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On Flat and Projective Envelopes*

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1. INTRODUCTION

All the rings will be associative with identity and the modules will be unital. If M is a right module we denote by M^* the dual module $\mathrm{Hom}_R(M, R)$ of M. If M is a right R-module, a flat preenvelope of M is a homomorphism $f\colon M \to F$ with F a flat right R-module such that for each homomorphism $g\colon M \to F'$ with F' a flat right R-module, there exists a homomorphism $h\colon F \to F'$ which completes the diagram, that is $h \circ f = g$. If, moreover, the only endomorphisms h of F such that $h \circ f = f$ are automorphisms, we say that f is a flat envelope of M. The flat envelope of a module, if it exists is unique up to isomorphism.

If we replace flat by injective in the above conditions, we get the usual concept of injective hull. The projective (pre)envelope can be defined in the same way. It is not known over which rings each module has a flat envelope. In [1, 2, 5, 10], some results about this question have been obtained. In this paper we continue the study of this problem and we get a characterization of rings over which every right module has a flat envelope satisfying an additional condition and the rings with projective envelope for each right module.

2. FLAT ENVELOPES

Unlike the injective hulls, the flat envelopes, when they exist, need not be monomorphisms. In fact, we have that every right R-module has a flat preenvelope which is an epimorphism (and hence it is a flat envelope) if

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and only if R is a left semihereditary ring. In particular, for Prüfer domains the flat envelope of each module is the canonical epimorphism to the quotient by its torsion submodule. In the cases in which the flat envelope is an epimorphism we have that the flat envelopes complete the diagrams in a unique way. In this sense we have:

PROPOSITION 2.1. *For a ring R the following conditions are equivalent*:

(i) *Every right R-module has a flat envelope which completes the diagrams in a unique way.*

(ii) *R is left coherent and the weak global dimension of R* ($WD(R)$) *is at most two.*

Proof. (i)$\Rightarrow$(ii) By [5, Prop. 5.1], R is left coherent. Let $0 \longrightarrow K \xrightarrow{\alpha} F_0 \xrightarrow{\beta} F_1$ be an exact sequence with F_0 and F_1 flat modules and $f: K \to H$ a flat envelope of K. There exists an unique homomorphism $h: H \to F_0$ such that $h \circ f = \alpha$, so $\beta \circ h \circ f = 0$ and hence by the uniqueness $\beta \circ h = 0$. Then we have an unique homomorphism $g: H \to K$ with $\alpha \circ g = h$ and so $\alpha \circ g \circ f = \alpha$ from which $g \circ f = 1_K$ and K is a flat module because it is a direct summand of H.

(ii)$\Rightarrow$(i) Let X be a finitely presented right R-module. By [8, Theorem 5.a], X^* is a flat left R-module and as a consequence of [4, Prop. 1] it is finitely presented and hence a projective module. Now, from [1, Prop. 1], we deduce that X has a flat envelope which, moreover, completes the diagrams in a unique way by [10, Prop. 2.10].

In these conditions, an argument similar to that used in [10, Theorem 2.11] allows us to obtain a flat envelope for each module which also completes the diagrams in an unique way. ∎

This is the situation in [5, Theorem 6.1], [10, Theorem 2.12] and [1, Theorem 9], where it is proved, by localization at prime ideals, that a commutative ring of finite weak global dimension is a coherent ring of weak global dimension at most two if and only if every module has a flat envelope.

The condition $WD(R) \leqslant 2$ is not necessary for the existence of flat envelopes. So, for instance, if R is a quasi-Frobenius ring, (QF for short), the class of injective modules coincides with the class of flat modules and hence every module has a flat envelope (its injective hull) and, if R is not a semisimple ring, then $WD(R) = \infty$. The QF rings are just the rings for which the injective hull of each module is its flat envelope. In [2, Theorem 12] the coherent self-injective commutative rings are characterized by the condition that each finitely presented module has a flat envelope which coincides with its injective hull.

In that paper it is proved that if R is a commutative ring with monomorphic flat envelopes for each module, then the localization $R_{\mathfrak{m}}$ of R at every maximal ideal $\mathfrak{m}$ of R is a QF ring. The converse of this result is not true, in fact there exist commutative rings which are locally QF but those are not even coherent. For instance, if we consider in the abelian group $E = (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$ the multiplication pointwise and we take $R = (\mathbb{Z}/4\mathbb{Z}) \oplus E$ with the usual adition and multiplication defined by $(m, e) \cdot (n, f) = (mn, mf + ne + ef)$, then R is a commutative ring which is not coherent ($\mathrm{Ann}_R(2, 0)$ is not a finitely generated ideal) and each localization of R at a maximal ideal is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. The following result is just a characterization of those rings.

Recall that an R-module X is said FP-injective if $\mathrm{Ext}^1_R(M, X) = 0$ for each finitely presented R-module M.

THEOREM 2.2. *Let R be a commutative ring. The following conditions are equivalent:*

(i) *Every R-module has a flat envelope which is a monomorphism.*

(ii) *For each maximal ideal* $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ *is either a field or a QF ring which is projective as an R-module.*

Proof. (i) $\Rightarrow$ (ii) We know that $R_{\mathfrak{m}}$ is a QF ring for each maximal ideal $\mathfrak{m}$ of R. If $F(\mathfrak{m})$ is a flat envelope of $\mathfrak{m}$, then $F(\mathfrak{m})$ is an ideal of R which contains $\mathfrak{m}$ (by [2, Corollary 10], $F(\mathfrak{m})$ is an essential extension of $\mathfrak{m}$). So we have either $F(\mathfrak{m}) = \mathfrak{m}$ or $F(\mathfrak{m}) = R$.

If $R_{\mathfrak{m}}$ is not a field, $\mathfrak{m}R_{\mathfrak{m}}$ is not flat and hence $\mathfrak{m}$ is not a flat R-module, so the inclusion of $\mathfrak{m}$ in R is a flat envelope of $\mathfrak{m}$.

Since $R_{\mathfrak{m}}$ is artinian there exists an integer n such that $(J(R_{\mathfrak{m}}))^n = (\mathfrak{m}R_{\mathfrak{m}})^n = (\mathfrak{m}^n)_{\mathfrak{m}} = 0$. Moreover, if $\mathfrak{m}' \in \mathrm{Max}(R)$, $\mathfrak{m}' \neq \mathfrak{m}$, $(\mathfrak{m}^n)_{\mathfrak{m}'} = R_{\mathfrak{m}'}$ and hence $\mathfrak{m}^n$ is flat. We claim that $\mathfrak{m}^n$ is finitely generated.

Indeed, let X be an FP-injective R-module, then we get an exact sequence

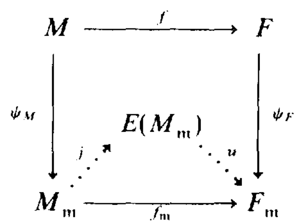
$$0 \to \mathrm{Hom}_R(R/\mathfrak{m}, X) \to \mathrm{Hom}_R(R, X) \to \mathrm{Hom}_R(\mathfrak{m}, X) \to \mathrm{Ext}^1_R(R/\mathfrak{m}, X) \to 0$$

where, furthermore, the last homomorphism factors through the zero module because the inclusion of $\mathfrak{m}$ in R is a flat preenvelope and X is a flat module (since every FP-injective R-module is a pure submodule of its injective hull ([1, Lemma 1.1]) and, in this case, every injective R-module is flat (see [2])). In this way, $\mathrm{Ext}^1_R(R/\mathfrak{m}, X) = 0$ for each FP-injective module X and hence $R/\mathfrak{m}$ is finitely presented [11, Prop. 1.11]. Then it results that $\mathfrak{m}$ is finitely generated and hence $\mathfrak{m}^n$ is also finitely generated so that $R/\mathfrak{m}^n$ is a projective finitely generated module. Now, it is easy to see that $R/\mathfrak{m}^n$ is isomorphic to $R_{\mathfrak{m}}$.

(ii) ⇒ (i) Let R^I be the product of copies of R indexed by a set I and $\mathfrak{m}$ a maximal ideal of R. If $R_{\mathfrak{m}}$ is a field, then obviously $(R^I)_{\mathfrak{m}}$ is flat. If $R_{\mathfrak{m}}$ is not a field, since $R_{\mathfrak{m}}$ is a finitely generated projective R-module, we have that $(R^I)_{\mathfrak{m}} \cong R^I \otimes R_{\mathfrak{m}} \cong (R \otimes R_{\mathfrak{m}})^I$ and so it is also in this case a flat module. Since flatness is a local property we deduce that R^I is a flat R-module and that R is a coherent ring.

On the other hand, as R is a pure submodule of $\prod_{\mathrm{Max}(R)} R_{\mathfrak{m}}$ and each $R_{\mathfrak{m}}$ is an injective module, R is self-FP-injective. We can deduce that every injective R-module is flat [4, Theorem 5] and hence each R-module has a flat preenvelope which is a monomorphism.

Now, let M be an R-module and $f\colon M \to F$ a (monomorphic) flat preenvelope. For each maximal ideal $\mathfrak{m}$ of R we have the commutative diagram



where ψ_M and ψ_F are the canonical homomorphisms and u is a monomorphism.

If $F^{\mathfrak{m}} = \varphi_F^{-1}(E(M_{\mathfrak{m}}))$, then $(F^{\mathfrak{m}})_{\mathfrak{m}} \cong E(M_{\mathfrak{m}})$ and if $\mathfrak{m}'$ is another maximal ideal of R we have $(F^{\mathfrak{m}})_{\mathfrak{m}'} \cong F_{\mathfrak{m}'}$ (the localization preserves pull-backs and $(E(M_{\mathfrak{m}}))_{\mathfrak{m}'} \cong (F_{\mathfrak{m}})_{\mathfrak{m}'} = 0$). So $F^{\mathfrak{m}}$ is a flat R-module and there exist $f^{\mathfrak{m}}\colon M \to F^{\mathfrak{m}}$ and $j^{\mathfrak{m}}\colon F^{\mathfrak{m}} \to F$ such that $j^{\mathfrak{m}} \circ f^{\mathfrak{m}} = f$.

We claim that if $F' = \bigcap_{\mathrm{Max}(R)} F^{\mathfrak{m}}$ and $f'\colon M \to F'$ is the homomorphism induced by $\{f^{\mathfrak{m}}\}$, then f' is a flat envelope of M.

Indeed, if $\mathfrak{m}$ is a maximal ideal of R such that $R_{\mathfrak{m}}$ is a field, then $(F')_{\mathfrak{m}}$ is obviously a flat module, and in fact, $(F')_{\mathfrak{m}} = M_{\mathfrak{m}}$ since, in this case, $M_{\mathfrak{m}}$ is its own injective hull. If $R_{\mathfrak{m}}$ is not a field, then it is finitely generated projective as R-module and hence the functor $-\otimes R_{\mathfrak{m}}$ commutes with intersections from which we get

$$(F')_{\mathfrak{m}} = \left(\bigcap F^{\mathfrak{m}'}\right)_{\mathfrak{m}} \cong \left(\bigcap F^{\mathfrak{m}'}\right) \otimes R_{\mathfrak{m}} \cong \bigcap (F^{\mathfrak{m}'} \otimes R_{\mathfrak{m}}) \cong \bigcap ((F^{\mathfrak{m}'})_{\mathfrak{m}})$$
$$= (F^{\mathfrak{m}})_{\mathfrak{m}} = E(M_{\mathfrak{m}})$$

which is flat as $R_{\mathfrak{m}}$-module because $R_{\mathfrak{m}}$ is a QF ring.

As $f\colon M \to F$ factors through $f'\colon M \to F'$, f' is a flat preenvelope of M and, moreover, for each maximal ideal $\mathfrak{m}$ of R $(F')_{\mathfrak{m}} = E(M_{\mathfrak{m}})$, so that if

h is an endomorphism of F' with $h \circ f' = f'$, we have that $h_{\mathfrak{m}} \circ f'_{\mathfrak{m}} = f'_{\mathfrak{m}}$ and hence $h_{\mathfrak{m}}$ is an isomorphism for each maximal ideal $\mathfrak{m}$ of R. Then h is an isomorphism and the proof is complete. ∎

3. Flat and Projective Envelopes

In the notation of [7], if Pf(R) denotes the full subcategory of the category of left R-modules whose objects are the finitely presented left R-modules and $D(R)$ is the Grothendieck category of additive functors from Pf(R) to abelian groups, we have the following characterization of the injective objects in this category.

Lemma 3.1 [7, Prop. 1.2]. *An object F of $D(R)$ is injective if and only if F is naturally equivalent to a functor $E \otimes -$ with E a right pure-injective module.*

On the other hand, we have:

Lemma 3.2. *Let R be a left coherent ring. Then every flat right R-module is pure-injective if and only if R is a right perfect ring.*

Proof. See [9, Prop. 1.4]. ∎

Now, we can prove:

Theorem 3.3. *Let R be a right perfect and left coherent ring. Then every right R-module has a flat envelope.*

Proof. Let M be a right R-module and $f: M \to F$ a flat preenvelope of M (which always exists because R is left coherent). From Lemma 3.2, F is a right pure-injective module and hence $F \otimes -$ is an injective object of $D(R)$ by Lemma 3.1. If $E \otimes -$ is the injective hull of the image functor $G = \mathrm{Im}(f \otimes -)$, then $E \cong E \otimes R$ is a direct summand of $F \cong F \otimes R$ and so it is a flat module. Moreover, f factors through $M \to G(R) \to E$, from which we obtain that if g is the above composition, it is a flat preenvelope of M. Now, each endomorphism h of E such that $h \circ g = g$ induces an endomorphism $h \otimes -$ of $E \otimes -$ in $D(R)$, whose restriction to G is the canonical inclusion of G in its injective hull and so $h \otimes -$ is an isomorphism in $D(R)$. In particular, h is an isomorphism, which proves that $g: M \to E$ is a flat envelope of M. ∎

Corollary 3.4. *If R is a left artinian ring, then every right R-module has a flat envelope.*

Note that, in the conditions of Theorem 3.3, since the right flat modules coincide with the right projective modules, the flat envelopes are also projective envelopes. More precisely we have:

PROPOSITION 3.5. *Every right R-module has a projective (pre)envelope if and only if R is a right perfect and left coherent ring.*

Proof. The conditions are sufficient as a consequence of Theorem 3.3.

Conversely, if every right R-module has a projective preenvelope, and M is a direct product of a family of right projective modules $\{P_i\}_I$ and $f: M \to P$ is a projective preenvelope, for each $i \in I$ there exists $g_i: P \to P_i$ such that $g_i \circ f$ is the canonical projection from M to P_i. If we take $g: P \to M$ the homomorphism induced by $\{g_i\}_I$, then $g \circ f = 1_M$. Thus M is a direct summand of P and hence it is projective. The result follows from [3, Theorem 3.3]. ∎

From [3, Theorem 3.4] we get:

COROLLARY 3.6. *If R is a commutative ring then every R-module has a projective (pre)envelope if and only if R is an artinian ring.*

To finish the paper, we include a method to find the projective envelope of a finitely generated right module over an arbitrary left artinian ring R.

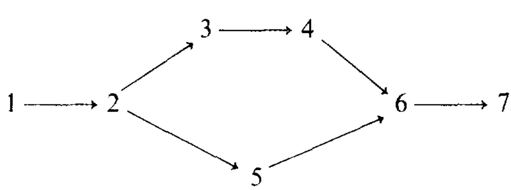
Let M be a finitely generated right module and $f: P \to M^*$ the projective cover of its dual module (which exists because R is left perfect). If $\varphi_M: M \to M^{**}$ is the canonical homomorphism, we claim that $f^* \circ \varphi_M: M \to P^*$ is a projective envelope of M.

Indeed, if $g: M \to Q$ is a homomorphism with Q a projective module which can be considered finitely generated, then g^* factors through f and hence it results that g can be factored through $f^* \circ \varphi_M$ and $f^* \circ \varphi_M$ is a projective preenvelope of M. Now, if $g \in \mathrm{End}(P^*)$ is such that $g \circ f^* \circ \varphi_M = f^* \circ \varphi_M$, then by dualizing we obtain that g^* is an isomorphism and finally that g is an isomorphism.

In the particular case that $wD(R) \leqslant 2$, we have that the projective envelope of a finitely generated right module M is the canonical homomorphism from M to its bidual M^{**}.

EXAMPLE. Let $R = KI$ be the incidence K-algebra whose (left) quiver is

I:



It is known that gl. $D(R)=2$ (it is not difficult to show that every simple left module S_i has projective dimension one except S_6, which has projective dimension two).

If X is a non-projective finitely generated indecomposable right R-module, which has a non-trivial projective envelope, then the transpose of X, TX (see [6, p. 70]) is an indecomposable left R-module of projective dimension two, and hence $X^* \cong (Re_2)^n$ for some $n \geqslant 1$, so the projective envelope of X is of the form $(e_2 R)^n$ being n dependent on the structure of X. Now, the projective envelope of each right R-module which has no non-zero projective direct summands is either zero or a direct sum of copies of $e_2 R$.

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