}) = \nabla\widetilde{\boldsymbol{\Theta}}(x)\widehat{\nabla}\widehat{\boldsymbol{\Theta}}(\hat{x}) = \nabla\widetilde{\boldsymbol{\Theta}}(x)\nabla\boldsymbol{\Theta}(x)^{-1} \quad \text{for almost all } \hat{x} = \boldsymbol{\Theta}(x) \in \widehat{\Omega},$$

the notation $\widehat{\nabla}$ indicating that differentiation is performed with respect to the variable $\hat{x}$. Hence $\det \widehat{\nabla}\widehat{\boldsymbol{\Phi}} > 0$ a.e. in $\widehat{\Omega}$ if in addition $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω.

By part (ii), there exists a constant $c_0(\boldsymbol{\Theta}) := \Lambda(\widehat{\Omega})$ with the following property: Given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω, there exists a matrix $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that the mapping $\widehat{\boldsymbol{\Phi}} = \widetilde{\boldsymbol{\Theta}} \circ \widehat{\boldsymbol{\Theta}}$ satisfies

$$\big\|\boldsymbol{R}\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\big\|_{L^2(\widehat{\Omega};\mathbb{M}^n)} \leqslant c_0(\boldsymbol{\Theta})\big\|\widehat{\nabla}\widehat{\boldsymbol{\Phi}}^T\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I}\big\|^{1/2}_{L^1(\widehat{\Omega};\mathbb{S}^n)}.$$

It is then easily seen that the assumed *injectivity* of the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ and the relation $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$ together imply that

$$\left\| \boldsymbol{R}\widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I} \right\|^2_{L^2(\widehat{\Omega};\mathbb{M}^n)} \geqslant c_1(\boldsymbol{\Theta}) \left\| \boldsymbol{R}\nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta} \right\|^2_{L^2(\Omega;\mathbb{M}^n)},$$

where $c_1(\boldsymbol{\Theta}) := \inf_{x\in\overline{\Omega}}\{|\nabla\boldsymbol{\Theta}(x)|^{-2} \det \nabla\boldsymbol{\Theta}(x)\} > 0$. Likewise, it is easily seen that

$$\left\| \widehat{\nabla}\widehat{\boldsymbol{\Phi}}^T \widehat{\nabla}\widehat{\boldsymbol{\Phi}} - \boldsymbol{I} \right\|_{L^1(\widehat{\Omega};\mathbb{S}^n)} \leqslant c_2(\boldsymbol{\Theta}) \left\| \nabla\widetilde{\boldsymbol{\Theta}}^T \nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^T \nabla\boldsymbol{\Theta} \right\|_{L^1(\Omega;\mathbb{S}^n)},$$

where $c_2(\boldsymbol{\Theta}) := \sup_{x\in\overline{\Omega}}\{|\nabla\boldsymbol{\Theta}(x)^{-T}||\nabla\boldsymbol{\Theta}(x)^{-1}| \det \nabla\boldsymbol{\Theta}(x)\} < \infty$. The announced inequality thus holds with $c(\boldsymbol{\Theta}) := c_0(\boldsymbol{\Theta}) c_1(\boldsymbol{\Theta})^{-1/2} c_2(\boldsymbol{\Theta})^{1/2}$.

(iv) *Let the assumptions on the set Ω and the mapping $\boldsymbol{\Theta}$ be as in part* (iii). *Then there exists a constant $C(\boldsymbol{\Theta})$ with the following property*: *Given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a matrix $\boldsymbol{R} = \boldsymbol{R}(\widehat{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\left\| \left(\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}\right) - \boldsymbol{\Theta} \right\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta}) \left\| \nabla\widetilde{\boldsymbol{\Theta}}^T \nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^T \nabla\boldsymbol{\Theta} \right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Let there be given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω. By part (iii), there exists a matrix $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that

$$\left\| \boldsymbol{R}\nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta} \right\|_{L^2(\Omega;\mathbb{M}^n)} \leqslant c(\boldsymbol{\Theta}) \left\| \nabla\widetilde{\boldsymbol{\Theta}}^T \nabla\widetilde{\boldsymbol{\Theta}} - \nabla\boldsymbol{\Theta}^T \nabla\boldsymbol{\Theta} \right\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

Let the vector $\boldsymbol{b} = \boldsymbol{b}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ be defined by

$$\boldsymbol{b} := \left(\int_\Omega \mathrm{d}x \right)^{-1} \int_\Omega \left(\boldsymbol{R}\widetilde{\boldsymbol{\Theta}}(x) - \boldsymbol{\Theta}(x) \right) \mathrm{d}x.$$

By the generalized Poincaré inequality, there exists a constant d such that, for all $\boldsymbol{\Psi} \in H^1(\Omega; \mathbb{R}^n)$,

$$\|\boldsymbol{\Psi}\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant d \left(\|\nabla\boldsymbol{\Psi}\|_{L^2(\Omega;\mathbb{M}^n)} + \left| \int_\Omega \boldsymbol{\Psi}(x)\,\mathrm{d}x \right| \right).$$

Applying this inequality to the mapping $\boldsymbol{\Psi} := (\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}) - \boldsymbol{\Theta}$ yields the desired conclusion, with $C(\boldsymbol{\Theta}) := d c(\boldsymbol{\Theta})$. □

The next two lemmas show that some classical definitions and properties pertaining to surfaces in $\mathbb{R}^3$ still hold under less stringent regularity assumptions than the usual ones (these definitions and properties are traditionally given and established under the assumptions that the immersions denoted $\boldsymbol{\theta}$ in Lemma 3.2 and $\tilde{\boldsymbol{\theta}}$ in Lemma 3.3 below belong to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$). For this reason, we shall continue to use the classical terminology, e.g., *surface* (for $\boldsymbol{\theta}(\overline{\omega})$ or $\tilde{\boldsymbol{\theta}}(\omega)$), *normal vector field* (for $\boldsymbol{a}_3$ or $\tilde{\boldsymbol{a}}_3$), *first*, *second*, and *third*, *fundamental forms* (for $(a_{\alpha\beta})$ or $(\tilde{a}_{\alpha\beta})$, $(b_{\alpha\beta})$ or $(\tilde{b}_{\alpha\beta})$, and $(c_{\alpha\beta})$ or $(\tilde{c}_{\alpha\beta})$), etc. If $y = (y_\alpha)$ designates the generic point in a domain ω in $\mathbb{R}^2$, we let $\partial_\alpha := \partial/\partial y_\alpha$.

Lemma 3.2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ be an immersion such that*

$$\boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^1\left(\overline{\omega}; \mathbb{R}^3\right) \quad \textit{where } \boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}.$$

Then the functions

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad b_{\alpha\beta} := -\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta, \quad b^\sigma_\alpha := a^{\beta\sigma} b_{\alpha\beta}, \quad \textit{and} \quad c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3,$$

where $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$, belong to the space $\mathcal{C}^0(\overline{\omega})$. Besides,

$$b_{\alpha\beta} = b_{\beta\alpha}.$$

Define the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\omega} \times \mathbb{R}; \mathbb{R}^3)$ by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \quad \textit{for all } (y, x_3) \in \overline{\omega} \times \mathbb{R}.$$

Then

$$\det \nabla \boldsymbol{\Theta}(y, x_3) = \sqrt{a(y)}\,\bigl\{1 - 2H(y)x_3 + K(y)x_3^2\bigr\} \quad \textit{for all } (y, x_3) \in \overline{\omega} \times \mathbb{R},$$

where the functions

$$a := \det(a_{\alpha\beta}) = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2, \qquad H := \frac{1}{2}\bigl(b_1^1 + b_2^2\bigr), \qquad K := b_1^1 b_2^2 - b_1^2 b_2^1$$

belong to the space $\mathcal{C}^0(\overline{\omega})$. Finally, let

$$(g_{ij}) := \nabla \boldsymbol{\Theta}^T \nabla \boldsymbol{\Theta}.$$

Then the functions $g_{ij} = g_{ji}$ belong to the space $\mathcal{C}^0(\overline{\omega} \times \mathbb{R})$ and they are given by

$$g_{\alpha\beta}(y, x_3) = a_{\alpha\beta}(y) - 2x_3 b_{\alpha\beta}(y) + x_3^2 c_{\alpha\beta}(y) \quad \textit{and} \quad g_{i3}(y, x_3) = \delta_{i3}$$

for all $(y, x_3) \in \overline{\omega} \times \mathbb{R}$.

Proof. Because the mapping $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ is an immersion, the symmetric matrices $(a_{\alpha\beta}(y))$ are positive-definite at all points $y \in \overline{\omega}$, the inverse matrices $(a^{\alpha\beta}(y))$ are well defined and also positive-definite at all points $y \in \overline{\omega}$, and the functions $a^{\alpha\beta}$ belong to the space $\mathcal{C}^0(\overline{\omega})$. Therefore the functions b_α^σ are well-defined and they also belong to the space $\mathcal{C}^0(\overline{\omega})$.

While the relations $b_{\alpha\beta} = b_{\beta\alpha}$ clearly hold if $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ (since $b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta$ in this case), this symmetry requires a proof under the present weaker regularity assumptions. Following [15], we first note to this end that the assumptions $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ and $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ imply that $-b_{\alpha\beta} = \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3 \in L^1_{\mathrm{loc}}(\omega)$, hence that $\partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3 \in \mathcal{D}'(\omega)$.

Given any $\varphi \in \mathcal{D}(\omega)$, let then U denote an open subset of $\mathbb{R}^2$ such that $\operatorname{supp}\varphi \subset U$ and $\overline{U}$ is a compact subset of ω. Denoting by ${}_{X'}\langle \cdot, \cdot \rangle_X$ the duality pairing between a topological vector space X and its dual X', we have

$${}_{\mathcal{D}'(\omega)}\langle \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3, \varphi \rangle_{\mathcal{D}(\omega)} = \int_\omega \varphi \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3 \,\mathrm{d}y = \int_\omega \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha (\varphi \boldsymbol{a}_3)\,\mathrm{d}y - \int_\omega (\partial_\alpha \varphi)\partial_\beta \boldsymbol{\theta} \cdot \boldsymbol{a}_3 \,\mathrm{d}y.$$

Observing that $\partial_\beta \boldsymbol{\theta} \cdot \boldsymbol{a}_3 = 0$ a.e. in ω and that

$$-\int_\omega \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha (\varphi \boldsymbol{a}_3)\,\mathrm{d}y = -\int_U \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha (\varphi \boldsymbol{a}_3)\,\mathrm{d}y = {}_{H^{-1}(U;\mathbb{R}^3)}\bigl\langle \partial_\alpha(\partial_\beta \boldsymbol{\theta}),\ \varphi \boldsymbol{a}_3 \bigr\rangle_{H_0^1(U;\mathbb{R}^3)},$$

we reach the conclusion that the expression ${}_{\mathcal{D}'(\omega)}\langle \partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3,\ \varphi \rangle_{\mathcal{D}(\omega)}$ is symmetric with respect to β and α since $\partial_{\beta\alpha}\boldsymbol{\theta} = \partial_{\alpha\beta}\boldsymbol{\theta}$ in $\mathcal{D}'(U; \mathbb{R}^3)$. Hence $\partial_\beta \boldsymbol{\theta} \cdot \partial_\alpha \boldsymbol{a}_3 = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{a}_3$ in $L^1_{\mathrm{loc}}(\omega)$, and the announced symmetry is established.

Because $\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 0$ (since $\boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1$), the classical *formula of Weingarten* $\partial_\alpha \boldsymbol{a}_3 = -b_\alpha^\sigma \boldsymbol{a}_\sigma$ still holds in the present case. The definition of the mapping $\boldsymbol{\Theta}$ shows that

$$\boldsymbol{g}_\alpha := \partial_\alpha \boldsymbol{\Theta} = (\boldsymbol{a}_\alpha + x_3 \partial_\alpha \boldsymbol{a}_3) \in \mathcal{C}^0\bigl(\overline{\omega} \times \mathbb{R}; \mathbb{R}^3\bigr), \qquad \boldsymbol{g}_3 := \partial_3 \boldsymbol{\Theta} = \boldsymbol{a}_3 \in \mathcal{C}^1\bigl(\overline{\omega} \times \mathbb{R}; \mathbb{R}^3\bigr),$$

hence that

$$\det \nabla \boldsymbol{\Theta} = (\boldsymbol{g}_1 \wedge \boldsymbol{g}_2) \cdot \boldsymbol{g}_3 = \bigl(\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 + x_3\{\boldsymbol{a}_1 \wedge \partial_2 \boldsymbol{a}_3 + \partial_1 \boldsymbol{a}_3 \wedge \boldsymbol{a}_2\} + x_3^2 \partial_1 \boldsymbol{a}_3 \wedge \partial_2 \boldsymbol{a}_3\bigr) \cdot \boldsymbol{a}_3.$$

The announced expression of the function $\det \nabla \boldsymbol{\Theta} \in \mathcal{C}^0(\overline{\omega} \times \mathbb{R})$ then follows from the formula of Weingarten and the relation $a = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2$. The announced expression of the functions $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j \in \mathcal{C}^0(\overline{\omega} \times \mathbb{R})$ follows from the relations $b_{\alpha\beta} = b_{\beta\alpha}$ and $\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 0$. □

Lemma 3.3. *Let ω be a domain in $\mathbb{R}^2$ and let there be given a mapping $\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3)$ such that $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq \boldsymbol{0}$ a.e. in ω, where $\tilde{\boldsymbol{a}}_\alpha := \partial_\alpha \tilde{\boldsymbol{\theta}}$, and such that*

$$\tilde{\boldsymbol{a}}_3 := \frac{\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|} \in H^1\bigl(\omega; \mathbb{R}^3\bigr).$$

Then the functions

$$\tilde{a}_{\alpha\beta} := \tilde{\boldsymbol{a}}_\alpha \cdot \tilde{\boldsymbol{a}}_\beta, \qquad \tilde{b}_{\alpha\beta} := -\partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \tilde{\boldsymbol{a}}_\beta, \qquad \tilde{c}_{\alpha\beta} := \partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \partial_\beta \tilde{\boldsymbol{a}}_3$$

are well defined a.e. in ω *and they belong to the space* $L^1(\omega)$. *Besides,*

$$\tilde{b}_{\alpha\beta} = \tilde{b}_{\beta\alpha} \quad \text{a.e. in } \omega.$$

Define the mapping $\widetilde{\boldsymbol{\Theta}} : \omega \times \mathbb{R} \to \mathbb{R}^3$ *by*

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \tilde{\boldsymbol{\theta}}(y) + x_3 \tilde{\boldsymbol{a}}_3(y) \quad \text{for almost all } (y, x_3) \in \omega \times \mathbb{R}.$$

Then $\widetilde{\boldsymbol{\Theta}} \in H^1(\omega \times]-\delta, \delta[; \mathbb{R}^3)$ *for any* $\delta > 0$. *Furthermore,*

$$\det \boldsymbol{\nabla} \widetilde{\boldsymbol{\Theta}}(y, x_3) = \sqrt{\tilde{a}(y)} \left\{1 - 2\widetilde{H}(y) x_3 + \widetilde{K}(y) x_3^2\right\}$$

for almost all $(y, x_3) \in \omega \times \mathbb{R}$, *where*

$$\tilde{a} := \det(\tilde{a}_{\alpha\beta}) = |\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|^2, \qquad \widetilde{H} := \frac{1}{2}\left(\tilde{b}_1^1 + \tilde{b}_2^2\right), \qquad \widetilde{K} := \tilde{b}_1^1 \tilde{b}_2^2 - \tilde{b}_1^2 \tilde{b}_2^1,$$
$$\tilde{b}_\alpha^\sigma := \tilde{a}^{\beta\sigma} \tilde{b}_{\alpha\beta}, \quad \text{and} \quad \left(\tilde{a}^{\alpha\beta}\right) := (\tilde{a}_{\alpha\beta})^{-1}.$$

Finally, let

$$(\tilde{g}_{ij}) := \boldsymbol{\nabla} \widetilde{\boldsymbol{\Theta}}^T \boldsymbol{\nabla} \widetilde{\boldsymbol{\Theta}} \quad \text{a.e. in } \omega \times \mathbb{R}.$$

Then the functions $\tilde{g}_{ij} = \tilde{g}_{ji}$ *belong to the space* $L^1(\omega \times]-\delta, \delta[)$ *for any* $\delta > 0$ *and they are given by*

$$\tilde{g}_{\alpha\beta}(y, x_3) = \tilde{a}_{\alpha\beta}(y) - 2x_3 \tilde{b}_{\alpha\beta}(y) + x_3^2 \tilde{c}_{\alpha\beta}(y) \quad \text{and} \quad \tilde{g}_{i3}(y, x_3) = \delta_{i3}$$

for almost all $(y, x_3) \in \omega \times \mathbb{R}$.

Proof. The assumptions made on the mapping $\tilde{\boldsymbol{\theta}}$ and on the vector field $\tilde{\boldsymbol{a}}_3$ clearly imply that the functions $\tilde{a}_{\alpha\beta}$, $\tilde{b}_{\alpha\beta}$, and $\tilde{c}_{\alpha\beta}$ are in the space $L^1(\omega)$. Because the symmetric matrices $(\tilde{a}_{\alpha\beta}(y))$ are positive-definite for almost all $y \in \overline{\omega}$, the inverse matrices $(\tilde{a}^{\alpha\beta}(y))$ are likewise positive-definite for almost all $y \in \overline{\omega}$, and thus the functions $\tilde{b}_\alpha^\sigma$ are well-defined a.e. in ω, like the functions $\tilde{a}$, $\widetilde{H}$, and $\widetilde{K}$ (however, these functions do not necessarily belong to the space $L^1(\omega)$).

Since the assumptions $\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3)$ and $\tilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbb{R}^3)$ again imply that $-\tilde{b}_{\alpha\beta} = \partial_\beta \tilde{\boldsymbol{\theta}} \cdot \partial_\alpha \tilde{\boldsymbol{a}}_3 \in L^1_{\text{loc}}(\omega)$, the relations $\tilde{b}_{\alpha\beta} = \tilde{b}_{\beta\alpha}$ hold a.e. in ω (see the proof of Lemma 3.2). Because $\partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \tilde{\boldsymbol{a}}_3 = 0$ a.e. in ω, the *formula of Weingarten* $\partial_\alpha \tilde{\boldsymbol{a}}_3 = -\tilde{b}_\alpha^\sigma \tilde{\boldsymbol{a}}_\sigma$ now holds a.e. in ω. The announced expressions of the function $\det \boldsymbol{\nabla} \widetilde{\boldsymbol{\Theta}}$, which is well-defined a.e. in $\omega \times \mathbb{R}$, and of the functions $\tilde{g}_{ij}$, which clearly belong to the space $L^1(\omega \times]-\delta, \delta[)$ for any $\delta > 0$, then follows from these observations. $\square$

If a mapping $\tilde{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ is a smooth immersion, the functions $\widetilde{H}$ and $\widetilde{K}$ simply represent the *mean*, and *Gaussian*, *curvatures* of the surface $\tilde{\boldsymbol{\theta}}(\omega)$. These functions are also given by

$$\widetilde{H} = \frac{1}{2}\left(\frac{1}{\widetilde{R}_1} + \frac{1}{\widetilde{R}_2}\right) \quad \text{and} \quad \widetilde{K} = \frac{1}{\widetilde{R}_1 \widetilde{R}_2},$$

where $\widetilde{R}_\alpha$ are the *principal radii of curvature* along the surface $\tilde{\boldsymbol{\theta}}(\omega)$ (with the usual convention that $|R_\alpha(y)|$ may take the value $+\infty$ at some points $y \in \omega$).

4. A nonlinear Korn inequality on a surface

We are now in a position to prove the announced *nonlinear Korn inequality on a surface*. The notations are the same as those in Lemmas 3.2 and 3.3.

Theorem 4.1. *Let there be given a domain* ω *in* $\mathbb{R}^2$, *an immersion* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ *such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$, *and* $\varepsilon > 0$.

Then there exists a constant $c(\boldsymbol{\theta}, \varepsilon)$ *with the following property*: *Given any mapping* $\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3)$ *such that* $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq \boldsymbol{0}$ *a.e. in* ω, $\tilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbb{R}^3)$, *and the principal radii of curvature* $\widetilde{R}_\alpha$ *of the surface* $\tilde{\boldsymbol{\theta}}(\omega)$ *satisfy*

$$\left|\widetilde{R}_\alpha\right| \geqslant \varepsilon \quad \text{a.e. in } \omega,$$

there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{R}^3$ *and a matrix* $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{O}^3_+$ *such that*

$$\begin{aligned}&\big\|\big(\boldsymbol{b} + \boldsymbol{R}\tilde{\boldsymbol{\theta}}\big) - \boldsymbol{\theta}\big\|_{H^1(\omega;\mathbb{R}^3)} + \varepsilon\|\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{H^1(\omega;\mathbb{R}^3)} \\ &\quad\leqslant c(\boldsymbol{\theta}, \varepsilon)\big\{\big\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)} + \varepsilon^{1/2}\big\|\big(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}\big)\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)} + \varepsilon\big\|(\tilde{c}_{\alpha\beta} - c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}\big\}.\end{aligned}$$

Proof. Given a mapping $\boldsymbol{\theta}$ satisfying the assumptions of Theorem 4.1, let the mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\omega} \times \mathbb{R}; \mathbb{R}^3)$ be constructed as in Lemma 3.2. Consequently,

$$\det \nabla\boldsymbol{\Theta}(y, x_3) = \sqrt{a(y)}\,\big\{1 - 2H(y)x_3 + K(y)x_3^2\big\} \quad \text{for all } (y, x_3) \in \overline{\omega} \times \mathbb{R},$$

by the same lemma. Since the functions a, H, and K are in the space $\mathcal{C}^0(\overline{\omega})$ and there exists $a_0 > 0$ such that $a(y) \geqslant a_0$ for all $y \in \overline{\omega}$, there exists a constant $\tilde{\delta}(\boldsymbol{\theta}) > 0$ such that $\det \nabla\boldsymbol{\Theta}(y, x_3) > 0$ for all $(y, x_3) \in \overline{\omega} \times [-\tilde{\delta}(\boldsymbol{\theta}), \tilde{\delta}(\boldsymbol{\theta})]$.

Given any mapping $\tilde{\boldsymbol{\theta}}$ satisfying the assumptions of Theorem 4.1, let the mapping $\widetilde{\boldsymbol{\Theta}} : \omega \times \mathbb{R} \to \mathbb{R}^3$ be constructed as in Lemma 3.3. By this lemma,

$$\det \nabla\widetilde{\boldsymbol{\Theta}}(y, x_3) = \sqrt{\tilde{a}(y)}\,\big\{1 - 2\widetilde{H}(y)x_3 + \widetilde{K}(y)x_3^2\big\}$$

for almost all $(y, x_3) \in \omega \times \mathbb{R}$. The assumption $|\widetilde{R}_\alpha| \geqslant \varepsilon$ a.e. in ω imply that $|\widetilde{H}| \leqslant 1/\varepsilon$ and $|\widetilde{K}| \leqslant 1/\varepsilon^2$ a.e. in ω. Hence there exists a constant $\tilde{c}$ such that

$$1 - 2\widetilde{H}(y)x_3 + \widetilde{K}(y)x_3^2 > 0 \quad \text{for almost all } (y, x_3) \in \omega \times \,]-\tilde{c}\varepsilon, \tilde{c}\varepsilon[.$$

Without loss of generality, we henceforth assume that $\varepsilon \leqslant 1$. Letting $\delta(\boldsymbol{\theta}) := \min\{\tilde{c}, \tilde{\delta}(\boldsymbol{\theta})\}$ and

$$\Omega = \Omega(\boldsymbol{\theta}, \varepsilon) := \omega \times \big]-\delta(\boldsymbol{\theta})\varepsilon, \delta(\boldsymbol{\theta})\varepsilon\big[,$$

noting that $\tilde{a} > 0$ a.e. in ω by assumption, we conclude that the restriction, still denoted $\widetilde{\boldsymbol{\Theta}}$ for convenience, of the mapping $\widetilde{\boldsymbol{\Theta}}$ to the set Ω belongs to the space $H^1(\Omega; \mathbb{R}^3)$ and satisfies $\det \nabla\widetilde{\boldsymbol{\Theta}} > 0$ a.e. in Ω on the one hand.

Since, on the other hand, the restriction, still denoted $\boldsymbol{\Theta}$ for convenience, of the mapping $\boldsymbol{\Theta}$ to the set $\overline{\Omega}$ belongs to the space $\mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$ and satisfies $\det \nabla\boldsymbol{\Theta} > 0$ in $\overline{\Omega}$, all the assumptions of Theorem 3.1 are satisfied. Therefore, given any $\varepsilon > 0$, there exists a constant $c_0(\boldsymbol{\theta}, \varepsilon)$ with the following property: Given any mapping $\tilde{\boldsymbol{\theta}}$ satisfying the assumptions of Theorem 4.1, there exist a vector $\boldsymbol{b} := \boldsymbol{b}(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{R}^3$ and a matrix $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, \varepsilon) \in \mathbb{O}^3_+$ such that

$$\big\|\big(\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}\big) - \boldsymbol{\Theta}\big\|_{H^1(\Omega;\mathbb{R}^3)} \leqslant c_0(\boldsymbol{\theta}, \varepsilon)\big\|(\tilde{g}_{ij} - g_{ij})\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^3)}.$$

In the remainder of this proof, we let $\delta := \delta(\boldsymbol{\theta})$ for conciseness. In order to get a lower bound of the left-hand side of this inequality in terms of $H^1(\omega; \mathbb{R}^3)$-norms of the mappings $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$, we simply note that, given any vector fields $\boldsymbol{u} \in L^2(\omega; \mathbb{R}^3)$ and $\boldsymbol{v} \in L^2(\omega; \mathbb{R}^3)$,

$$\int_\Omega \big|\boldsymbol{u}(y) + x_3\boldsymbol{v}(y)\big|^2\,\mathrm{d}x = 2\delta\varepsilon\int_\omega \big|\boldsymbol{u}(y)\big|^2\,\mathrm{d}y + \frac{2}{3}\delta^3\varepsilon^3\int_\omega \big|\boldsymbol{v}(y)\big|^2\,\mathrm{d}y,$$

since $\int_\Omega x_3(\boldsymbol{u}(y) \cdot \boldsymbol{v}(y))\,\mathrm{d}x = 0$. Consequently,

$$\int_\Omega \big|\big(\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}\big) - \boldsymbol{\Theta}\big|^2\,\mathrm{d}x = 2\delta\varepsilon\int_\omega \big|\big(\boldsymbol{b} + \boldsymbol{R}\tilde{\boldsymbol{\theta}}\big) - \boldsymbol{\theta}\big|^2\,\mathrm{d}y + \frac{2}{3}\delta^3\varepsilon^3\int_\omega |\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3|^2\,\mathrm{d}y,$$

and

$$\begin{aligned}&\int_\Omega \sum_i \big|\boldsymbol{R}\partial_i\widetilde{\boldsymbol{\Theta}} - \partial_i\boldsymbol{\Theta}\big|^2\,\mathrm{d}x \\ &\quad= \int_\Omega \bigg\{\sum_\alpha \big|\boldsymbol{R}\partial_\alpha\tilde{\boldsymbol{\theta}} - \partial_\alpha\boldsymbol{\theta} + x_3(\boldsymbol{R}\partial_\alpha\tilde{\boldsymbol{a}}_3 - \partial_\alpha\boldsymbol{a}_3)\big|^2 + |\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3|^2\bigg\}\,\mathrm{d}x \\ &\quad= 2\delta\varepsilon\int_\omega \sum_\alpha \big|\partial_\alpha\big(\boldsymbol{R}\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\big)\big|^2\,\mathrm{d}y + 2\delta\varepsilon\int_\omega |\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3|^2\,\mathrm{d}y + \frac{2}{3}\delta^3\varepsilon^3\int_\omega \sum_\alpha \big|\partial_\alpha(\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3)\big|^2\,\mathrm{d}y.\end{aligned}$$

There thus exists a constants $c_1(\boldsymbol{\theta})$ such that

$$\big\|\big(\boldsymbol{b}+\boldsymbol{R}\widetilde{\boldsymbol{\Theta}}\big)-\boldsymbol{\Theta}\big\|_{H^1(\Omega;\mathbb{R}^3)} \geqslant c_1(\boldsymbol{\theta})\varepsilon^{1/2}\big\{\big\|\big(\boldsymbol{b}+\boldsymbol{R}\tilde{\boldsymbol{\theta}}\big)-\boldsymbol{\theta}\big\|_{H^1(\omega;\mathbb{R}^3)}+\varepsilon\|\boldsymbol{R}\tilde{\boldsymbol{a}}_3-\boldsymbol{a}_3\|_{H^1(\omega;\mathbb{R}^3)}\big\}.$$

In order to get an upper bound of the $L^1(\Omega;\mathbb{S}^3)$-norm of the matrix field $(\tilde{g}_{ij}-g_{ij})$ in terms of $L^1(\omega;\mathbb{S}^2)$-norms of the fundamental forms of surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ and $\boldsymbol{\theta}(\overline{\omega})$, we again resort to Lemmas 3.2 and 3.3, which imply that

$$\tilde{g}_{\alpha\beta}-g_{\alpha\beta}=(\tilde{a}_{\alpha\beta}-a_{\alpha\beta})-2x_3\big(\tilde{b}_{\alpha\beta}-b_{\alpha\beta}\big)+x_3^2(\tilde{c}_{\alpha\beta}-c_{\alpha\beta}) \quad \text{a.e. in } \Omega,$$
$$\tilde{g}_{i3}-g_{i3}=0 \quad \text{a.e. in } \Omega.$$

Given a matrix field $\boldsymbol{F}^\sharp := (f^\sharp_{\alpha\beta}) \in L^1(\omega;\mathbb{S}^2)$, define the matrix field $\boldsymbol{F}=(f_{ij}) \in L^1(\Omega;\mathbb{S}^3)$ by letting $f_{\alpha\beta}(y,x_3)=f^\sharp_{\alpha\beta}(y)$ and $f_{i3}(y,x_3)=0$ for almost all $(y,x_3)\in\Omega$. Then it is easily seen that

$$\|\boldsymbol{F}\|_{L^1(\Omega;\mathbb{S}^3)}=2\delta\varepsilon\big\|\boldsymbol{F}^\sharp\big\|_{L^1(\omega;\mathbb{S}^2)}.$$

Combining these observations, we conclude that there exists a constant $c_2(\boldsymbol{\theta})$ such that

$$\big\|(\tilde{g}_{ij}-g_{ij})\big\|^{1/2}_{L^1(\Omega;\mathbb{S}^3)} \leqslant c_2(\boldsymbol{\theta})\varepsilon^{1/2}\big\{\big\|(\tilde{a}_{\alpha\beta}-a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon^{1/2}\big\|\big(\tilde{b}_{\alpha\beta}-b_{\alpha\beta}\big)\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon\big\|(\tilde{c}_{\alpha\beta}-c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}\big\}.$$

The announced inequality then follows with $c(\boldsymbol{\theta},\varepsilon):=c_0(\boldsymbol{\theta},\varepsilon)c_1(\boldsymbol{\theta})^{-1}c_2(\boldsymbol{\theta})$. □

The essence of the inequality established above can thus be summed up as follows: *Given any family of surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ whose principal radii of curvature stay uniformly away from zero, the $H^1(\omega;\mathbb{R}^3)$-distance between the two surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ and $\boldsymbol{\theta}(\overline{\omega})$ and between their normal vector fields $\tilde{\boldsymbol{a}}_3$ and $\boldsymbol{a}_3$ is "controlled" by the $L^1(\omega;\mathbb{S}^2)$-distance between their three fundamental forms* (recall that the principal radii of curvature of such "admissible" surfaces $\tilde{\boldsymbol{\theta}}(\omega)$ are possibly understood in a generalized sense, viz., as the inverses of the eigenvalues of the associated matrices $(\tilde{b}^\beta_\alpha)$). Naturally, the $H^1(\omega;\mathbb{R}^3)$-distance between the surfaces is only measured *up to properly isometrically equivalent surfaces*, since such surfaces share the same fundamental forms.

5. Some consequences

Define the set (the notations are those of Lemma 3.3)

$$H^1_\sharp\big(\omega;\mathbb{R}^3\big):=\big\{\tilde{\boldsymbol{\theta}}\in H^1\big(\omega;\mathbb{R}^3\big);\ \tilde{\boldsymbol{a}}_1\wedge\tilde{\boldsymbol{a}}_2\neq\boldsymbol{0} \text{ a.e. in } \omega, \tilde{\boldsymbol{a}}_3\in H^1\big(\omega;\mathbb{R}^3\big)\big\}.$$

Then two mappings $\hat{\boldsymbol{\theta}}\in H^1_\sharp(\omega;\mathbb{R}^3)$ and $\tilde{\boldsymbol{\theta}}\in H^1_\sharp(\omega;\mathbb{R}^3)$ are said to be *properly isometrically equivalent* if there exist a vector $\boldsymbol{b}\in\mathbb{R}^3$ and a matrix $\boldsymbol{R}\in\mathbb{O}^3_+$ such that

$$\hat{\boldsymbol{\theta}}(y)=\boldsymbol{b}+\boldsymbol{R}\tilde{\boldsymbol{\theta}}(y) \quad \text{for almost all } y\in\omega,$$

and, by extension, the *surfaces* $\hat{\boldsymbol{\theta}}(\omega)$ and $\tilde{\boldsymbol{\theta}}(\omega)$ are also said to be *properly isometrically equivalent*. Note that, while *the fundamental forms of properly isometrically equivalent surfaces are clearly equal a.e. in ω*, the converse does *not* hold in general. The converse *does* hold, however, if one of the mappings is in $\mathcal{C}^1(\overline{\omega})$ and its associated normal vector field is also in $\mathcal{C}^1(\overline{\omega})$ (see Ciarlet and Mardare [15, Theorem 3]).

One application of Theorem 4.1 is then the following result of *sequential continuity for surfaces*:

Corollary 5.1. *Let $(a_{\alpha\beta}),(b_{\alpha\beta}),(c_{\alpha\beta})$ denote the three fundamental forms of a surface $\boldsymbol{\theta}(\overline{\omega})$, where $\boldsymbol{\theta}\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ is an immersion satisfying $\boldsymbol{a}_3\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$. Let $\boldsymbol{\theta}^k\in H^1_\sharp(\omega;\mathbb{R}^3)$, $k\geqslant 1$, be a sequence of mappings with the following properties: There exists a constant $\varepsilon>0$ such that the principal radii of curvature R^k_α of the surfaces $\boldsymbol{\theta}^k(\omega)$ satisfy*

$$\big|R^k_\alpha\big|\geqslant\varepsilon>0 \quad \textit{a.e. in } \omega \textit{ for all } k\geqslant 1,$$

and (with self-explanatory notations)

$$\big(a^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(a_{\alpha\beta}), \quad \big(b^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(b_{\alpha\beta}), \quad \big(c^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(c_{\alpha\beta}) \quad \textit{in } L^1\big(\omega;\mathbb{S}^2\big).$$

Then there exist mappings $\hat{\boldsymbol{\theta}}^k \in H^1_\sharp(\omega;\mathbb{R}^3)$ *that are properly isometrically equivalent to the mappings* $\boldsymbol{\theta}^k$, $k \geqslant 1$, *such that*

$$\hat{\boldsymbol{\theta}}^k \underset{k\to\infty}{\longrightarrow} \boldsymbol{\theta} \quad \text{and} \quad \hat{\boldsymbol{a}}_3^k \underset{k\to\infty}{\longrightarrow} \boldsymbol{a}_3 \quad \text{in } H^1(\omega;\mathbb{R}^3).$$

Proof. The proof is an immediate consequence of the inequality established in Theorem 4.1. □

A significant strengthening of the regularity assumptions regarding the convergence of the first and second fundamental forms yields another result of *sequential continuity for surfaces*, this time without any assumptions on their third fundamental forms nor on their principal radii of curvature.

Corollary 5.2. *Let* $(a_{\alpha\beta})$ *and* $(b_{\alpha\beta})$ *denote the first and second fundamental forms of a surface* $\boldsymbol{\theta}(\overline{\omega})$, *where* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ *is an immersion satisfying* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$. *Let* $\boldsymbol{\theta}^k \in H^1_\sharp(\omega;\mathbb{R}^3)$, $k \geqslant 1$, *be a sequence of mappings such that* (*with self-explanatory notations*) $a^k_{\alpha\beta} \in L^\infty(\omega)$, $b^k_{\alpha\beta} \in L^\infty(\omega)$, *and*

$$\left(a^k_{\alpha\beta}\right) \underset{k\to\infty}{\longrightarrow} (a_{\alpha\beta}) \quad \text{and} \quad \left(b^k_{\alpha\beta}\right) \underset{k\to\infty}{\longrightarrow} (b_{\alpha\beta}) \quad \text{in } L^\infty(\omega;\mathbb{S}^2).$$

Then there exist mappings $\hat{\boldsymbol{\theta}}^k \in H^1_\sharp(\omega;\mathbb{R}^3)$ *that are properly isometrically equivalent to the mappings* $\boldsymbol{\theta}^k$, $k \geqslant 1$, *such that*

$$\hat{\boldsymbol{\theta}}^k \underset{k\to\infty}{\longrightarrow} \boldsymbol{\theta} \quad \text{and} \quad \hat{\boldsymbol{a}}_3^k \underset{k\to\infty}{\longrightarrow} \boldsymbol{a}_3 \quad \text{in } H^1(\omega;\mathbb{R}^3).$$

Proof. The notations used in this proof should be self-explanatory. The above assumptions imply the following properties: The third fundamental forms $(c^k_{\alpha\beta}) = (a^{\sigma\tau,k} b^k_{\alpha\tau} b^k_{\sigma\beta})$ of the surfaces $\boldsymbol{\theta}^k(\omega)$ are also in $L^\infty(\omega;\mathbb{S}^2)$, they satisfy

$$\left(c^k_{\alpha\beta}\right) \underset{k\to\infty}{\longrightarrow} (c_{\alpha\beta}) \quad \text{in } L^\infty(\omega;\mathbb{S}^2),$$

and the eigenvalues of matrices $(b^{\sigma,k}_\alpha)$ converge in $L^\infty(\omega)$ to the eigenvalues of the matrix (b^σ_α) as $k \to \infty$. This last property implies that there exists $\varepsilon > 0$ such that $|R^k| \geqslant \varepsilon$ for all $k \geqslant 1$. The conclusion is then another consequence of the Korn inequality of Theorem 4.1. □

The Korn inequality of Theorem 4.1 can also be recast as one involving *distances in metric spaces*. To this end, define the quotient set

$$\dot{H}^1_\sharp(\omega;\mathbb{R}^3) = H^1_\sharp(\omega;\mathbb{R}^3)/\mathcal{R},$$

where $(\boldsymbol{\chi},\boldsymbol{\theta}) \in \mathcal{R}$ means that $\boldsymbol{\chi} \in H^1_\sharp(\omega;\mathbb{R}^3)$ and $\boldsymbol{\theta} \in H^1_\sharp(\omega;\mathbb{R}^3)$ are properly isometrically equivalent, and let $\dot{\boldsymbol{\theta}}$ denote the equivalence class of $\boldsymbol{\theta} \in H^1_\sharp(\omega;\mathbb{R}^3)$ *modulo* $\mathcal{R}$. Since the norm $\|\cdot\|_{H^1(\omega;\mathbb{R}^3)}$ is invariant under the action of $\mathbb{O}^3_+$ (in the sense that $\|\boldsymbol{Q}\boldsymbol{\theta}\|_{H^1(\omega;\mathbb{R}^3)} = \|\boldsymbol{\theta}\|_{H^1(\omega;\mathbb{R}^3)}$ for any $\boldsymbol{Q} \in \mathbb{O}^3_+$ and any $\boldsymbol{\theta} \in H^1(\omega;\mathbb{R}^3)$), the mapping $d : \dot{H}^1_\sharp(\omega;\mathbb{R}^3) \times \dot{H}^1_\sharp(\omega;\mathbb{R}^3) \to \mathbb{R}$ defined by

$$d\big(\dot{\tilde{\boldsymbol{\theta}}},\dot{\boldsymbol{\theta}}\big) := \inf_{\boldsymbol{b}\in\mathbb{R}^3,\ \boldsymbol{R}\in\mathbb{O}^3_+} \left\{ \big\|\big(\boldsymbol{b} + \boldsymbol{R}\tilde{\boldsymbol{\theta}}\big) - \boldsymbol{\theta}\big\|_{H^1(\omega;\mathbb{R}^3)} + \|\boldsymbol{R}\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3\|_{H^1(\omega;\mathbb{R}^3)} \right\}$$

is a *distance* on the quotient set $\dot{H}^1_\sharp(\omega;\mathbb{R}^3)$. In terms of this distance, the inequality of Theorem 4.1 then becomes:

Corollary 5.3. *Let there be given a domain* ω *in* $\mathbb{R}^2$, *an immersion* $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ *such that* $\boldsymbol{a}_3 \in \mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$, *and* $\varepsilon > 0$. *Then there exists a constant* $\dot{c}(\boldsymbol{\theta},\varepsilon)$ *with the following property*: *Given any mapping* $\tilde{\boldsymbol{\theta}} \in H^1_\sharp(\omega;\mathbb{R}^3)$ *such that* $|\widetilde{R}_\alpha| \geqslant \varepsilon$ *a.e. in* ω,

$$d\big(\dot{\tilde{\boldsymbol{\theta}}},\dot{\boldsymbol{\theta}}\big) \leqslant \dot{c}(\boldsymbol{\theta},\varepsilon)\left\{ \big\|(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)} + \varepsilon^{1/2}\big\|\big(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}\big)\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)} + \varepsilon\big\|(\tilde{c}_{\alpha\beta} - c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)} \right\}. \qquad \square$$

6. The linear Korn inequality on a surface revisited

To begin with, we observe that the nonlinear Korn inequality on a surface established in Theorem 4.1 may be equivalently restated as follows, thanks to the invariance of the norm $\|\cdot\|_{H^1(\omega;\mathbb{R}^3)}$ under the action of the group $\mathbb{O}^3_+$. Given an immersion $\boldsymbol{\theta}\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ such that $\boldsymbol{a}_3\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ and $\varepsilon>0$, there exists a constant $c(\boldsymbol{\theta},\varepsilon)$ with the following property: Given any mapping $\tilde{\boldsymbol{\theta}}\in H^1(\omega;\mathbb{R}^3)$ such that $\tilde{\boldsymbol{a}}_1\wedge\tilde{\boldsymbol{a}}_2\neq\boldsymbol{0}$ a.e. in ω, $\tilde{\boldsymbol{a}}_3\in H^1(\omega;\mathbb{R}^3)$, and the principal radii of curvature $\widetilde{R}_\alpha$ of the surface $\tilde{\boldsymbol{\theta}}(\omega)$ satisfy

$$\big|\widetilde{R}_\alpha\big|\geqslant\varepsilon\quad\text{a.e. in }\omega,$$

there exist a vector $\boldsymbol{a}=\boldsymbol{a}(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},\varepsilon)\in\mathbb{R}^3$ and a matrix $\boldsymbol{Q}=\boldsymbol{Q}(\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},\varepsilon)\in\mathbb{O}^3_+$ such that

$$\begin{aligned}&\big\|\tilde{\boldsymbol{\theta}}-(\boldsymbol{a}+\boldsymbol{Q}\boldsymbol{\theta})\big\|_{H^1(\omega;\mathbb{R}^3)}+\varepsilon\|\tilde{\boldsymbol{a}}_3-\boldsymbol{Q}\boldsymbol{a}_3\|_{H^1(\omega;\mathbb{R}^3)}\\&\quad\leqslant c(\boldsymbol{\theta},\varepsilon)\big\{\big\|(\tilde{a}_{\alpha\beta}-a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon^{1/2}\big\|\big(\tilde{b}_{\alpha\beta}-b_{\alpha\beta}\big)\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon\big\|(\tilde{c}_{\alpha\beta}-c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}\big\}.\end{aligned}$$

To shed more light on this inequality, we now compare it with its *linear* counterpart, the genesis of which we first briefly review.

Let ω be a domain in $\mathbb{R}^2$ and let there be given an immersion $\boldsymbol{\theta}\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$ such that $\boldsymbol{a}_3\in\mathcal{C}^1(\overline{\omega};\mathbb{R}^3)$. The *"linear" Korn's inequality on a surface* then asserts the existence of a constant $c_0(\boldsymbol{\theta})$ such that

$$\begin{aligned}&\big\{\|\tilde{\boldsymbol{\eta}}\|^2_{H^1(\omega;\mathbb{R}^3)}+\big\|\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}})\big\|^2_{H^1(\omega;\mathbb{R}^3)}\big\}^{1/2}\\&\quad\leqslant c_0(\boldsymbol{\theta})\big\{\|\tilde{\boldsymbol{\eta}}\|^2_{L^2(\omega;\mathbb{R}^3)}+\big\|\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}})\big\|^2_{L^2(\omega;\mathbb{R}^3)}+\big\|\big(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)}+\big\|\big(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)}\big\}^{1/2}\end{aligned}$$

for all vector fields

$$\tilde{\boldsymbol{\eta}}\in\widetilde{\boldsymbol{V}}(\omega):=\big\{\tilde{\boldsymbol{\eta}}\in H^1\big(\omega;\mathbb{R}^3\big);\ \boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}})\in H^1\big(\omega;\mathbb{R}^3\big)\big\},$$

where

$$\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}}):=-(\partial_\alpha\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_3)a^{\alpha\beta}\boldsymbol{a}_\beta,\qquad\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}):=\frac{1}{2}(\partial_\beta\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\alpha+\partial_\alpha\tilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\beta)\in L^2(\omega),$$

$$\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}):=-\big(\partial_\beta\tilde{\boldsymbol{\eta}}\cdot\partial_\alpha\boldsymbol{a}_3+\partial_\alpha\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}})\cdot\boldsymbol{a}_\beta\big)\in L^2(\omega),$$

and the vectors $\boldsymbol{a}_i$ are defined as in Lemma 3.2 in terms of the immersion $\boldsymbol{\theta}$ (the notation $\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}})$ will be justified later). Under the assumption that $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$, this inequality was first proved by Bernadou and Ciarlet [4] and was later given a simpler proof by Ciarlet and Miara [19] (see also Bernadou, Ciarlet and Miara [5]). The regularity assumption on the immersion $\boldsymbol{\theta}$ was weakened to that considered here by Le Dret [24] (see also Blouza and Le Dret [6]).

The linear Korn inequality is the basis of the *existence theorems in linear shell theory* (see, e.g., [7] or [9]). In this context, the surface $\boldsymbol{\theta}(\overline{\omega})$ is the *middle surface* of a *linearly elastic shell*, the vector fields $\tilde{\boldsymbol{\eta}}\in\widetilde{\boldsymbol{V}}(\omega)$ are *displacement fields of the surface* $\boldsymbol{\theta}(\overline{\omega})$, and the matrix fields $(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))\in\boldsymbol{L}^2(\omega)$ and $(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))\in\boldsymbol{L}^2(\omega)$ are respectively the *linearized change of metric*, and *linearized change of curvature, tensors* associated with such displacement fields. Let

$$\mathbf{Rig}^{\mathrm{lin}}(\omega)=\big\{\tilde{\boldsymbol{\eta}}\in\widetilde{\boldsymbol{V}}(\omega);\ \gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})=\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})=0\text{ in }\omega\big\}$$

denote the space of *infinitesimal rigid displacement of the surface* $\boldsymbol{\theta}(\overline{\omega})$. Then this space can be equivalently defined as (see [3])

$$\mathbf{Rig}^{\mathrm{lin}}(\omega)=\big\{\tilde{\boldsymbol{\eta}}\in\widetilde{\boldsymbol{V}}(\omega);\ \tilde{\boldsymbol{\eta}}=\boldsymbol{a}+\boldsymbol{b}\wedge\boldsymbol{\theta}\text{ for some }\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3\big\}.$$

Given any displacement field $\tilde{\boldsymbol{\eta}}\in\widetilde{\boldsymbol{V}}(\omega)$ of the surface $\boldsymbol{\theta}(\overline{\omega})$, let

$$\tilde{\boldsymbol{\theta}}:=(\boldsymbol{\theta}+\tilde{\boldsymbol{\eta}})\in H^1\big(\omega;\mathbb{R}^3\big)$$

denote the associated *deformation* of the surface $\boldsymbol{\theta}(\overline{\omega})$, and *assume in addition that* $\tilde{\boldsymbol{a}}_1\wedge\tilde{\boldsymbol{a}}_2\neq\boldsymbol{0}$ a.e. in ω *and*

$$\tilde{\boldsymbol{a}}_3:=\frac{\tilde{\boldsymbol{a}}_1\wedge\tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1\wedge\tilde{\boldsymbol{a}}_2|}\in H^1\big(\omega;\mathbb{R}^3\big);$$

in other words, the mappings $\tilde{\boldsymbol{\theta}}$ precisely satisfy the assumptions of Theorem 4.1. Let

$$(a_{\alpha\beta}) = (\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta) \in \boldsymbol{L}^2(\omega) \quad \text{and} \quad (b_{\alpha\beta}) = (-\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta) \in \boldsymbol{L}^2(\omega),$$

and

$$(\tilde{a}_{\alpha\beta}) = (\tilde{\boldsymbol{a}}_\alpha \cdot \tilde{\boldsymbol{a}}_\beta) \in \boldsymbol{L}^2(\omega) \quad \text{and} \quad (\tilde{b}_{\alpha\beta}) = (-\partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \tilde{\boldsymbol{a}}_\beta) \in \boldsymbol{L}^2(\omega),$$

respectively denote the first and second fundamental forms of the surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $\tilde{\boldsymbol{\theta}}(\omega)$. Then it is well known (see, e.g., [7] or [9]) that the tensors $(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))$ and $(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))$ can also be defined as

$$\big(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big) = \left(\frac{1}{2}[\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\text{lin}}\right) \quad \text{and} \quad \big(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big) = \big([\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\text{lin}}\big),$$

where $[\ldots]^{\text{lin}}$ denotes the linear part with respect to $\tilde{\boldsymbol{\eta}}$ in the expression $[\ldots]$. In the same vein, it can also be easily verified that

$$\boldsymbol{\Delta a}_3(\tilde{\boldsymbol{\eta}}) = [\tilde{\boldsymbol{a}}_3 - \boldsymbol{a}_3]^{\text{lin}}.$$

Finally, define the *quotient space*

$$\dot{\tilde{\boldsymbol{V}}}(\omega) := \widetilde{\boldsymbol{V}}(\omega)/\mathbf{Rig}^{\text{lin}}(\omega),$$

and let $\|\cdot\|_{\dot{\tilde{\boldsymbol{V}}}(\omega)}$ denote the associated quotient norm. Arguing as in [12], it can then be shown that the above linear Korn inequality is *equivalent* to the following *Korn inequality in the quotient space* $\dot{\tilde{\boldsymbol{V}}}(\omega)$: There exists a constant $c_1(\boldsymbol{\theta})$ such that

$$\|\dot{\tilde{\boldsymbol{\eta}}}\|_{\dot{\tilde{\boldsymbol{V}}}(\omega)} \leqslant c_1(\boldsymbol{\theta})\big\{\big\|\big(\gamma_{\alpha\beta}(\dot{\tilde{\boldsymbol{\eta}}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)} + \big\|\big(\rho_{\alpha\beta}(\dot{\tilde{\boldsymbol{\eta}}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)}\big\}^{1/2}$$

for all $\dot{\tilde{\boldsymbol{\eta}}} \in \dot{\tilde{\boldsymbol{V}}}(\omega)$. Thanks to the definition of the quotient norm and to the specific form taken by the infinitesimal rigid displacements of the surface $\boldsymbol{\theta}(\overline{\omega})$, this inequality can be immediately recast as follows: Given any vector field $\tilde{\boldsymbol{\eta}} \in \widetilde{\boldsymbol{V}}(\omega)$, there exist vectors $\boldsymbol{a} = \boldsymbol{a}(\tilde{\boldsymbol{\eta}}, \boldsymbol{\theta}) \in \mathbb{R}^3$ and $\boldsymbol{b} = \boldsymbol{b}(\tilde{\boldsymbol{\eta}}, \boldsymbol{\theta}) \in \mathbb{R}^3$ such that

$$\begin{aligned}&\big\{\big\|\tilde{\boldsymbol{\eta}} - (\boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta})\big\|^2_{H^1(\omega;\mathbb{R}^3)} + \big\|\boldsymbol{\Delta a}_3\big(\tilde{\boldsymbol{\eta}} - (\boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta})\big)\big\|^2_{H^1(\omega;\mathbb{R}^3)}\big\}^{1/2}\\ &\quad \leqslant c_1(\boldsymbol{\theta})\big\{\big\|\big(\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)} + \big\|\big(\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)}\big\}^{1/2}.\end{aligned}$$

In terms of deformation of surfaces and fundamental forms, the *linear Korn inequality on a surface* thus asserts the existence of a constant $c_1(\boldsymbol{\theta})$ with the following property: *Given any* deformation $\tilde{\boldsymbol{\theta}} = (\boldsymbol{\theta} + \tilde{\boldsymbol{\eta}})$ of the surface $\boldsymbol{\theta}(\overline{\omega})$ such that $\tilde{\boldsymbol{\eta}} \in \widetilde{\boldsymbol{V}}(\omega)$, $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq \boldsymbol{0}$ a.e. in ω, and $\tilde{\boldsymbol{a}}_3 \in \boldsymbol{H}^1(\omega)$, there exist vectors $\boldsymbol{a} = \boldsymbol{a}(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) \in \mathbb{R}^3$ and $\boldsymbol{b} = \boldsymbol{b}(\tilde{\boldsymbol{\theta}}, \boldsymbol{\theta}) \in \mathbb{R}^3$ such that

$$\begin{aligned}&\big\{\big\|\tilde{\boldsymbol{\theta}} - (\boldsymbol{a} + \boldsymbol{\theta} + \boldsymbol{b} \wedge \boldsymbol{\theta})\big\|_{H^1(\omega;\mathbb{R}^3)} + \big\|\boldsymbol{\Delta a}_3\big(\tilde{\boldsymbol{\theta}} - (\boldsymbol{a} + \boldsymbol{\theta} + \boldsymbol{b} \wedge \boldsymbol{\theta})\big)\big\|_{H^1(\omega;\mathbb{R}^3)}\big\}^{1/2}\\ &\quad \leqslant c_1(\boldsymbol{\theta})\big\{\big\|\big([\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\text{lin}}\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)} + \big\|\big([\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\text{lin}}\big)\big\|^2_{L^2(\omega;\mathbb{S}^2)}\big\}^{1/2}.\end{aligned}$$

This last inequality provides the *essence of the linear Korn inequality on a surface*: *The* $H^1(\omega;\mathbb{R}^3)$*-distance between the deformed surface* $\tilde{\boldsymbol{\theta}}(\omega)$ *and the surface* $\boldsymbol{\theta}(\overline{\omega})$ *and the* $H^1(\omega;\mathbb{R}^3)$*-norm of the linearized difference between their normal vector fields* $\tilde{\boldsymbol{a}}_3$ *and* $\boldsymbol{a}_3$ *are "controlled" by the* $L^2(\omega;\mathbb{S}^2)$*-norms of the linearized change of metric, and change of curvature, tensors associated with the vector field* $\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}$.

As expected, the distance between the two surfaces is only measured *up to infinitesimal rigid displacements of the surface* $\boldsymbol{\theta}(\overline{\omega})$, since these are precisely those whose associated matrix fields $([\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\text{lin}})$ and $([\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\text{lin}})$ vanishes (this indeterminacy would no longer hold if the displacements fields $\tilde{\boldsymbol{\eta}}$ were subjected to appropriate boundary conditions, such as those of clamping along a portion γ_0 of $\partial\omega$ satisfying *length* $\gamma_0 > 0$; cf. [3, Theorem 4.1] and [7, Theorem 2.6-3]). In the same spirit, the term $\boldsymbol{Q\theta}$ appearing in the nonlinear inequality is replaced by the term $\boldsymbol{\theta} + \boldsymbol{b} \wedge \boldsymbol{\theta}$ in the linear inequality. This replacement simply reflects that the matrix $\boldsymbol{Q} \in \mathbb{O}^3_+$ is close to the identity matrix if the displacement vector field $\tilde{\boldsymbol{\eta}}$ is small (it is well known that the tangent space to the manifold $\mathbb{O}^3_+$ at the identity matrix coincides with the space of all antisymmetric matrices of order three; cf., e.g., Avez [1]).

Recast in this way, the *"linear" Korn inequality on a surface thus appears as a natural linearization of the nonlinear Korn inequality on a surface*, as rewritten at the beginning of this section.

This is obvious for their right-hand sides, where the matrix fields $(\tilde{a}_{\alpha\beta} - a_{\alpha\beta})$ and $(\tilde{b}_{\alpha\beta} - b_{\alpha\beta})$ are replaced by their linearized fields $([\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\mathrm{lin}}) = (\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))$ and $([\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\mathrm{lin}}) = (\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}}))$ (that the $L^1(\omega; \mathbb{S}^2)$-norm is replaced by $L^2(\omega; \mathbb{S}^2)$-norm is no surprise, since each norm corresponds to the regularity of the mappings $\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}$ respectively assumed in the nonlinear and linearized cases).

This is true, *albeit* less evident, for their left-hand sides. As shown by Ciarlet and Mardare [15], the underlying reason is that the set

$$\boldsymbol{M}(\omega) := \{\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3);\, \tilde{a}_{\alpha\beta} = a_{\alpha\beta} \text{ a.e. in } \omega,\, \tilde{\boldsymbol{a}}_3 \in H^1(\omega; \mathbb{R}^3),\, \tilde{b}_{\alpha\beta} = b_{\alpha\beta} \text{ a.e. in } \omega\}$$

is a submanifold (of dimension 6) of the space $H^1(\omega; \mathbb{R}^3)$, and furthermore, the space $\mathbf{Rig}^{\mathrm{lin}}(\omega)$ (also of dimension 6) is nothing but the *tangent space* $T_{\boldsymbol{\theta}}\boldsymbol{M}(\omega)$ at $\boldsymbol{\theta}$ to $\boldsymbol{M}(\omega)$. In other words,

$$T_{\boldsymbol{\theta}}\boldsymbol{M}(\omega) = \{\tilde{\boldsymbol{\eta}} \in H^1(\omega; \mathbb{R}^3);\;\; \tilde{\boldsymbol{\eta}} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta} \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

Finally, that the linearized tensor field $([\tilde{c}_{\alpha\beta} - c_{\alpha\beta}]^{\mathrm{lin}})$ does not appear in the right-hand side of the linear Korn inequality is no surprise: it is an easy matter to show that the $L^2(\omega; \mathbb{S}^2)$-norm of this linearized tensor field is controlled by the sum of the $L^2(\omega; \mathbb{S}^2)$-norms of the linearized tensor fields $([\tilde{a}_{\alpha\beta} - a_{\alpha\beta}]^{\mathrm{lin}})$ and $([\tilde{b}_{\alpha\beta} - b_{\alpha\beta}]^{\mathrm{lin}})$.

7. Concluding remarks

The nonlinear inequality established in this paper has potential applications to differential geometry and to nonlinear shell theory. From the viewpoint of differential geometry, the continuity result implied by this inequality is a mathematical expression of a natural idea: If the fundamental forms of two surfaces in $\mathbb{R}^3$ are close, then the two surfaces are also close (up to proper isometries, of course). While the previous results in this direction involved topologies of spaces of continuously differentiable mappings (see [8,18] and [27]), the present result can be considered as a genuine improvement over these, inasmuch as the norms used for evaluating the distance between the fundamental forms and surfaces are "weaker".

From the viewpoint of nonlinear shell theory, this inequality also represents a first step toward considering the fundamental forms of the unknown deformed surface as the primary unknowns. But, unlike in the linear case [12], much further work is clearly needed before a satisfactory existence theory can be developed along these lines.

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