JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 179, 120–134 (1993)

Approximation of Dissipative Hereditary Systems*

RICHARD C. MACCAMY

Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213-3890

Submitted by Thomas S. Angell

Received April 23, 1991

1. INTRODUCTION

The idea of this paper was suggested to the author during work on artificial boundary techniques for exterior scattering problems. The underlying theme is this. One has an evolution equation containing effects that are non-local in space and/or time and these effects produce dissipation. The non-local effects also produce serious numerical difficulties. The question is can one approximate with equations which are more local, hence easier to handle, while preserving the dissipation.

In the scattering problems the effects are both spatially and temporally non-local. They are introduced artificially as a numerical device to reduce the problems to finite domains, [1], [4] and [5]. Here we consider systems in which the non-local effect is only temporal but in which that effect is part of the model. Such models have been very useful in control theory, viscoelasticity and heat flow.

This paper is a very modest first effort. We consider a simple model equation for which we can give precise, but non-trivial, dissipativity results. We indicate some applications and possible extensions in Section 5.

Our equation is typical of the hereditary models which have been successfully studied. The equations can be nonlinear but the memory effect is linear. Specifically we consider equations of the form,

$$\dot{u}(t) + L_a[g(u)]^{\cdot}(t) = f(t), \qquad t > 0 \quad u(0) = u_0$$

$$L_a[\zeta](t) = \int_0^t a(t-\tau)\,\zeta(\tau))\,d\tau \tag{E_a}$$

We study this problem in a familiar Banach space setting with g possibly non-linear but monotone and coercive.

* This work was supported by the National Science Foundation under Grant DMS-90-01012.

0022-247X/93 $5.00

We call (E_a) a *hereditary* system. We say it is *dissipative* relative to a class $\mathscr{F}$ if for any u_0 and $f \in \mathscr{F}$ there is a unique solution $u(t)$ for all $t > 0$ and there is a unique $u_\infty = U^\infty[f, u_0]$ such that (in some sense) $u(t) \to u_\infty$ as $t \to \infty$.

There exist numerical approximation methods for equations like (E_a), [9], [11]. These are accurate but very complicated for reasons which are shown in Section 4. When $a(t) \equiv 1$ (E_a) reduces to a differential equation,

$$\dot{u}(t) + g(u(t)) = f(t), \qquad t > 0 \quad u(0) = u_0. \qquad (E_1)$$

In the setting we use the theory for (E_1) is very well known, [6], and one can obtain numerical results quite easily by using simple time stepping, [3].

We will give conditions on g and a which insure dissipativity for (E_a). These results follow rather directly from work in [2], [7] and [8]. Our main goal is to approximate (E_a) with a low order ordinary differential equation. Numerically it will have about the same simplicity as (E_1). We want this equation to preserve the dissipation exactly so that we capture the long time behavior. We also want it to capture the short time behavior. Finally we want to be able to form our approximating equation with very little specific knowledge about the kernel a.

The price for all the simplicity is, of course, that our scheme may be very crude at intermediate times. We do not have any very good theorems about the error in our approximation, a defect which also occurs in the scattering theory results. As a partial check we present a numerical example in Section 4. We find the results there most intriguing. Our approximation contains a free parameter γ and the outcomes are very sensitive to the choice of γ. In Section 2 we suggest two possible choices, based on rather vague arguments. In Section 4 we find that the first choice gives considerable error at intermediate times while the second gives striking accuracy for all t.

Our idea is extremely simple. We use well known Volterra equation ideas to show that dissipativity is controlled by properties of the Laplace transform $\hat{a}(s)$. Next we follow scattering theory ideas and find a low order Padé approximation $\hat{b}$ which agrees with $\hat{a}$ for large and small s and preserves the dissipativity condition in the transform domain. Then we replace a in (E_a) by the inverse transform b of $\hat{b}$. The rational character of $\hat{b}$ implies that (E_b) is equivalent to a differential equation.

The author wishes to make two acknowledgments. Preliminary work on the approximation idea was carried out in a Master's thesis by Petros Hadjicostas for the linear scalar problem of a viscoelastic fiber. Assistance on the numerical computations was provided by Daniel Burkett.

2. Statement of Results

Our general setting is familiar. We have a separable, reflexive Banach space V continuously imbedded in a Hilbert space H. We imbed H in the dual V' of V by $\langle h, v\rangle = (h, v)_H$. We have a map g from V into V', with $g(0) = 0$, and we assume there are constants $p \geqslant 2$, $M > 0$ and $m > 0$ such that,

$$\|g\|_{V'} \leqslant M(1 + \|u\|_V^{p-1}), \ \langle g(u) - g(v), u - v\rangle \geqslant m \|u - u\|_V^p. \qquad \text{(G)}$$

We suppose $u_0 \in H$ and that f has the form,

$$f(t) = f_\infty + F(t), F \in L_q(0, \infty : V'), p^{-1} + q^{-1} = 1. \qquad (\mathrm{F}_1)$$

Under these conditions the following result is standard [6] and its proof is essentially the same as Theorem 2 below.

Theorem 1. (i) *There is a unique* u_∞ *such that* $g(u_\infty) = f_\infty$

(ii) (E_1) *has a unique (generalized) solution* u *for all* $t > 0$ *with*

$$\dot{u} \in L_2(0, \infty : V'), u - u_\infty \in L_p(0, \infty : V). \qquad (2.1)$$

Remark 2.1. The second of equations (2.1) is our decay result. It suggests that $u(t) \to u_\infty$ in V as $t \to \infty$ and, in fact, implies that result if one has additional smoothness on u.

We aim at an analogous result for (E_a) if (G) holds and we need conditions on a. These conditions are rather technical and are given in the Laplace transform domain. Let us make a definition.

Definition 1. Let $\Gamma = \{s : s = \xi + i\eta, \xi > 0\}$. The space $\hat{L}_m$ is the set of all $\hat{\varphi}$ such that:

(i) $\hat{\varphi}(s) = \varphi_\infty s^{-1} + \hat{\Phi}(s), \varphi_\infty \geqslant 0, \hat{\Phi} \in C^{(3)}(\bar{\Gamma})$

(ii) $\hat{\Phi}$ is analytic in Γ and $\hat{\varphi}(s)$ is real for s real and positive

(iii) $\Phi(s) = \Phi_0 s^{-1} + \cdots + \Phi_m s^{-m-1} + 0(s^{-m-2})$ as $s \to \infty$ in $\bar{\Gamma}$

(iv) $\hat{\Phi}^{(j)}(s) = 0(s^{-2})$ as $s \to \infty$ $j = 1, 2, 3$.

Functions $\hat{\varphi}$ in $\hat{L}_m$ are the Laplace transforms of functions $\varphi(t) = \varphi_\infty + \Phi(t)$ with properties which we describe at the end of the section. First let us state our results. We make two assumptions on a.[1] The first is technical:

$$a \text{ has a Laplace transform } \hat{a} \text{ in } \hat{L}_4 \qquad (\mathrm{A.1})$$

[1] We assume here that a is a scaler function. We comment on the extension to the case where $a(t)$ is a family of linear operators in Section 5.

Proposition 1 at the end of the section shows that (A.1) implies $a \in C^{(2)}[0, \infty)$. Our second condition on a is a familiar one in Volterra equation theory and is the key to dissipativity.

$$a(0) > 0, \ \dot{a}(0) < 0, \ \operatorname{Re} \hat{a}(i\eta) > 0 \quad \forall \eta. \tag{A.2}$$

Remark 2.2. It is known that a sufficient condition for (A.2) is $(-1)^k a^{(k)}(t) > 0$, $k = 0, 1, 2$. A prototype is $e^{-\alpha t}$, $\alpha > 0$. There are, however, oscillatory functions which satisfy (A.2), for instance $e^{-\alpha t} \cos \beta t$, $\alpha > 0$, $\beta > 0$

We also need an additional condition on f:

$$F \in L_1(0, \infty : V'), \int_0^\infty \left(\int_t^\infty \|F(\tau)\|_{V'} \, d\tau \right)^2 dt < \infty. \tag{F$_2$}$$

We assume (G), (A_1), (A_2), (F_1), and (F_2) hold.

Dissipativity

THEOREM 2. (i) *If $a_\infty > 0$ there is a unique $u_\infty = U_a^\infty [f, u_0]$ such that*

$$a_\infty g(u_\infty) = f_\infty. \tag{2.2}$$

(ii) *If $a_\infty = 0$ and $f_\infty = 0$ there is a unique $u_\infty = U_a^\infty [f, u_0]$ such that*

$$\hat{A}(0) u_\infty + g(u_\infty) = u_0 + \int_0^\infty F(\tau) \, d\tau. \tag{2.3}$$

(iii) *In either case* (E_a) *has a unique (generalized) solution for all $t > 0$ with,*

$$\dot{u} \in L_2(0, \infty : V'), \ u - u_\infty \in L_p(0, \infty : V). \tag{2.4}$$

Remark 2.3. Note that the damping effect with $a_\infty > 0$ is stronger than when $a_\infty = 0$.

Approximation

THEOREM 3. *Define* $\hat{b}(s) = a_\infty s^{-1} + \hat{B}(s)$ *where*

$$\hat{B}(s) = (s^2 + Rs + \gamma)^{-1} (A(0) s + \gamma \hat{A}(0)), \ R = (\gamma \hat{A}(0) - \dot{A}(0))/A(0) \tag{2.5}$$

Then, for any $\gamma > 0$, $\hat{b}$ is the transform of a function b satisfying the hypothesis of Theorem 2 *with $b_\infty = a_\infty$ and*

$$b(0) = a(0), \ \dot{b}(0) = \dot{a}(0), \ \hat{B}(0) = \hat{A}(0) \tag{2.6}$$

We will explain and verify the following result in the next section.

THEOREM 4. *If $a_\infty = 0$ ($a_\infty > 0$) (E_b) is formally equivalent to a second (third) order differential equation.*

The idea now is to solve (E_b) giving u_b and hope that it is close to the solution u_a of (E_a). A first remark in this direction is u_a and u_b will agree for long time. Indeed we have $U_a^{\infty}[f, u_0] = U_b^{\infty}[f, u_0]$ so that $u_b - u_a \in L_p(0, \infty : V)$. Under some additional assumptions they will also agree for short time. Suppose $u_0 \in V$ and g is differentiable at u_0. Then from the equation one has formally,

$$\begin{aligned} u_a(0) &= u_b(0) = u_0, \quad \dot{u}_a(0) = \dot{u}_b(0) = f(0) - a(0)\, g(u_0) \\ \ddot{u}_a(0) &= \ddot{u}_b(0) = \dot{f}(0) - a(0)\, g'(u_0)[f(0) - a(0)\, g(u_0)] - \dot{a}(0)\, g(u_0) \end{aligned} \tag{2.7}$$

If u_a and u_b are smooth enough that (2.1) is valid then we have

$$u_b(t) - u_a(t) = o(t^2) \qquad \text{as} \quad t \to 0 \tag{2.8}$$

Remark 2.4. For each of the two functions in Remark 2.2 the approximate equation (E_b) is exact.

The parameter γ is so far free. We indicate two possible choices.

Choice I. So far we have $a(0) = b(0)$, $\dot{a}(0) = \dot{b}(0)$. An obvious choice is to try to make $\ddot{a}(0) = \ddot{b}(0)$. It will follow from Proposition 1 that this will be so if,

$$\gamma = (\dot{A}(0)^2 - \ddot{A}(0)\, A(0))/(A(0)^2 + \dot{A}(0)\, \hat{A}(0)) \tag{2.9}$$

The argument of the preceding paragraph can then be extended to show that formally $u_a(t) - u_b(t) = o(t^3)$.

Choice II. Here we try to make b agree with a for large t instead of small. Suppose we know that a decays exponentially, $a(t) = 0(e^{-\alpha t})$ and we can choose $\gamma = \gamma_{II}$ so that

$$Re(-R + \sqrt{R^2 - 4\gamma_{II}})/2 = \alpha \tag{2.10}$$

and make $b(t)$ have the same exponential decay rate as $a(t)$.

Remark 2.5. The choice γ_I need not give a positive value and hence may not be usable. It is of interest to note that it is always positive for a commonly used class of kernels, namely,

$$A(t) = \int_0^\infty e^{-\lambda t} \varphi(\lambda)\, d\lambda, \quad \hat{A}(s) = \int_0^\infty \frac{\varphi(\lambda)}{\lambda + s}\, d\lambda$$

If $\varphi(\lambda) > 0$ and $\varphi(\lambda)$ is suitably restricted for λ near 0 and ∞ one can see that this satisfies our hypotheses. Under these conditions it is easy to see from Schwarz's lemma that both numerator and denominator in (2.9) are negative.

Remark 2.6. It is to be emphasized that our procedures use very little detailed information about $a(t)$. We need only a_∞, $a(0)$, $\dot{a}(0)$, $\hat{A}(0)$ in general and $\ddot{a}(0)$ or α if we use choices I or II. It is not difficult to imagine fairly simple experiments which would determine these quantities and g.

Remark 2.7. It will be clear from later calculations that we could take for $\hat{b}$ a higher order Padé approximation of $\hat{a}$. This would presumably, increase the accuracy but would produce higher order equations and require a knowledge of more derivatives of a at $t=0$.

The question of how well u_b approximates u_a for all t is a difficult one. (The same is true in artificial boundary theory). In Theorem 5 of the next section we give an estimate for the error $e = u_b - u_a$ in terms of the data. The constant in this estimate, however, depends on the nature of a and is difficult to compute so the result is not very useful. The numerical results in Section 4 indicate that the error is very sensitive to γ. We find that the choice (2.9) yields a very crude result at intermediate times while the choice (2.10) gives quite accurate results.

We comment on the meaning of $\hat{L}_m$. The idea is that a $\hat{\varphi}$ in L_m is the Laplace transform of a function φ determined by the inversion integral:

$$\varphi = \varphi_\infty + L^{-1}[\Phi], L^{-1}[\Phi](t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{i\eta t} \Phi(i\eta)\, d\eta. \quad (2.11)$$

PROPOSITION 1. (i) *Suppose* $\hat{\varphi} \in \hat{L}_1$. *Then* $\varphi(t) = \varphi_\infty + L^{-1}[\hat{\Phi}](t)$ *defines a function* φ *such that*:

$$\varphi \in C^{(1)}[0, \infty), \varphi(0) = \varphi_0, \dot{\varphi}(0) = \varphi_1, \varphi(t), \dot{\varphi}(t) = 0(t^{-3}) \quad as \quad t \to \infty.$$

(ii) *Suppose* $a \in C^{(4)}[0, \infty), a(t) = a_\infty + A(t)$ *with* $a_\infty > 0$ and $A^{(k)} \in L_1(0, \infty), k \leqslant 4$. *Put* $\alpha_j(t) = t^j A(t)$ *and suppose* $\alpha_j^{(k)} \in L_1(0, \infty)$ *for* $j \leqslant 3$, $k \leqslant 2$. *Then* $\hat{a} \in L_2$ *with* $a_j = a^{(j)}(0)\, j = 0, 1, 2$.

This result is not very difficult once one overcomes the notation. We observe first that the functions $\psi_j(t) = t^j e^{-t}$ have transforms $\hat{\psi}_j(s) = j!\,(s+1)^{-j-1}$. If $\hat{\varphi}$ is in $\hat{L}_1$ we can rewrite (iii) of Definition 1 as

$$\hat{\Phi} = \Phi_0 \hat{\psi}_0 + (\Phi_1 + \Phi_0)\, \hat{\psi}_1 (s) + \hat{\psi}(s) \qquad \hat{\psi}(s) = 0(s^{-3})$$

Thus $L^{-1}[\hat{\Phi}](t) = e^{-t}(\Phi_0 + (\Phi_1 + \Phi_0)\, t) + L^{-1}[\hat{\Phi}]$. It follows that $L^{-1}[\hat{\Phi}] \in C^{(1)}[0, \infty)$ with $L^{-1}[\hat{\Phi}](0) = \Phi_0$, $L^{-1}[\hat{\psi}](0) = k_1$. Next we observe that the conditions on $\hat{\Phi}$ yield the following result after three intergation by parts:

$$\Phi(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\eta t} \hat{\Phi}(i\eta)\, d\eta = -(2\pi)^{-1} t^{-3} \int_{-\infty}^{+\infty} e^{i\eta t} \hat{\Phi}'''(i\eta)\, d\eta$$

This shows that indeed $\Phi(t) = 0(t^{-3})$. One can perform a similar calculation to estimate $\dot{\Phi}$. This indicates the proof of (i).

The proof of (ii) is tedious but straightforward. First note that $t^3 A(t) \in L_1(0, \infty)$ implies $\hat{A} \in C^{(3)}(\bar{\Gamma})$. Next we integrate by parts to obtain,

$$\hat{A}(s) = A(0)\, s^{-1} + \dot{A}(0)\, s^{-2} + \ddot{A}(0)\, s^{-3} + \left(\dddot{A}(0)\, s^{-4} + s^{-4} \int_0^{\infty} e^{-st} A^{(4)}(t)\, dt \right)$$

The quantity in parenthesis is $0(s^{-4})$ hence this gives (iii) of Definition 1. A similar calculation starting from $\hat{A}^{(j)}(s) = \int_0^\infty e^{-st} \alpha_j(t)\, dt$ yields (iv).

3. VERIFICATION

We assume here, without loss of generality, that $a(0) = 1$. We transform (E_a) by a device from [8]. Define the function $k_a(t)$ by

$$L_a[k_a]^{\cdot}(t) = -\dot{a}(t) \qquad t > 0. \tag{3.1}$$

Then one can verify (E_a) is equivalent to

$$\dot{u}(t) + L_{k_a}[u]^{\cdot}(t) + g(u(t)) = \Phi_a[f, u_0](t) \qquad t > 0 \quad u(0) = u_0$$
$$\Phi_a[f, u_0(t)] = f(t) + L_{k_a}[f](t) + k_a(t)\, u_0 \tag{E_a'}$$

This shows that (E_a) is really just (E_1) perturbed by a linear memory operator. (See the comment in Section 5.)

The following result is the key to our proof.

LEMMA 1. *Suppose a is as in theorem* 2 and $p^{-1} + q^{-1} = 1$. *Then*:

(i) $k_a \in C^{(1)}[0, \infty)$, $k_a(0) = -\dot{a}(0)$

(ii) $k_a(t) = k_a\infty + K_a(t)$ *where*

$$K_a^{(j)} \in L_1(0, \infty) \cap L_q(0, \infty), \int_0^\infty \left(\int_t^\infty |K_a(\tau)|\, dr \right)^q dt < \infty.$$

(iii) *If* $a_\infty > 0$, $k_a^\infty = 0$, $\hat{K}_a(0) = a_\infty^{-1} - 1$. *If* $a_\infty = 0$ $\quad k_a^\infty = \hat{A}(0)^{-1}$.

(iv) *For any* T, $\int_0^T (u(t), L_{k_a}[u]^{\cdot}(t))_H\, dt \geq 0$.

Proof. From (3.1) one finds for the transform $s\hat{a}(s)\hat{k}_a(s) = -s\hat{a}(s) + 1$ or $\hat{k}_a(s) = (s\hat{a}(s))^{-1} - 1$. We have $\hat{a} \in \hat{L}_2$. If $a_\infty > 0$ $s\hat{a}(s)$ cannot vanish in $\bar{\Gamma}$ and $\hat{k}_a \in C^{(3)}(\bar{\Gamma})$. If $a_\infty = 0$ we have $\hat{k}_a(s) = \hat{A}(0)^{-1} s^{-1} + \hat{K}_a(s)$ where $\hat{K}_a \in C^{(3)}(\bar{\Gamma})$. Since $\hat{a} \in L_2$ we can use the expansion (iii) of Definition 1 for $\hat{a}$ to obtain,

$$\hat{k}_a(s) = k_0 s^{-1} + k_1 s^{-2} + 0(s^{-3}),\ k_0 = -\dot{a}(0)$$

This yields (iii) of Definition 1 for $\hat{k}_a$ and (iv) of that definition $\hat{k}_a$ is inherited from (iv) for $\hat{a}$. Thus $\hat{k}_a \in \hat{L}_1$ and we can apply Proposition 1 to conclude $k_a \in C^1[0, \infty)$, $k_a(0) = -\dot{a}(0)$. From (A_2) we have

$$-\eta \operatorname{Im} \hat{k}_a(i\eta) = (\operatorname{Re} \hat{a}(i\eta))/[(\operatorname{Im} \hat{a}(i\eta))^2 + (\eta \operatorname{Re} \hat{a}(i\eta))^2] > 0$$

Conclusion (iv) of Lemma 1 then follows from a result in [7].

We use Lemma 1 to decompose Φ_a as $\Phi_a[f, u_0] = \Phi_a^\infty[f, u_0] + \Psi_a[f, u_0]$ where,

$$\Phi_a^\infty[f, u_0] = k_a^\infty \left(u_0 + \int_0^\infty F(\tau)\,d\tau\right) = \hat{A}(0)^{-1}\left(u_0 + \int_0^\infty F(\tau)\,d\tau\right),$$

$$\text{if} \quad a_\infty = 0, f_\infty = 0, \tag{3.2}$$

$$\Psi_a[f, u_0](t) = F(t) - k_a^\infty \int_t^\infty F(\tau)\,d\tau + L_{k_a}[F](t) + k_a(t)\,u_0$$

$$\Phi_a[f, u_0] = (1 + \hat{K}_a(0)) f_\infty = f_\infty / a_\infty$$

$$\text{if} \quad a_\infty = 0 \tag{3.3}$$

$$\Phi_a[f, u_0](t) = F(t) + L_{K_a}[F](t) + K_a(t)\,u_0$$

Proof of Theorem 3.2. We use (3.2) and (3.3) to rewrite equation (2.2) and (2.3) as

$$\begin{aligned} g(u_\infty) &= \Phi_a^\infty[f, u_0] \quad &\text{if} \quad a_\infty > 0 \\ k_a^\infty u_\infty + g(u_\infty) &= \Phi_a^\infty[f, u_0] \quad &\text{if} \quad a_\infty = 0 \end{aligned} \tag{3.4}$$

In view of condition (G) and the fact that $k_a^\infty > 0$ for $a_\infty = 0$ we see that the operators on the left side are monotone and coercive and the existence of unique solutions is a standard result.

We now subtract (3.4) from (E_a). We put $w(t) = u(t) - u_\infty$ and $G(w) = g(w + u_\infty) - g(u_\infty)$. We obtain then,

$$\begin{aligned} \dot{w}(t) + L_{k_a}[w]^{\cdot}(t) + G(w(t)) &= J_a[f, u_0] :\equiv: \Phi_a[f, u_0] - K_a(t)\,u_\infty \\ w(0) &= u_0 - u_\infty \end{aligned} \tag{3.5}$$

We note that condition (G) implies $\langle G(w), w\rangle \geq m\,\|w\|_V^p$. Thus if we multiply (3.5) by $w(t)$ and integrate from 0 to T we can use (iv) of Lemma 1 to argue that there is a $c>0$ such that for any $T>0$

$$\|w\|_{L_\infty(0,T:H)} + \|w\|_{L_p(0,T:V)} \leq c[\|u_0-u_\infty\|_H + \|J_a[f,u_0]\|_{L_q(0,T:V')}] \quad (3.6)$$

From Lemma 1 we have $K_a(t)=O(t^{-3})$ and this together with (F_2) shows that $J_a[f,u_0]\in L_q(0,\infty:V')$. Thus (3.6) shows that there is a constant $C[f_0,u_0]$ such that for any $T>0$

$$\|w\|_{L_\infty(0,T:H)} + \|w\|_{L_p(0,T:V)} \leq C[f_0,u_0] \quad (3.7)$$

If we assume that we have a solution then (3.7) yields the estimates (2.4). We can also use (3.7) to establish the existence of a solution.

The argument is almost the same as for (E_1) (see [6]) and we merely sketch it. We choose a family V^n of finite dimensional subspaces of V which approximate V as $n\to\infty$. Then for a fixed $T>0$ we find a sequence

$$w^n\in C^{(1)}([0,T):V^n) \text{ such that for any } v^n\in V^n,$$
$$(\dot w^n(t),v_n)_H + (L_a[w^n]\cdot(t),v_n)_H + \langle g(w^n(t)),v^n\rangle = \langle J_a[f,u_0](t),v^n\rangle$$
$$(w^n(0),v^n)_H = (u_0-u_\infty,v^n)_H \quad (3.8)$$

For these the estimate (3.6) holds with a constant independent of n and T. It follows that the w^n exist and that they are bounded in $L_\infty((0,T):H)$ and in $L_p((0,T):V)$. Then we can extract a subsequence $u^{n'}$ which converges to u weakly in $L_p((0,T):H)$ and weak star in $L_\infty((0,T):H)$. Since L_a is linear one can then use standard monotone operator theory to argue that u is a generalized solution.

Proof of Theorem 3.3. We observe first that the quantity R in (2.5) is positive. It follows that, if $\gamma>0$, $\hat B\in C^\infty(\bar\Gamma)$ and $\hat B$ will be in $\hat L_m$ for any m. It is the transform of $B(t)$ an exponentially decaying function. An easy calculation gives (2.6).[2] Moreover we have,

$$\begin{aligned}\operatorname{Re}\hat B(i\eta) &= (\gamma\hat A(0)(\gamma-\eta^2)+R\eta^2)((\gamma-\eta^2)^2+R^2\eta^2)^{-1}\\ &= (\gamma^2\hat A(0)-\dot a(0)\,\eta^2)((\gamma-\eta^2)^2+R^2\eta^2)^{-1}>0\end{aligned}$$

Remark 3.1. An easy calculation shows that if $a_\infty=0$, R_b always has the simple form,

$$K_b(t)=me^{-\alpha t},\ \alpha=\gamma\hat A(0)/A(0),\ M=-[(\dot A(0)/A(0)^2)+\hat A(0)^{-1}] \quad (3.9)$$

[2] The formula (2.9) is obtained by equating the coefficient of s^{-3} is the expansion for $\hat B$ to $\ddot a(0)$.

Proof of Theorem 4. The statement is rather vague. Let us explain what it means in the case $a_\infty = 0$. We proceed formally. We transform (E_b), divide by s and multiply by $(s^2 + Rs + \gamma)$ to obtain,

$$s^2\hat{u}(s) + Rs\hat{u}(s) + \gamma\hat{u}(s) + A(0)\, s\hat{g}(u)(s) + \gamma\hat{A}(0)\, \hat{g}(u)(s)$$
$$= s\hat{F}(s) + R\hat{F}(s) + \gamma \frac{\hat{F}(s)}{s} + su_0 + Ru_0 + \frac{\gamma u_0}{s} \tag{3.10}$$

Now for any smooth function $\chi(t)$ we have $s\hat{\chi}(s) = \hat{\dot{\chi}} + \chi(0)$ and $s^2\hat{\chi}(s) = \hat{\ddot{\chi}} + s\chi(0) + \dot{\chi}(0)$. We use these to rewrite (3.10) in the form,

$$\hat{\ddot{u}}(s) + R\hat{\dot{u}}(s) + \gamma\hat{u}(s) + A(0)\, \hat{\dot{g}}(u)(s) + \gamma\hat{A}(0)\, \hat{g}(u)(s)$$
$$= (F(0) - \dot{u}(0) - A(0)\, g(u_0)) + \hat{\dot{F}}(s) + R\hat{F}(s) + \frac{\gamma F(s)}{s} + \frac{\gamma u_0}{s}. \tag{3.11}$$

(The term $Su_0 + Ru_0$ is canceled by terms from the left). If the equation holds at $t = 0$ the term in parenthesis is zero and (3.11) translated back to the time domain yields

$$\ddot{u}(t) + R\dot{u}(t) + \gamma u(t) + A(0)\, g(u(t))^{\cdot} + \gamma\hat{A}(0)\, g(u(t))$$
$$= \dot{F}(t) + RF(t) + \gamma \int_0^t F(\tau)\, dr + u_0$$
$$u(0) = u_0 \qquad \dot{u}(0) = F(0) - A(0)\, g(u_0) \tag{3.12}$$

For the above calculation to make sense we need the following additional assumptions:

$$u_0 \in V,\ g(u) \text{ differentiable}$$
$$u \in C^{(2)}((0, \infty) : V') \cap C^{(1)}([0, \infty) : V') \cap C^{(1)}((0, \infty) : V) \tag{3.13}$$

If (3.13) holds then u is a solution of (E_b) if and only if u is a solution of (3.13).

The same type of argument can be applied when $a_\infty = 0$ but is a little more complicated. The resulting differential equation is third order and has the form

$$\dddot{u} + R\ddot{u} + \gamma\dot{u} + a(0)\, \dot{g}(\dot{u}) + (a_\infty R + \gamma\hat{A}(0))\, \dot{g}(u) + \gamma g(u) \qquad \mathfrak{J}\ddot{f} + R\dot{f} + \gamma$$
$$u(0) = u_0,\ \dot{u}(0) = f(0) - a(0)\, g(u_0)$$
$$\ddot{u}(0) = \dot{f}(0) - a(0)\, g'(u_0)[\dot{u}(0) - \dot{a}(0)\, g(u_0)] \tag{3.14}$$

For this to be equivalent (E_b) one needs g to be twice differentiable and for u to have more smoothness.

We can use (E'_a) and (E'_b) to obtain a little more information about the error in our approximation. Let u_a and u_b be the solutions of (E_a) and (E_b) and put $e = u_b - u_a$. For a fixed a and γ let us define a quantity $\theta_{a,\gamma}$ by,

$$\theta_{a,\gamma} = \|K_b - K_a\|_{L_2(0,\infty)} + \|K_b - K_a\|_{L_1(0,\infty)} + \|\dot{K}_b - \dot{K}_a\|_{L_1(0,\infty)} \tag{3.15}$$

Then we have the following result.

THEOREM 5. *There exists a constant C depending only on the data f, u_0 such that*

$$\|e\|_{L_\infty(0,\infty;H)} + \|e\|_{L_p(0,\infty;V)} \leqslant C\theta_{a,\gamma} \tag{3.16}$$

Proof. Put $H(e, t) = g(e + u_a(t)) - g(u_a(t))$. Then if we subtract (E'_a) from (E'_b) we obtain,

$$\begin{aligned} \dot{e}(t) + L_b[e]^{\cdot}(t) + H(e(t), t) &= \Phi_b[f, m_0] - \Phi_a[f, u_0] \\ &\quad + L_{k_a}[u_a]^{\cdot}(t) - L_{k_b}[u_b]\cdot(t) :\equiv: E(t) \qquad e(0) = 0 \end{aligned} \tag{3.17}$$

From (3.2) and (3.3) we have,

$$\begin{aligned} E(t) &= (K_b(t) - K_a(t))\, u_0 + \int_0^t (K_b(t-\tau) - K_a(t-\tau))\, F(\tau)\, d\tau \\ &\quad + \int_0^t (\dot{K}_a(t-\tau) - \dot{K}_a(t-\tau))(u_a(\tau) - u_\infty)\, d\tau \end{aligned} \tag{3.18}$$

We observe that (G) yields $\langle H(e, t), e\rangle \geqslant m\, \|e\|_V^p$ for any t. Thus we can repeat our earlier energy estimate. This will bound e in terms of $\|E\|_{Lq(0,\infty;V')}$. (3.18) together with our earlier estimates for $u - u_\infty$ in terms of the data yields (3.16).

The estimate (3.16) requires a knowledge of $\theta_{a,\gamma}$ and this is difficult to achieve. It depends on γ but it also depends in a very complicated way, on the frequency spectrum of a.

4. A NUMERICAL EXAMPLE

We consider (E_a) on $V = H = V' = R$ with

$$g(u) = u + u^3, \quad a(t) = e^{-t} + e^{-t} \cos 2t. \tag{4.1}$$

One can verify that all the conditions are satisfied. Here we can integrate (E_a) to obtain an integral equation which we write, with a new meaning for f, as

$$u(t) + \int_0^t a(t-\tau)\, g(u(\tau))\, d\tau = f(t) \qquad t > 0 \tag{4.2}$$

We considered this equation first for $f(t) \equiv 1$ and solved it by a simple scheme. Let $t_k = kh$ $h = 0, 1, \ldots$. Then we approximate the integral in (4.2) by the trapezoid rule to obtain a set of approximate values u_k of $u(t_k)$. The equations are:

$$u_{k+1} + \frac{h}{2} a(0)\, g(u_{k+1})$$

$$= f(t_k) - \frac{h}{2} a(t_{k+1})\, g(1) - h \sum_{1}^{k-1} a(t_{k+1} - jh)\, g(u_j),\ k \geqslant 0 \tag{4.3}$$

One can show that this is an $0(h^2)$ scheme. We solved it with a small enough h to get essentially an exact solution.

The formula (4.3) shows the numerical problem with (E_a). The sum on the right must be computed at each step and the number of terms in it increases with k. On infinite dimensional spaces this presents a major problem.

For our special problem we find $R = 0.6\gamma + 1$ and the associated differential equation is,

$$\ddot{u}(t) + R\dot{u}(t) + \gamma u(t) + g'(u(t))\, \dot{u}(t) + \tfrac{3}{2}\gamma g(u(t)) = \gamma$$

$$u(0) = 1 \qquad \dot{u}(0) = -a(0)\, g(1) = -4$$

The steady state limit is given by $1.2g(u_\infty) = 1$.

Our first attempt was to choose γ according to formula (2.9), that is $\gamma = 5$. The results are plotted in Fig. 1a. We see, as the theory indicates that short and long time behavior is fairly good but there are significant errors for intermediate t values. We then tried γ_H from (2.10) and the results are given in Fig. 1b. We see a significant improvement.

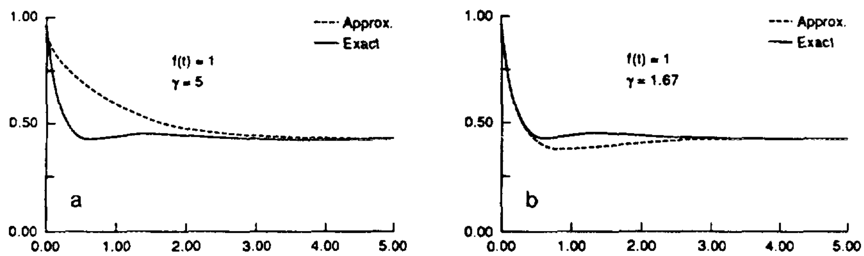

FIGURE 1

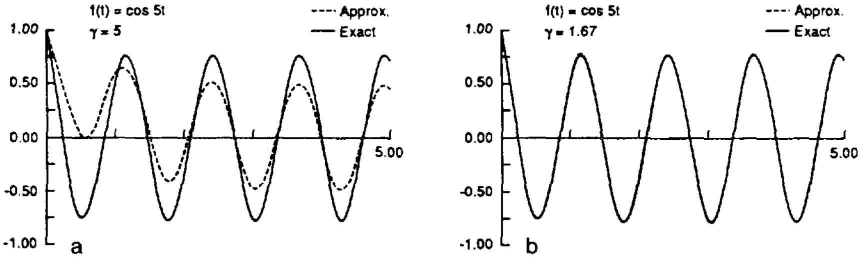

FIGURE 2

The results presented in Fig. 2. are even more intriguing. One can show, under our hypotheses, that if the forcing term $f(t)$ tends to an ω periodic limit then so will the solution. There is however no prior reason why the periodic limits for (E_a) and (E_b) should be the same unless ω is very small. We solved (E_a) and (E_b) with $f(t) = \cos 5t$ on the right side. The results are plotted in Figs. 2a and 2b for $\gamma = 5$ and $5/3$. For the second choice the exact and approximate solutions are nearly indistinguishable.

We have tried other examples with similar results but, as yet, do not have a very good explanation.

5. *Applications and Extensions*

Feedback Control. Suppose one has a linear hereditary input -output device that is $u(t) = L_a[Z](t)$. One has a system in which there is an external input φ and the state variable Z is controlled by u according to the rule $Z(t) = \varphi(t) - g(u(t))$. Then one has

$$u(t) = L_a[\varphi] - L_a[g(u)](t). \tag{5.1}$$

Differentiation leads to (E_a) with $f(t) = (L_a[\varphi]^{\cdot}(t)$. In circut problems one would expect a to be exponentially decreasing so that in this case we want $a_\infty = 0$.

One Dimensional Heat Flow. Suppose one has one dimensional heat flow in a rod with x position and t time. Let u, e, q, f denote temperature, internal energy heat flux and heat supply. If the bar occupies $0 < x < L$ the balance of energy is,

$$e_t(x, t) = -\sigma_x(x, t) + f(x, t) \tag{5.2}$$

Suppose the bar is homogeneous and assume that e is proportional to u, $e(x, t) = cu(x, t)$ $c > 0$. A non-hereditary model would be to assume that $\sigma(x, t) = \psi(u_x(x, t))$. Equation (5.2) is then a parabolic problem if $\psi'(\xi) > 0$.

One can obtain a hereditary model by assuming

$$\sigma(x, t) = -\alpha\psi(u_x(x, t)) - L_\beta[\chi(u_x(x, \cdot)] \tag{5.3}$$

A special case occurs if $\chi(\xi) = \psi(\xi)$. Then one can take $a(t) = \alpha(t) + \beta(t)$ and (5.3) inserted into (5.2) yields (E_a). One needs boundary conditions, say $u(0, t) = u(L, t) = 0$. Suppose for instance that $\psi(\xi) = -(r + s\xi^2)\xi$, $r, s > 0$. Then we take $H = L_2(0, L)$, $V = W_0^{1,4}(0, L)$ and

$$\langle g[u], v \rangle = -\int_0^L \psi(u_x(x, t))\, v(x)\, dx$$

Our condition (G) is then satisfied.

Note that one can get two essentially different models by making $a(t) = a_\infty + A(t)$ with $a_\infty > 0$ or $a_\infty = 0$.

Extensions

For applications in feedback control it would be desirable to have the theory on $\mathbb{R}^n$ with $a(t)$ a family of matrices. We can extend our dissipation theory almost unchanged if a is replaced by a family of symmetric linear operators. We can also give a formal extension of the approximate kernel b. What is difficult to check the dissipativity of b. Currently we can do this only of the $a(t)$ all commute. For control theory this may not be realistic. It will, however, be true for the system on $\mathbb{R}^n$ arising when one applies Galerkin methods to (5.2) with scalar function a.

For the heat flow model it would be physically more realistic to have different nonlinearities ψ and χ in (5.3). We could still make our formal approximation for b but we cannot use the inversion device of Section three so the theory is incomplete.

If one removes the derivative on $L_a[g(u)]$ in (E_a) one obtains the equation,

$$\dot{u}(t) + L_a[g(u)](t) = f(t) \tag{5.4}$$

This equation is studied in [8]. When $a \in L_a(0, \infty)$ it is a model for heat flow in materials with memory with finite propagation speed. When $a(t) = a_\infty + A(t)$, $A \in L_1(0, \infty)$ it is a special model of viscoelasticity [10]. For both cases we can present a formal approximation theory but there remain serious answered questions.

References

1. A. Barry, J. Bielak, and R. C. MacCamy, On absorbing boundary conditions for wave propagation, *J. Comput. Phys.* **79** (1988), 449–468.
2. M. G. Crandell and J. A. Nohel, An abstract functional differential equation and a related nonlinear Volterna equation, *Israel J. Math.* **29** (1978), 313-328.

3. J. DOUGLAS AND T. DUPONT, Galerkin methods for parabolic equations, *SIAM J. of Numer. Anal.* **7** (1970), 575–626.
4. B. ENGQUIST AND L. HALPERN, Far field boundary conditions for computation over long time, *Appl. Numer. Math.* **4** (1988), 21–45.
5. B. ENGQUIST AND A. MAJDA, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.* (1977), 629–651.
6. J. L. LIONS, "Quelques méthodes de résolutions des problèmes aux limites non linéaires," Dunod Gauthiers-Villars, Paris, 1969.
7. R. C. MACCAMY, Nonlinear Volterra equations on Hilbert space, *J. Differential Equations* **10** (1974), 373–398.
8. R. C. MACCAMY, An integro-differential equation with applications in heat flow, *Quart. Appli. Math.* **35** (1977), 1–19.
9. R. C. MACCAMY AND P. WEISS, Numerical solution of Volterra integral equations, *Nonlinear Anal.* **3** (1979), 677–695.
10. M. RENARDY, W. J. HRUSA, AND J. A. NOHEL, "Mathematical Problems in Viscoelasticity," Wiley, New York, 1987.
11. I. H. Sloan and V. Thomee, Time discretization of integro-differential equations of parabolic type, *SIAM J. Numer. Anal.* **23** (1986), 1052–1061.