Extremal Problems and Coverings of the Space

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The aim of this paper is to prove that whenever the $n$-dimensional Euclidean space is covered by $r$ sets, $t$ is a positive integer $n > n_0(t)$, and $r = n^t$ then there is one of the sets within which all the distances are realized. We obtain this result by proving for natural numbers $n, k, l, k \geq 3l, n \geq n_0(k)$, and a family $\mathcal{F}$ of $k$-element subsets of an $n$-set such that no two members of $\mathcal{F}$ have intersection of cardinality $l$ then $|\mathcal{F}| < c(k)(\binom{n}{k-l-1})$, $c(k)$ being a constant depending only on $k$.

1. INTRODUCTION

Let $E^n$ denote the $n$-dimensional Euclidean space. We say that all the distances are realized in the set $D \subset E^n$ if for any real number $y$ we can find $P, R \in D$ such that their distance equals $y$.

In 1944 and 1945 H. Hadwiger [8, 9] proved that if $E^n$ is covered by $n + 1$ closed sets then there is one within which all the distances are realized.

Let us denote by $s(n)$ the minimal number such that $E^n$ can be covered by $s(n)$ sets (not necessarily measurable) such that there is none among the sets within which all the distances are realized.


They also proved $s(n) \leq (3 + o(1))^n$.

In 1977 D. G. Larman [10] improved the lower bound to $\frac{(n-1)(n-2)(n-3)}{178} 200$.

In this paper we prove the following theorem.

THEOREM 1. For a natural number $t$, and if $n$ is sufficiently large with respect of $t$, then

$$s(n) > n^t. \quad (1)$$

Though (1) is a great improvement on the lower bound of Larman, it is far from the upper bound, and even from a lower bound conjectured by Larman [10]: $s(n) \approx \frac{1}{3}(\binom{3n}{n})^{3n/4}$.

Theorem 1 follows from the following theorem.

THEOREM 2. Suppose we have natural numbers $n, k, l$ and a family $\mathcal{F}$ of $k$ element subsets of an $n$-set, such that $k \geq 3l, n \geq n_0(k)$ and for $F_1, F_2 \in \mathcal{F}, |F_1 \cap F_2| \neq l$ holds. Then for an appropriate constant $c = c(k)$ we have

$$|\mathcal{F}| \leq c(k)\binom{n-l-1}{k-l-1}.$$

2. The Proof of Theorem 1 using Theorem 2

Theorem 2 in Larman, Rogers [11] states the following:

Suppose in $E^n$ there exists a set of $M$ points such that in every subset of it containing more than $D$ points we can find two points at distance $1$.

Then $s(n) \geq M/D$.

Let us take all the points in $E^n$ which have $0$ in $n-k$ coordinates and $\delta$ in the remaining $k$ coordinates. We have $M = \binom{n}{\delta}$.
Let \( D = c(k)(\binom{n-k}{l-1}) \). Then in view of Theorem 2 among any \( D + 1 \) points we can find two, say \( x, y \), such that they agree in exactly \( l \) non-zero coordinates. Consequently their distance is \( \delta \sqrt{2(k-1)} \). Hence choosing \( \delta = \left( \frac{\sqrt{2(k-1)}}{} \right)^{-1} \) we deduce

\[
s(n) \geq \left( \binom{n}{k} \right) c(k) \binom{n-l-1}{k-l-1} > c'(k) n^{l+1} > n^l, \quad \text{for } n \geq n_0(k, l).
\]

As \( k \) was arbitrary and \( l \) can be chosen as large as \( \lfloor k/3 \rfloor \), the statement of Theorem 1 follows.

### 3. Remark to Theorem 2

In the case \( l = 0 \), the Erdös-Ko-Rado theorem [3] settles the problem:

\( |\mathcal{F}| \leq \binom{n-1}{k-1} \) with equality holding iff \( \mathcal{F} \) consists of all the \( k \)-subsets containing a fixed element (\( n > 2k \)).

For the case \( l = 1 \), \( n > n_0(k), k \geq 4 \)

\( |\mathcal{F}| \leq \binom{n-2}{k-2} \) was conjectured by P. Erdős and V. T. Sós (see [2]) and it was proved by the author in [5]. He also proved that equality holds iff \( \mathcal{F} \) consists of all the subsets containing 2 fixed elements.

In [2] Erdős conjectures for the general case, \( n > n_0(k) \)

\[
|\mathcal{F}| \leq \max \left\{ \binom{n-l-1}{k-l-1}, \binom{n}{l} \right\}.
\]

At the end of this paper we give an argument yielding for \( k \geq 3l+2 \)

\[
|\mathcal{F}| \leq \left( 1 + o(1) \right) \binom{n-l-1}{k-l-1}.
\]

Theorem 2 will be a special case of the following theorem.

**Theorem 3.** Let \( X \) be a finite set of \( n \) elements and \( \mathcal{F} \) be a family of \( k \)-element subsets of \( X \). Suppose \( n > n_0(k) \) and let \( l, r \) be natural numbers satisfying \( k > l > r > 0 \). If for all \( F_1, F_2 \in \mathcal{F} \),

\[
|F_1 \cap F_2| \notin (l-r, l]
\]

then

\[
|\mathcal{F}| \leq c(k, l, r) n^{\max(k-l-1, l-r+1 + (l-k+1)/(r+1))},
\]

where \((a, b)\) is the set of integers \( q, a < q \leq b \), \([y]\) is the greatest integer not exceeding \( y \), and \( c(k, l, r) \) is a constant depending on \( k, l, r \).

The more general problem, for all \( F_1 \neq F_2 \in \mathcal{F} |F_1 \cap F_2| \in L \), where \( L \) is a given subset of \( \{0, 1, \ldots, k-1\} \) was considered by Ray-Chaudhury and Wilson [12] and Deza, Erdős and Frankl [1], however for this special case (3) gives a much better bound.

The proof of Theorem 3 will depend heavily on a method developed in [6].

### 4. The Proof of Theorem 3

We make a recurrent construction.

Let us set \( \mathcal{F}_0 = \mathcal{F} \) and let \( D_1 \) be a set of maximal cardinality such that there exist \( k+1 \) different members \( F_1, \ldots, F_{k+1} \) of \( \mathcal{F} \) satisfying \( F_i \cap F_j = D_1 \) for \( 1 \leq i < j \leq k+1 \) (i.e., \( D_1 \) is the kernel of the \( \Delta \)-system \( \{F_1, \ldots, F_{k+1}\} \)).
Let us define
\[ \mathcal{E}_1 = \{(F, D_1) | F \in \mathcal{F}_0, D_1 \subset F\}, \]
\[ \mathcal{F}_1 = \mathcal{F}_0 - \{F \in \mathcal{F}, D_1 \subset F\}. \]

Let us continue this way, i.e., if \( \mathcal{E}_i, \mathcal{F}_i \) are defined the let \( D_{i+1} \) be a set of maximal cardinality such that there exist \( F_1, \ldots, F_{k+1} \in \mathcal{F}, F_1 \neq F_2 \) and \( F_j \cap F_j = D_{i+1} \) for \( 1 \leq j < j' \leq k + 1 \). Now let us set
\[ \mathcal{E}_{i+1} = \{(F, D_{i+1}) | F \in \mathcal{F}_i, D_{i+1} \subset F\} \cup \mathcal{E}_i \]
\[ \mathcal{F}_{i+1} = \mathcal{F}_i - \{F \in \mathcal{F} | D_{i+1} \subset F\}. \]

Let this procedure stop at the \( q \)-th step, i.e., \( \mathcal{F}_q \) doesn't contain a \( \Delta \)-system of \( k + 1 \) members. Hence, by a result of Erdős and Rado [4]
\[ |\mathcal{F}_q| < k!k^k. \] (4)

Obviously
\[ |\mathcal{F}_j| + |\mathcal{E}_i| = |\mathcal{F}| \text{ for } 1 \leq j \leq q. \]

In [6] it is proved that
\[ |\mathcal{F}_j - \mathcal{F}_{j+1}| \leq k!k^n \] (5)
for \( 0 \leq j \leq q - 1 \).

Let us set \( \mathcal{E} = \mathcal{E}_q \) and
\[ \mathcal{E}^i = \{(F, D) \in \mathcal{E} | |D| = k - i\}. \]

Now we need the following proposition which was proved in [6].

**PROPOSITION.** Let \((F, D), (F', D') \in \mathcal{E}\), then
\[ |D \cap D'| \notin (l-r, l]. \] (6)

Moreover if \( F - D = F' - D' \), then
\[ |D \cap D'| \notin (l-r - |F - D|, l - |F - D|]. \] (7)

Now we proceed with the proof of Theorem 3 by applying induction on \( k \). For \( k \leq 2 \) the statement of the theorem is obvious. As \( \mathcal{E} = \bigcup_{i=1}^k \mathcal{E}^i \), in view of (4), either \( |\mathcal{F}| = |\mathcal{E}| + |\mathcal{F}_q| < (k + 1)!k^k \), and we are done, or there exists \( 1 \leq i \leq k \) such that
\[ |\mathcal{E}^i| \geq \frac{|\mathcal{F}|}{k + 1}. \] (8)

In the second case let us choose \( i \) to satisfy (8).

We distinguish three cases.

(a) \( i \leq r \).

Then in view of (6) \( k - i > l \).

The inequality (8) implies that among the \( \binom{n}{i} \) \( i \)-subsets of \( X \) we may find one, say \( C \), such that setting \( \mathcal{D}_C = \{D | (F, D) \in \mathcal{E}^i, F - D = C\} \), we have
\[ |\mathcal{D}_C| \geq \frac{|\mathcal{F}|}{(k + 1)} \binom{n}{i}. \] (9)

The proposition implies that for \( D, D' \in \mathcal{D}_C \) we have
\[ |D \cap D'| \in (l - r - i, l]. \] (10)
Now the induction hypothesis yields:

$$|\mathcal{F}| \leq (k + 1)(n \choose i)|C| \leq (k + 1)(n \choose i)c(k - i, l, r - i)n^\max(k - i - l, l - r - i + 1 + \frac{k - l - i - 1}{r + 1})$$

$$\leq c(k, l, r)n^\max(k - l, l - r + 1 + \frac{k - l - 1}{r + 1}),$$

as required.

(b) \(i \geq r + 1, k - i > l\).

In this case let us set

$$\mathcal{D} = \{D | (F, D) \in \mathcal{C}^l\}.$$

Then \(\mathcal{D}\) is a family of \((k - i)\)-subsets of \(X\), and in view of (6) for \(D, D' \in \mathcal{D}\) we have

$$|D \cap D'| \leq (l - r, l].$$

(11)

On the other hand (5) implies

$$|\mathcal{D}| \geq (\mathcal{F})/(k + 1)!k^k \cdot n.$$

(12)

Hence applying the induction hypothesis we obtain

$$|\mathcal{F}| \leq |\mathcal{D}| \cdot n \cdot (k + 1)!k^k \leq n \cdot (k + 1)!k^k c(k - i, l, r) \cdot n^\max(k - i - l - 1, l - r + 1 + \frac{k - l - i - 1}{r + 1})$$

$$\leq c(k, l, r)n^\max(k - l, l - r + 1 + \frac{k - l - 1}{r + 1}),$$

and we are done.

(c) \(k - i \leq l\).

Then (6) yields

$$k - i \leq l - r.$$

Setting again

$$\mathcal{D} = \{D | (F, D) \in \mathcal{C}^l\},$$

(13) implies

$$|\mathcal{D}| \leq \binom{n}{l - r},$$

and in view of (12)

$$|\mathcal{F}| \leq ((k + 1)!k^k/(l - r)!) n^{l - r + 1}$$

completing the proof of Theorem 3.

**Corollary 1.** Let \(\mathcal{F}\) be a family of \(k\) element subsets of an \(n\)-set, and suppose (2) is satisfied. If \(n > n_0(k),\)

$$k \geq l\left(2 + \frac{1}{r}\right), \quad l > r > 0,$$

then

$$|\mathcal{F}| \leq c(k, l, r)n^{k - l - 1}.$$

**5. Strengthening the Bound of Theorem 3**

Let us suppose now \(r = 1, k \geq 3l + 2\). We want to prove that for any \(\varepsilon > 0, n > n_c(k, \varepsilon)\) and \(\mathcal{F}\) as in Theorem 3.

$$|\mathcal{F}| \leq (1 + \varepsilon)\binom{n - l - 1}{k - l - 1}. \quad (14)$$

The proof is very technical therefore we only sketch it.
Let \( \mathcal{F} \) be a family of maximal size. Let us proceed as with the proof of Theorem 3. We claim that for an arbitrary positive \( \delta \), and for any \( 2 \leq i \leq k \)
\[
|\mathcal{C}^i| < \delta |\mathcal{F}|/(k + 1).
\]
Indeed, otherwise to \( \mathcal{C} \) case (b) or (c) of the proof applies yielding
\[
|\mathcal{F}| < \frac{(k + 1)}{\delta} c^r(k)n^{k-l-2} < \binom{n-l-1}{k-l-1} \quad \text{for } n > n_0(k),
\]
contradicting the maximal choice of \( \mathcal{F} \).

Now we set
\[
Y = \left\{ y \in X | |D_{(y)}| > \frac{\delta}{2n} |\mathcal{F}| \right\}.
\]
(Recall \( D_{(y)} = \{D | (F, D) \in \mathcal{C}^1, F - D = \{y\}\} \).) Then
\[
\sum_{y \in Y} |D_{(y)}| > (1-\delta)|\mathcal{F}|. \tag{15}
\]
For \( y \in Y \), in view of the proposition, for all \( D, D' \in D_{(y)} \)
\[
|D \cap D'| \leq (l-2, l).
\]
Now we proceed in the same way with each particular \( D_{(y)}, y \in Y \).
After \( l \) steps we obtain a collection \( \mathcal{G} \) or ordered \( l \)-tuples \( \langle y_1, y_2, \ldots, y_l \rangle \), such that
\[
|D_{(y_1, y_2, \ldots, y_l)}| > |\mathcal{F}| \left/ \left( \frac{2n}{\delta} \right)^l \right. \tag{16}
\]
Moreover for \( D, D' \in D_{(y_1, \ldots, y_l)} \)
\[
|D \cap D'| \in [0, l]
\]
or equivalently
\[
|D \cap D'| \geq l + 1. \tag{17}
\]
In view of Theorem 2 of [7] (16) and (17) imply that there exists an \((l+1)\)-element subset of \( X - \{y_1, \ldots, y_l\} \), say \( G_0(y_1, \ldots, y_l) \), which is contained in every member of \( D_{(y_1, \ldots, y_l)} \) \((n > n_0(k, \delta))\). (By choosing \( \delta \) sufficiently small we may obtain
\[
|\mathcal{F}| - \sum_{(y_1, \ldots, y_l) \in \mathcal{G}} |D_{(y_1, \ldots, y_l)}| < (\varepsilon/2)|\mathcal{F}|. \tag{18}
\]
Now we assert that for \( \langle y_1, \ldots, y_l \rangle, \langle y_1, \ldots, y_l \rangle \in \mathcal{G} \) and \( D \in D_{(y_1, \ldots, y_l)}, D' \in D_{(y_1, \ldots, y_l)} \)
\[
(D - G(y_1, \ldots, y_l)) \cup \{y_1, \ldots, y_l\} \neq (D' - G(y_1, \ldots, y_l)) \cup \{y_1, \ldots, y_l\}. \tag{19}
\]
Then of course,
\[
\sum_{(y_1, \ldots, y_l) \in \mathcal{G}} |D_{(y_1, \ldots, y_l)}| \leq \binom{n}{k-l-1}
\]
If we proved (19) then (18) yields (for \( \varepsilon < 1/2, n > n_0(k, \varepsilon) \))
\[
|\mathcal{F}| < \left( 1/\left( 1 - \frac{\varepsilon}{2} \right) \right) \binom{n}{k-l-1} < \left( 1 + \varepsilon \right) \binom{n-l-1}{k-l-1},
\]
as desired. So all we are left to do is proving (19). Suppose it fails for some \( Y = \{y_1, \ldots, y_l\} \)
\[
Y' = \{y_1', \ldots, y_l'\}, \quad D, D', G = G(y_1, \ldots, y_l), \quad G' = G(y_1', \ldots, y_l').
\]
Let \(|G \cap G'| = l - i| \), obviously \(0 \leq i \leq l\). By the definition of the \(D_C\)’s \(G\) is the kernel of a \(\Delta\)-system of cardinality \(k + 1: \{A_1, A_2, \ldots, A_{k+1}\}\) with \(A_j \in D_{\{y_1, \ldots, y_l\}}\), \(1 \leq j \leq k + 1\). Hence we may choose a \(j, 1 \leq j \leq k + 1\) in such a way that

\[|A_j \cap (D' \cup Y')| = l - i.\]

But in this case

\[|(A_j \cup \{y_1, \ldots, y_l\}) \cap (D' \cup Y')| = l,\]

a contradiction since both these sets are in \(\mathcal{F}\).

REFERENCES


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