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Parameter free induction and provably total computable functions

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Abstract

We study the classes of computable functions that can be proved to be total by means of parameter free Σ_n and Π_n induction schemata, $I\Sigma_n^-$ and $I\Pi_n^-$, over Kalmar elementary arithmetic. We give a positive answer to a question, whether the provably total computable functions of $I\Pi_2^-$ are exactly the primitive recursive ones, and show that the class of such functions for $I\Sigma_1 + I\Pi_2^-$ coincides with the class of doubly recursive functions of Peter. We also characterize provably total computable functions of theories of the form $I\Pi_{n+1}^-$ and $I\Sigma_n + I\Pi_{n+1}^-$ for all $n \geqslant 1$, in terms of the fast growing hierarchy.

These results are based on a precise characterization of $I\Sigma_n^-$ and $I\Pi_n^-$ in terms of reflection principles and conservation results for local reflection principles obtained by techniques of modal provability logic. Using similar ideas we show that $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. boolean combinations of Σ_{n+1} sentences, for $n \geqslant 1$, and obtain a number of results on the strength of bounded number of instances of parameter free induction schemata and complexity of their axiomatization. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we shall deal with first-order theories containing basic Kalmar elementary arithmetic *EA* or, equivalently, $I\Delta_0 + \mathrm{Exp}$ (cf. [11]). We are interested in the general question how various ways of formal reasoning correspond to models of computation. This kind of analysis is traditionally based on the concept of provably total computable function (p.t.c.f.) of a theory. A somewhat older term for the same notion,

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introduced in the 1950s by Kreisel, is "provably (total) recursive function". Given a theory T containing EA, a function $f(\boldsymbol{x})$ is called *provably total computable in T*, iff there is a Σ_1 formula $\phi(\boldsymbol{x}, y)$, sometimes called the *specification* of f, that defines the graph of f in the standard model of arithmetic and such that

$$T \vdash \forall \boldsymbol{x} \exists ! y \phi(\boldsymbol{x}, y).$$

Note that the existence of a Σ_1 formula defining the graph of f in the standard model is a necessary and sufficient condition for the computability of f. Provable totality of a function provides additional information to the simple fact of computability. Thus, the class of p.t.c.f. of T, denoted $\mathscr{F}(T)$, is one of the most interesting characteristics of T, which measures its power of reasoning about the termination of computations or 'computational strength'.

We are going to analyze from this point of view the role of parameters involved in applications of the principle of mathematical induction. Parameter free induction schemata have been introduced and investigated in a number of works by Kaye, Paris, and Dimitracopoulos [13], Adamowicz and Bigorajska [1], Ratajczyk [20], Kaye [12], and others. $I\Sigma_n^-$ is the theory axiomatized over EA by the schema of induction

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x),$$

for Σ_n formulas $A(x)$ containing no other free variables but x, and $I\Pi_n^-$ is similarly defined. [2]

It is known that the schemata $I\Sigma_n^-$ and $I\Pi_n^-$ have a very different behavior from their parametric counterparts $I\Sigma_n$ and $I\Pi_n$. In particular, for $n \geqslant 1$, $I\Sigma_n^-$ and $I\Pi_n^-$ are not finitely axiomatizable, and $I\Sigma_n^-$ is strictly stronger than $I\Pi_n^-$, in fact, stronger than $I\Sigma_{n-1} + I\Pi_n^-$. (In contrast, it is known that $I\Sigma_n$ is a finitely axiomatizable theory, and $I\Sigma_n$ and $I\Pi_n$ are deductively equivalent.) Furthermore, it is known that $I\Sigma_n$ is a conservative extension of $I\Sigma_n^-$ w.r.t. Σ_{n+2} sentences, whereas $I\Sigma_n^-$ has an axiomatization of strictly lower arithmetical complexity, namely, by boolean combinations of Σ_{n+1} sentences [13].

In contrast, nontrivial conservation results for $I\Pi_n^-$ for $n>1$, were unknown. In particular, it was unknown, if the provably total recursive functions of $I\Pi_2^-$ coincide with the primitive recursive ones (communicated by R. Kaye). The case of $I\Pi_1^-$ (over PA^-) was treated in [13], where the authors showed that Π_2 consequences of that theory are contained in EA, cf. also [7].

In this paper we prove that the p.t.c.f. of $I\Pi_2^-$ are exactly the primitive recursive functions (Corollary 4.1). Moreover, we show that $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. boolean combinations of Σ_{n+1} sentences, for $n \geqslant 1$ (Theorem 2). In particular, this allows us to characterize p.t.c.f. of the theories $I\Pi_{n+1}^-$ and $I\Sigma_n + I\Pi_{n+1}^-$ for any $n \geqslant 1$ (Theorems 8 and 9 in Section 6).

[2] This definition differs from the one in [13] in that we work over EA, rather than over the weaker theories $I\Delta_0$ or PA^-. Since $I\Sigma_1^-$ in the sense of [13] obviously contains EA, the two definitions are equivalent for $n \geqslant 1$ in Σ case, and for $n \geqslant 2$ in Π case.

Notice that our characterization of $\mathscr{F}(I\Pi_2^-)$ is similar to a well-known theorem of Parsons [18] (independently proved by Mints and Takeuti) stating that $\mathscr{F}(I\Sigma_1)$ coincides with the class of primitive recursive functions, too. However, the relationship between these two results is nontrivial, because the theories $I\Pi_2^-$ and $I\Sigma_1$ are incomparable in strength (neither is included in the other). In fact, it is easy to see that the theory $I\Sigma_1 + I\Pi_2^-$ has a larger class of p.t.c.f. than the class of primitive recursive functions. This can be seen from the following characteristic example.

The well-known *Ackermann function* $Ack(x)$ is defined by double recursion as follows. $Ack(x) := g(x, x)$, where

Ack is known to grow faster than any primitive recursive function (cf. [22]). The graphs of g and *Ack* can be naturally defined by Σ_1 formulas, for which one can also verify in *EA* the inductive definition clauses above. In order to show that *Ack* is total we prove that the two-argument function $g(x, n)$ is total. A natural proof of the statement $\forall n \forall x \exists y \; g(x, n) = y$ goes by induction on n. Notice that the corresponding induction formula is Π_2 and parameter free. However, in order to verify the induction step one must argue that

$$\forall x \exists y \; g(x, n) = y \rightarrow \forall x \exists y \; g(x, n+1) = y.$$

This statement is provable by a subordinate Σ_1 induction on x with a parameter n. In other words, the usual argument for the totality of Ackermann function is formalizable in $I\Sigma_1 + I\Pi_2^-$. Our result shows that *any* correct argument for the totality of *Ack* formalizable in Peano arithmetic must involve parameters or induction formulas outside the class Π_2.

In Theorem 9 below we show that $\mathscr{F}(I\Sigma_1 + I\Pi_2^-)$ actually coincides with the class of *doubly-recursive functions* of Peter (cf. [22]). This class can also be characterized as the class corresponding to the ordinal ω^2 of the extended Grzegorczyk (or Fast Growing) hierarchy and thus involves functions growing much faster than the Ackermann function. It is well known that $\mathscr{F}(I\Pi_2)$ is the class of *multiply-recursive functions*, that is, corresponds in the same sense yet to a bigger ordinal ω^ω.

The above example of a natural pair of theories capturing the same class of computable functions, whose union captures a much bigger class, opens the question whether there may exist in general a unique 'most natural' arithmetical theory corresponding to a given computation model. For the case of primitive recursion $I\Sigma_1$ is generally held to be such a theory. Now we are confronted with the question, if Σ_1 induction with parameters is more natural than Π_2 induction without parameters. Our answer to this (admittedly, somewhat philosophical) dilemma is that there is more to each of these two theories, than their computational content. Apart from the primitive recursion mechanism, both of them involve some more complex principles of reasoning. Taken together these principles complement each other in a way that significantly increases their class of p.t.c.f..

The proofs of our results are based on a characterization of parameter free induction schemata in terms of reflection principles and (generalizations of) the conservation results for local reflection principles obtained in [3] using methods of provability logic. In our opinion, such a relationship presents an independent interest, especially because this seems to be the first occasion when *local* reflection principles naturally arise in the study of fragments of arithmetic. Using the techniques of reflection principles we also obtain a number of other results, in particular, sharp characterizations of the strength of bounded number of instances of parameter free induction schemata and some corollaries on the complexity of their axiomatization.

We shall also essentially rely on the results from [4] characterizing the closures of arbitrary arithmetical theories extending *EA* under Σ_n and Π_n induction *rules*. In fact, the results of this paper show that much of the unusual behaviour of parameter free induction schemata can be explained by their tight relationship with the theories axiomatized by induction rules.

The results of Sections 3 and 4 of this paper appeared in [5].

2. Preliminaries

We shall work in the language of Peano Arithmetic enriched by a binary predicate symbol of inequality. Bounded or Δ_0 formulas in this language are those, all of whose quantifier occurrences have the form $\forall x\ (x \leqslant t \rightarrow A(x))$ or $\exists x\ (x \leqslant t \wedge A(x))$, where t is a term not involving x. In *EA* a function symbol for exponention function 2^x can be introduced [11]; $\Delta_0(\exp)$ formulas are bounded formulas in the extended language. Σ_n and Π_n formulas are prenex formulas obtained from the bounded ones by n alternating blocks of similar quantifiers, starting from '$\exists$' and '$\forall$', respectively. $\mathscr{B}(\Sigma_n)$ denotes the class of boolean combinations of Σ_n formulas. Σ_n^{st} and Π_n^{st} denote the classes of Σ_n and Π_n sentences. *St* denotes the class of all arithmetical sentences. EA^+ denotes the extension of *EA* by a natural Π_2 axiom stating that the iterated exponentiation function is total, or $I\Delta_0 + \mathrm{Supexp}$ in the terminology of [11, 27]. *PRA* denotes the standard first order Primitive Recursive Arithmetic.

Next, we establish some useful terminology and notation concerning rules in arithmetic (cf. also [4]). We say that a *rule* is a set of *instances*, that is, expressions of the form

$$\frac{A_1,\ldots,A_n}{B},$$

where $A_1,\ldots,A_n$ and B are formulas. Derivations using rules are defined in the standard way; $T + R$ denotes the closure of a theory T under a rule R and first order logic. $[T,R]$ denotes the closure of T under *unnested applications* of R, that is, the theory axiomatized over T by all formulas B such that, for some formulas $A_1,\ldots,A_n$

derivable in T, $A_1,\dots,A_n/B$ is an instance of R. $T\equiv U$ means that theories T and U are deductively equivalent, i.e., have the same set of theorems.

A rule R_1 is *derivable* from R_2 iff, for every theory T containing EA, $T+R_1\subseteq T+R_2$. A rule R_1 is *reducible* to R_2 iff, for every theory T containing EA, $[T,R_1]\subseteq[T,R_2]$. R_1 and R_2 are *congruent* iff they are mutually reducible (denoted $R_1\cong R_2$). For a theory U containing EA we say that R_1 and R_2 are *congruent modulo* U, iff for every extension T of U, $[T,R_1]\equiv[T,R_2]$.

Induction rule is defined as follows:

$$\mathrm{IR}\colon\ \frac{A(0),\forall x(A(x)\rightarrow A(x+1))}{\forall x A(x)}.$$

Whenever we impose a restriction that $A(x)$ only ranges over a certain subclass Γ of the class of arithmetical formulas, this rule is denoted Γ-IR. The theory $EA+\Sigma_n$-IR will also be denoted $I\Sigma_n^R$. In general, we allow parameters to occur in A, however the following lemma holds.

Lemma 2.1. Π_n-IR *is reducible to parameter free* Π_n-IR. Σ_n-IR *is reducible to parameter free* Σ_n-IR.

Proof. An application of IR for a formula $A(x,a)$ can obviously be reduced to the one for $\forall z A(x,z)$, and this accounts for the Π_n case.

On the other hand, if $A(x,y,a)$ is Π_{n-1}, then an application of Σ_n-IR for the formula $\exists y A(x,y,a)$ is reducible, using the standard coding of sequences available in EA, to the one for $\exists y A'(x,y)$, where

$$A'(x,y):=\forall i\leqslant x A((i)_0,(y)_i,(i)_1).$$

Indeed, assume that

$$T\vdash\exists y A(0,y,a) \tag{1}$$

and

$$T\vdash\forall x\ (\exists y\ A(x,y,a)\rightarrow\exists y\ A(x+1,y,a)). \tag{2}$$

Then by (1) and the monotonicity of the coding of sequences, $T\vdash\exists y A'(0,y)$. For a proof of

$$T\vdash\forall x\ (\exists y A'(x,y)\rightarrow\exists y' A'(x+1,y')),$$

assume $\forall i\leqslant x A((i)_0,(y)_i,(i)_1)$. If $(x+1)_0=0$, then by (1) there is an element z such that $A(0,z,(x+1)_1)$, and we can take for y' the sequence $y*\langle z\rangle$ ($*$ denotes concatenation). If $(x+1)_0>0$, then the code of the pair $p:=\langle(x+1)_0-1,(x+1)_1\rangle$ is strictly less than $x+1$, and thus, by the induction hypothesis, there is a $z=(y)_p$ such that $A((x+1)_0-1,z,(x+1)_1)$. From (2) it follows that for some z' one has $A((x+1)_0,z',(x+1)_1)$. Hence, for y' one can take the sequence $y*\langle z'\rangle$. $\square$

Reflection principles, for a given r.e. theory T containing EA, are defined as follows. The *uniform* reflection principle is the schema

$$\mathrm{RFN}_T\colon\ \forall x\ (\mathrm{Prov}_T(\ulcorner A(\dot{x})\urcorner)\rightarrow A(x)),\quad A(x) \text{ a formula,}$$

where $\mathrm{Prov}_T(\cdot)$ denotes a canonical provability predicate for T.

The *local* reflection principle is the schema

$$\mathrm{Rfn}_T\colon\ \mathrm{Prov}_T(\ulcorner A\urcorner)\rightarrow A,\quad A \text{ a sentence.}$$

Partial reflection principles are obtained from the above schemata by imposing a restriction that A belongs to one of the classes Γ of the arithmetic hierarchy (denoted $\mathrm{Rfn}_T(\Gamma)$ and $\mathrm{RFN}_T(\Gamma)$, respectively). It is known that, due to the existence of partial truthdefinitions, the schema $\mathrm{RFN}_T(\Pi_n)$ is equivalent to a single Π_n sentence over EA. In particular, $\mathrm{RFN}_T(\Pi_1)$ is equivalent to the consistency assertion Con_T for T. See [24, 14, 3] for some basic information about reflection principles. In addition we note the following facts: $EA^+\equiv EA+\mathrm{RFN}_{EA}(\Pi_2)$ [27, 4], and $I\Sigma_n\equiv EA+\mathrm{RFN}_{EA}(\Pi_{n+2})$, for all $n\geqslant 1$ [15, 17, 11].

We shall also consider the following *metareflection rule*:

$$\mathrm{RR}(\Pi_n)\colon\ \frac{P}{\mathrm{RFN}_{EA+P}(\Pi_n)}.$$

We let Π_m-$\mathrm{RR}(\Pi_n)$ denote the above rule with the restriction that P is a Π_m sentence. Main results (Theorems 1, 2–3) of [4] can then be reformulated as follows.

Proposition 2.1. 1. Π_n-IR $\cong\Pi_{n+1}$-$\mathrm{RR}(\Pi_n)$, for $n>1$;
2. Π_1-IR $\cong\Pi_2$-$\mathrm{RR}(\Pi_1)$ (modulo EA^+).

Proposition 2.2. 1. Σ_1-IR $\cong\Pi_2$-$\mathrm{RR}(\Pi_2)$;
2. Σ_n-IR $\cong\Pi_{n+1}$-$\mathrm{RR}(\Pi_{n+1})$ (modulo $I\Sigma_{n-1}$,), for $n>1$.

Since $[EA,\Sigma_n\text{-IR}]$ contains $I\Sigma_{n-1}$, the second claim of this proposition implies that the rules Π_{n+1}-$\mathrm{RR}(\Pi_{n+1})$ and Σ_n-IR are interderivable, for all $n\geqslant 1$. Also notice that Propositions 2.1 and 2.2 imply the following result of Parsons [19]: $I\Sigma_n^R\equiv I\Pi_{n+1}^R$, for all $n\geqslant 1$.

3. Characterizing parameter free induction by reflection principles

Having in mind the exact correspondence between parametric induction schemata and uniform reflection principles over EA, it seems natural to conjecture that parameter free induction should correspond to parameter free, that is, *local* reflection principles. However, it is also well known that local reflection schemata per se are too weak: e.g., Rfn_{EA} is contained in the extension of EA by the set of all true Π_1 sentences, yet none of the schemata $I\Pi_n^-$ for $n>1$ satisfies this property. It turns out that in order to obtain a sharp characterization of parameter free induction one has to relativize the provability operator.

For $n \geqslant 1$, $\Pi_n(\mathbf{N})$ denotes the set of all true Π_n sentences. $\mathrm{True}_{\Pi_n}(x)$ denotes a canonical truthdefinition for Π_n sentences, that is, a Π_n formula naturally defining the set of Gödel numbers of $\Pi_n(\mathbf{N})$ sentences in EA. $\mathrm{True}_{\Pi_n}(x)$ provably in EA satisfies Tarski satisfaction conditions (cf. [11]), and therefore, for every formula $A(x_1,\dots,x_n) \in \Pi_n$,

$$EA \vdash A(x_1,\dots,x_n) \leftrightarrow \mathrm{True}_{\Pi_n}(\ulcorner A(\dot{x}_1,\dots,\dot{x}_n)\urcorner). \qquad (*)$$

Tarski's truth lemma $(*)$ is formalizable in EA, in particular,

$$EA \vdash \forall s \in \Pi_n^{st}\mathrm{Prov}_{EA}(s \mathrel{\dot{\leftrightarrow}} \ulcorner \mathrm{True}_{\Pi_n}(\dot{s})\urcorner), \qquad (**)$$

where Π_n^{st} is a natural elementary definition of the set of Gödel numbers of Π_n sentences in EA. We also assume w.l.o.g. that

$$EA \vdash \forall x\ (\mathrm{True}_{\Pi_n}(x) \to x \in \Pi_n^{st}).$$

Let T be an r.e. theory containing EA. A provability predicate for the theory $T + \Pi_n(\mathbf{N})$ can be naturally defined, e.g., by the following Σ_{n+1} formula:

$$\mathrm{Prov}_T^{\Pi_n}(x) := \exists s\ (\mathrm{True}_{\Pi_n}(s) \wedge \mathrm{Prov}_T(s \mathrel{\dot{\to}} x)).$$

Lemma 3.1. 1. *For each Σ_{n+1} formula $A(x_1,\dots,x_n)$,*

$$EA \vdash A(x_1,\dots,x_n) \to \mathrm{Prov}_T^{\Pi_n}(\ulcorner A(\dot{x}_1,\dots,\dot{x}_n)\urcorner).$$

2. $\mathrm{Prov}_T^{\Pi_n}(x)$ *satisfies Löb's derivability conditions in T:*
   (a) $T \vdash A \Rightarrow T \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner A\urcorner)$;
   (b) $T \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner A \to B\urcorner) \to (\mathrm{Prov}_T^{\Pi_n}(\ulcorner A\urcorner) \to \mathrm{Prov}_T^{\Pi_n}(\ulcorner B\urcorner))$;
   (c) $T \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner A\urcorner) \to \mathrm{Prov}_T^{\Pi_n}\left(\ulcorner \mathrm{Prov}_T^{\Pi_n}(\ulcorner A\urcorner)\urcorner\right)$.

Proof. Statement 1 follows from $(*)$. Statement 2 follows from Statement 1, Tarski satisfaction conditions, and is essentially well known (cf. [25]). □

We define

$$\mathrm{Con}_T^{\Pi_n} := \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner 0=1\urcorner),$$

$$\mathrm{Rfn}_T^{\Pi_n} := \{\mathrm{Prov}_T^{\Pi_n}(\ulcorner \phi\urcorner) \to \phi \mid \phi \in St\},$$

$$\mathrm{Rfn}_T^{\Pi_n}(\Sigma_m) := \{\mathrm{Prov}_T^{\Pi_n}(\ulcorner \sigma\urcorner) \to \sigma \mid \sigma \in \Sigma_m^{st}\}.$$

For $n = 0$ all these schemata coincide, by definition, with their nonrelativized counterparts.

Lemma 3.2. *For all $n \geqslant 0$, $m \geqslant 1$, the following schemata are deductively equivalent over EA*:
(i) $\mathrm{Con}_T^{\Pi_n} \equiv \mathrm{RFN}_T(\Pi_{n+1})$;
(ii) $\mathrm{Rfn}_T^{\Pi_n}(\Sigma_m) \equiv \{P \to \mathrm{RFN}_{T+P}(\Pi_{n+1}) \mid P \in \Pi_m^{st}\}$.

Proof. (i) Observe that, using (∗∗),

$$EA \vdash \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner 0=1 \urcorner) \leftrightarrow \neg\exists s(\mathrm{True}_{\Pi_n}(s) \wedge \mathrm{Prov}_T(s \dot{\to} \ulcorner 0=1 \urcorner))$$
$$\leftrightarrow \forall s(\mathrm{Prov}_T(\dot{\neg} s) \to \neg\mathrm{True}_{\Pi_n}(s))$$
$$\leftrightarrow \forall s(\mathrm{Prov}_T(\ulcorner \neg\mathrm{True}_{\Pi_n}(\dot{s}) \urcorner) \to \neg\mathrm{True}_{\Pi_n}(s)).$$

The latter formula clearly follows from $\mathrm{RFN}_T(\Sigma_n)$, but it also implies $\mathrm{RFN}_T(\Sigma_n)$, and hence $\mathrm{RFN}_T(\Pi_{n+1})$, by (∗).
(ii) By formalized Deduction theorem,

$$EA \vdash \mathrm{Con}_{T+P}^{\Pi_n} \leftrightarrow \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner \neg P \urcorner). \tag{3}$$

Hence, over *EA*,

$$\mathrm{Rfn}_T^{\Pi_n}(\Sigma_m) \equiv \{\mathrm{Prov}_T^{\Pi_n}(\ulcorner S \urcorner) \to S \mid S \in \Sigma_m^{st}\}$$
$$\equiv \{P \to \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner \neg P \urcorner) \mid P \in \Pi_m^{st}\}$$
$$\equiv \{P \to \mathrm{RFN}_{T+P}(\Pi_{n+1}) \mid P \in \Pi_m^{st}\} \quad \text{by (3) and (i).} \qquad \square$$

Theorem 1. For $n \geqslant 1$,
(i) $I\Sigma_n^- \equiv EA + \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+1})$;
(ii) $I\Pi_{n+1}^- \equiv EA + \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2})$;
(iii) $EA^+ + I\Pi_1^- \equiv EA^+ + \mathrm{Rfn}_{EA}(\Sigma_2) \equiv EA^+ + \mathrm{Rfn}_{EA^+}(\Sigma_2)$.

Proof. All statements are proved similarly, respectively, relying upon Propositions 2.2 and 2.1, so we shall only elaborate the proof of the first one. For the inclusion ($\subseteq$) we have to derive

$$A(0) \wedge \forall x\,(A(x) \to A(x+1)) \to \forall x A(x),$$

for each Σ_n formula $A(x)$ with the only free variable x. Let P denote the Π_{n+1} sentence (logically equivalent to) $A(0) \wedge \forall x\ (A(x) \to A(x+1))$. Then, by external induction on n it is easy to see that, for each n, $EA+P \vdash A(\bar{n})$. This fact is formalizable in *EA*, therefore

$$EA \vdash \forall x\, \mathrm{Prov}_{EA+P}(\ulcorner A(\dot{x}) \urcorner). \tag{4}$$

By Lemma 3.2 we conclude that

$$EA + \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+1}) + P \vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1})$$
$$\vdash \forall x\,(\mathrm{Prov}_{EA+P}(\ulcorner A(\dot{x}) \urcorner) \to A(x))$$
$$\vdash \forall x A(x) \quad \text{by (4).}$$

It follows that $EA + \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+1}) \vdash P \to \forall x A(x)$, as required.

For the inclusion ($\supseteq$) we observe that, for any Π_{n+1} sentence P, the theory $I\Sigma_n^- + P$ contains $P + \Sigma_n$-IR by Lemma 2.1, and hence

$$I\Sigma_n^- + P \vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1}),$$

by Proposition 2.2. It follows that

$$I\Sigma_n^- \vdash P \rightarrow \mathrm{RFN}_{EA+P}(\Pi_{n+1}),$$

and Lemma 3.2(ii) yields the result. □

4. Analyzing $I\Pi_n^-$

The following theorem and its Corollary 4.1 are the main results of this paper.

Theorem 2. *For any $n \geqslant 1$, $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. $\mathscr{B}(\Sigma_{n+1})$ sentences.*

Proof. The result follows from Theorem 1 and the following relativized version of Theorem 1 of [3]. □

Theorem 3. *For any $n \geqslant 0$, $T + \mathrm{Rfn}_T^{\Pi_n}$ is conservative over $T + \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1})$ w.r.t. $\mathscr{B}(\Sigma_{n+1})$ sentences.*

Proof. The proof of this theorem makes use of a purely modal logical lemma concerning Gödel–Löb provability logic GL (cf., e.g. [8, 25]). Recall that GL is formulated in the language of propositional calculus enriched by a unary modal operator $\Box$. The expressions $\Diamond\phi$ and $\Box^+\phi$ are the standard abbreviations for $\neg\Box\neg\phi$ and $\phi \wedge \Box\phi$, respectively. Axioms of GL are all instances of propositional tautologies in this language together with the following schemata:
L1. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$;
L2. $\Box\phi \rightarrow \Box\,\Box\phi$;
L3. $\Box(\Box\phi \rightarrow \phi) \rightarrow \Box\phi$.

Rules of GL are *moduls ponens* and $\phi \vdash \Box\phi$ (necessitation).

By an *arithmetical realization* of the language of GL we mean any function $(\cdot)^*$ that maps propositional variables to arithmetical sentences. For a modal formula ϕ, $(\phi)_T^*$ denotes the result of substituting for all the variables of ϕ the corresponding arithmetical sentences and of translating $\Box$ as the provability predicate $\mathrm{Prov}_T(\ulcorner\cdot\urcorner)$. Under this interpretation, axioms L1, L2 and the necessitation rule can be seen to directly correspond to the three Löb's derivability conditions, and axiom L3 is the formalization of Löb's theorem. It follows that, for each modal formula ϕ, $GL \vdash \phi$ implies $T \vdash (\phi)_T^*$, for every realization $(\cdot)^*$ of the variables of ϕ. The opposite implication, for the case of a Σ_1 sound theory T, is also valid; this is the content of the important *arithmetical completeness theorem* for GL due to Solovay (cf. [8]).

For us it will also be essential that GL is sound under the interpretation of $\Box$ as a *relativized* provability predicate. For an arithmetical realization $(\cdot)^*$, we let $(\phi)_{T+\Pi_n(N)}^*$

denote the result of substituting for all the variables of ϕ the corresponding arithmetical sentences and of translating $\Box$ as $\mathrm{Prov}_T^{\Pi_n}(\ulcorner\cdot\urcorner)$. The following lemma is a corollary of Lemma 3.1 and the fact that (formalized) Löb's theorem for relativized provability follows by the usual fixed-point argument from the derivability conditions.

Lemma 4.1. *If* $GL \vdash \phi$, *then* $T \vdash (\phi)^*_{T+\Pi_n(N)}$, *for every arithmetical realization* $(\cdot)^*$ *of the variables of* ϕ.

The opposite implication, that is, the arithmetical completeness of GL w.r.t. the relativized provability interpretation is also well known (cf. [25]). Yet, below we do not use this fact.

The following crucial lemma is a modification of a similar lemma in [3].

Lemma 4.2. *Let modal formulas* Q_i *be defined as follows*:

$$Q_0 := p, \qquad Q_{i+1} := Q_i \vee \Box Q_i,$$

where p *is a propositional variable. Then, for any variables* $p_0, \dots, p_m$,

$$GL \vdash \Box^+ \left(\bigwedge_{i=0}^{m} (\Box p_i \to p_i) \to p \right) \to \left(\bigwedge_{i=0}^{m} (\Box Q_i \to Q_i) \to p \right).$$

Proof. Rather than exhibiting an explicit proof of the formula above, we shall argue semantically, using a standard Kripke model characterization of GL.

Recall that a *Kripke model* for GL is a triple $(W, R, \Vdash)$, where

1. W is a finite nonempty set;
2. R is an irreflexive partial order on W;
3. $\Vdash$ is a forcing relation between elements (*nodes*) of W and modal formulas such that

$$x \Vdash \neg\phi \Leftrightarrow x \nVdash \phi,$$

$$x \Vdash (\phi \to \psi) \Leftrightarrow (x \nVdash \phi \text{ or } x \Vdash \psi),$$

$$x \Vdash \Box\phi \Leftrightarrow \forall y \in W\ (xRy \Rightarrow y \Vdash \phi).$$

Theorem 4 on p. 95 of [8] (originally proved by Segerberg) states that a modal formula is provable in GL, iff it is forced at every node of any Kripke model of the above kind. This provides a useful criterion for showing provability in GL.

Consider any Kripke model $(W, R, \Vdash)$ in which the conclusion $(\bigwedge_{i=0}^{m}(\Box Q_i \to Q_i) \to p)$ is false at a node $x \in W$. This means that $x \nVdash p$ and $x \Vdash \Box Q_i \to Q_i$, for each $i \leqslant m$. An obvious induction on i then shows that $x \nVdash Q_i$ for all $i \leqslant m+1$, in particular, $x \nVdash Q_{m+1}$.

Unwinding the definition of Q_i we observe that in W there is a sequence of nodes

$$x = x_{m+1} R x_m R \dots R x_0$$

such that, for all $i \leqslant m+1$, $x_i \nVdash Q_i$. Since R is irreflexive and transitive, all x_i's are pairwise distinct. Moreover, it is easy to see by induction on i that, for all i,

$$GL \vdash p \to Q_i.$$

Hence, for each $i \leqslant m+1$, $x_i \nVdash p$.

Now, we notice that each formula $\Box p_i \to p_i$ can be false at no more than one node of the chain $x_{m+1}, \dots, x_0$. Therefore, by Pigeon-hole Principle, there must exist a node z among the $m+2$ nodes x_i such that

$$z \Vdash \bigwedge_{i=0}^{m} (\Box p_i \to p_i) \wedge \neg p.$$

In case z coincides with $x = x_{m+1}$ we have

$$x \nVdash \bigwedge_{i=0}^{m} (\Box p_i \to p_i) \to p.$$

In case $z = x_i$, for some $i \leqslant m$, we have xRz by transitivity of R, and thus

$$x \nVdash \Box \left(\bigwedge_{i=0}^{m} (\Box p_i \to p_i) \to p \right).$$

This shows that the formula in question is forced at every node of any Kripke model; hence it is provable in GL. $\Box$

Lemma 4.3. *For any $n \geqslant 0$, the following schemata are deductively equivalent over EA:*

$$\mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1}) \equiv \mathrm{Rfn}_T^{\Pi_n}(\mathscr{B}(\Sigma_{n+1})).$$

Proof. We prove that

$$EA + \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1}) \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner \phi \urcorner) \to \phi$$

for any boolean combination of Σ_{n+1} sentences ϕ. The formula ϕ is equivalent to a formula of the form $\bigwedge_{i=1}^{n} (\pi_i \vee \sigma_i)$, for some sentences $\pi_i \in \Pi_{n+1}$ $\sigma_i \in \Sigma_{n+1}$. Since the provability predicate $\mathrm{Prov}_T^{\Pi_n}(\ulcorner \cdot \urcorner)$ commutes with conjunction, it is sufficient to derive in $EA + \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1})$ the formulas

$$\mathrm{Prov}_T^{\Pi_n}(\ulcorner \pi_i \vee \sigma_i \urcorner) \to (\pi_i \vee \sigma_i),$$

for each i. By Lemma 3.1

$$\begin{aligned} \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner \pi_i \vee \sigma_i \urcorner) \wedge \neg \pi_i &\to \mathrm{Prov}_T^{\Pi_n}(\ulcorner \neg \pi_i \urcorner) \\ &\to \mathrm{Prov}_T^{\Pi_n}(\ulcorner \sigma_i \urcorner) \\ &\to \sigma_i, \end{aligned}$$

using $\mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1})$. Hence,

$$EA + \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+1}) \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner \pi_i \vee \sigma_i \urcorner) \to (\pi_i \vee \sigma_i). \qquad \Box$$

Now we complete our proof of Theorems 2 and 3. Assume $T + \mathrm{Rfn}_T^{\Pi_n} \vdash A$, where A is a $\mathscr{B}(\Sigma_{n+1})$ sentence. Then there are finitely many instances of relativized local reflection that imply A, that is, for some arithmetical sentences $A_0, \dots, A_m$, we have

$$T \vdash \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner A_i \urcorner) \to A_i) \to A.$$

Since the relativized provability predicate satisfies Löb's derivability conditions, we also obtain

$$T \vdash \mathrm{Prov}_T^{\Pi_n}\left(\ulcorner \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner A_i \urcorner) \to A_i) \to A \urcorner\right).$$

Considering an arithmetical realization $(\cdot)^*$ that maps the variable p to the sentence A and p_i to A_i, for each i, by Lemma 4.2 we conclude that

$$T \vdash \bigwedge_{i=0}^{m} (\mathrm{Prov}_T^{\Pi_n}(\ulcorner B_i \urcorner) \to B_i) \to A,$$

where B_i denote the formulas $(Q_i)^*_{T+\Pi_n(N)}$. Now we observe that if $A \in \mathscr{B}(\Sigma_{n+1})$, then for all i, $B_i \in \mathscr{B}(\Sigma_{n+1})$. Hence

$$T + \mathrm{Rfn}_T^{\Pi_n}(\mathscr{B}(\Sigma_{n+1})) \vdash A,$$

which yields Theorem 3 by Lemma 4.3. Theorem 2 follows from Theorem 3 and the observation that the schema $\mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2})$ corresponding to $I\Pi_{n+1}^-$ is actually weaker than the full $\mathrm{Rfn}_{EA}^{\Pi_n}$. □

It is obvious, e.g., since $I\Sigma_1^-$ contains $I\Sigma_1^R$, that all primitive recursive functions are provably total recursive in $I\Sigma_1^-$ and $I\Pi_2^-$. Moreover, since $I\Sigma_1^-$ is contained in $I\Sigma_1$, by Parsons' theorem all p.t.c.f. of $I\Sigma_1^-$ are primitive recursive. The following corollary strengthens this result and gives a positive answer to a question by R. Kaye.

Corollary 4.1. *Provably total recursive functions of $I\Pi_2^-$ are exactly the primitive recursive ones.*

Proof. Follows from $\mathscr{B}(\Sigma_2)$ conservativity of $I\Pi_2^-$ over $I\Sigma_1^-$. □

By a similar argument we obtain the following.

Corollary 4.2. *Provably total recursive functions of $I\Pi_{n+1}^-$ are the same as those of $I\Sigma_n$ and $I\Sigma_n^-$.*

Proof. Follows from Theorem 2 and the fact that $I\Sigma_n$ is Σ_{n+2} conservative over $I\Sigma_n^-$ [13]. □

Remark 4.1. Perhaps somewhat more naturally, conservation results for relativized local reflection principles can be stated modally within a certain bimodal system GLB due to Japaridze, with the operators $\Box$ and $\boxdot$, that describes the joint behaviour of the usual and the relativized provability predicate (cf. [8]). Using a suitable Kripke model characterization of GLB, one can semantically prove that

$$GLB \vdash \boxdot\left(\bigwedge_{i=0}^{m} \boxdot p_i \to p_i\right) \to p) \to \Box\left(\bigwedge_{i=0}^{m} (\boxdot Q_i \to Q_i) \to p\right),$$

where the formulas Q_i are now understood w.r.t. the modality $\boxdot$, and this yields Theorem 3 almost directly.

5. Further conservation and axiomatization results

The characterization of parameter free induction in terms of reflection principles (Theorem 1) actually reveals other interesting information about these schemata.

The following theorem, which is a corollary of a relativized version of another conservation result for local reflection principles (due, essentially, to Goryachev [10]), gives a characterization of Π_{n+1} consequences of $I\Sigma_n^-$ and $I\Pi_{n+1}^-$. For the case of $I\Sigma_n^-$ a related characterization of p.t.c.f. is given in [1, 20]. On the other hand, the paper [13] also contains a related conservation result for $I\Pi_1^-$ w.r.t. Π_1 sentences ($I\Pi_1^-$ is formulated over PA^-).

Let T be an r.e. theory containing EA. For a fixed $n \geqslant 1$, we define a sequence of theories $(T)_i^n$ as follows:

$$(T)_0^n := T; \quad (T)_{i+1}^n := (T)_i^n + \mathrm{RFN}_{(T)_i^n}(\Pi_n); \quad (T)_\omega^n := \bigcup_{i \geqslant 0} (T)_i^n.$$

Theorem 4. *For any* $n \geqslant 1$,

(i) *The theory axiomatized over EA by arbitrary m instances of* $I\Pi_{n+1}^-$ *is* Π_{n+1} *conservative over* $(EA)_m^{n+1}$.

(ii) $I\Pi_{n+1}^-$ is Π_{n+1} *conservative over* $(EA)_\omega^{n+1}$.

Proof. Statement (ii) follows from (i). The proof of (i) relies on the fact that our characterization of parameter free induction schemata in terms of reflection principles respects the number of instances of these schemata.

Lemma 5.1. *For every instance B of* $I\Pi_{n+1}^-$ *there is a* Π_{n+2} *sentence P such that* $P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1})$ *implies B over EA. Vice versa, for every such P there is an instance B of* $I\Pi_{n+1}^-$ *such that* $EA + B$ *proves* $P \to \mathrm{RFN}_{EA+P}(\Pi_{n+1})$.

Proof. This is easy to check by inspection of our proof of Theorem 1. For the 'vice versa' part we employ Proposition 2.1 (1) stating that

$$[EA + P, \Pi_{n+1}\text{-IR}] \vdash \mathrm{RFN}_{EA+P}(\Pi_{n+1}).$$

Also notice that any finite number of *unnested* applications of Π_{n+1}-IR can be obviously merged into a single one, which, in turn, is reducible to a single instance of $I\Pi^-_{n+1}$. □

Remark 5.1. A similar statement holds for $I\Sigma^-_n$, but the 'vice versa' part only holds over $I\Sigma_{n-1}$. In general one seems to need $m+1$ instances of $I\Sigma^-_n$ in order to derive m instances of the corresponding reflection schema (the first one is used to derive $I\Sigma_{n-1}$).

Let $\bot$ denote the boolean constant 'falsum'.

Lemma 5.2. $GL \vdash \Box^+\neg\bigwedge_{i=0}^m(\Box p_i \to p_i) \to \Box^{m+1}\bot$.

Proof. By Lemma 4.2 we have

$$GL \vdash \Box^+\left(\bigwedge_{i=0}^{m}(\Box p_i \to p_i) \to p\right) \to \left(\bigwedge_{i=0}^{m}(\Box Q_i \to Q_i) \to p\right).$$

Then, substituting in the above formula $\bot$ for p, observe that

$$GL \vdash Q_i(p/\bot) \leftrightarrow \Box^i\bot,$$

and therefore

$$GL \vdash \bigwedge_{i=0}^{m}(\Box Q_i(p/\bot) \to Q_i(p/\bot)) \leftrightarrow \neg\Box^{m+1}\bot. \qquad \Box$$

The following lemma is a relativization of Goryachev's theorem [10].

Lemma 5.3. *The theory axiomatized over T by any m instances of* $\mathrm{Rfn}_T^{\Pi_n}$ *is* Π_{n+1} *conservative over* $(T)_m^{n+1}$.

Proof. Let U be a theory axiomatized over T by m instances of relativized local reflection, say $\mathrm{Prov}_{EA}^{\Pi_n}(\ulcorner A_i\urcorner) \to A_i$, for $i<m$. Let A be a Π_{n+1} sentence such that $U \vdash A$. Then we have

$$T \vdash \neg A \to \neg\bigwedge_{i=0}^{m-1}(\mathrm{Prov}_{EA}^{\Pi_n}(\ulcorner A_i\urcorner) \to A_i)$$

and, by Löb's derivability conditions,

$$T \vdash \mathrm{Prov}_T^{\Pi_n}(\ulcorner\neg A\urcorner) \to \mathrm{Prov}_T^{\Pi_n}\left(\ulcorner\neg\bigwedge_{i=0}^{m-1}(\mathrm{Prov}_T^{\Pi_n}(\ulcorner A_i\urcorner) \to A_i)\urcorner\right).$$

By Lemma 5.2 we then obtain

$$\begin{aligned} T \vdash (\neg\Box^m\bot)^*_{T+\Pi_n(\mathcal{N})} &\to (A \vee \neg\mathrm{Prov}_T^{\Pi_n}(\ulcorner\neg A\urcorner)) \\ &\to A, \end{aligned}$$

by Lemma 3.1 (1). Statement (i) of Lemma 3.2 implies that, for all i,

$$(T)_i^{n+1} \vdash (\neg\Box^i\bot)^*_{T+\Pi_n(N)},$$

therefore $(T)_m^{n+1} \vdash A$. □

Theorem 4 (i) obviously follows from Lemmas 5.1 and 5.3. □

Remark 5.2. The first statement of Theorem 4 is also valid for $n=0$, but only over EA^+ rather than EA. A proof is similar, using Theorem 1 (iii). For EA a similar characterization can be obtained using bounded cut-rank provability a là Wilkie and Paris [27], cf. also [4].

The following corollary was first proved model-theoretically in [13].

Corollary 5.1. *For $n \geqslant 1$, neither $I\Sigma_n^-$, nor $I\Pi_{n+1}^-$ is finitely axiomatizable.*

Proof. If any of these theories were, then its Π_{n+1} consequences would be contained in $(EA)_m^{n+1}$ for some finite m. But this is impossible, since $I\Sigma_n^-$ obviously contains $(EA)_\omega^{n+1}$. □

This corollary can be strengthened by using the following generalization of Theorem 4.

Theorem 5. *Let T be an extension of EA by finitely many Π_{n+2} sentences, $n \geqslant 1$. Then*

(i) *The extension of T by any m instances of $I\Pi_{n+1}^-$ is Π_{n+1} conservative over $(T)_m^{n+1}$.*

(ii) *$T + I\Pi_{n+1}^-$ is Π_{n+1} conservative over $(T)_\omega^{n+1}$.*

Proof. By formalized Deduction theorem it is easy to see that for the given T

$$T \vdash \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+2}) \;\leftrightarrow\; \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2}).$$

Hence, by Theorem 1,

$$\begin{aligned} T + I\Pi_{n+1}^- &\equiv T + \mathrm{Rfn}_{EA}^{\Pi_n}(\Sigma_{n+2}) \\ &\equiv T + \mathrm{Rfn}_T^{\Pi_n}(\Sigma_{n+2}). \end{aligned}$$

Lemma 5.3 then implies the second claim of the theorem. (The fact that $T + I\Pi_{n+1}^-$ contains $(T)_\omega^{n+1}$ follows from Proposition 2.1 and Lemma 2.1.) The first claim of the theorem is obtained from the first part of Theorem 4 in a similar manner. □

Corollary 5.2. *No consistent extension of $I\Pi_{n+1}^-$ by Π_{n+2} sentences is finitely axiomatizable.*

Proof. Suppose, on the contrary, that there is such an extension. We may assume w.l.o.g. that it has the form $T+U$, for some m instances U of $I\Pi^-_{n+1}$, where T is a finite Π_{n+2} axiomatized extension of EA. Then, by Theorem 5, Π_{n+1} consequences of $T+U$ are provable in $(T)^{n+1}_m$ for some finite m. Yet, by the second claim of the same theorem,

$$T + I\Pi^-_{n+1} \vdash \mathrm{RFN}_{(T)^{n+1}_m}(\Pi_{n+1}).$$

The latter formula is Π_{n+1} and unprovable in $(T)^{n+1}_m$. □

We also obtain the following statement.

Theorem 6. *$I\Pi^-_{n+1}$ is not contained in any consistent extension of EA by an r.e. set of Π_{n+2} sentences.*

Proof. By Theorem 1 $I\Pi^-_{n+1}$ contains the schema $\mathrm{Rfn}^{\Pi_n}_{EA}(\Sigma_{n+2})$ and thus the weaker schema $\mathrm{Rfn}_{EA}(\Sigma_{n+2})$. The result follows by the well-known Unboundedness theorem for local reflection (cf. [14, 3]) stating that no consistent Π_m axiomatized r.e. extension of EA contains $\mathrm{Rfn}_{EA}(\Sigma_m)$. □

Corollary 5.3. $I\Pi^-_{n+1} \not\subseteq I\Sigma^R_{n+1}$.

Notice that the complexity of the natural axiomatization of $I\Pi^-_{n+1}$ is Σ_{n+2}, and $I\Sigma^-_n$ has the complexity $\mathscr{B}(\Sigma_{n+1})$. We have the following variant of the Unboundedness theorem for $\mathrm{Rfn}^{\Pi_n}_T(\Sigma_{n+1})$.

Lemma 5.4. *$\mathrm{Rfn}^{\Pi_n}_T(\Sigma_{n+1})$ is not contained in any consistent extension of T by finitely many $\mathscr{B}(\Sigma_{n+1})$ sentences.*

Proof. By Lemma 4.3 the schemata $\mathrm{Rfn}^{\Pi_n}_T(\Sigma_{n+1})$ and $\mathrm{Rfn}^{\Pi_n}_T(\mathscr{B}(\Sigma_{n+1}))$ are equivalent over EA. If the latter is contained in $T+\phi$, where $\phi \in \mathscr{B}(\Sigma_{n+1})$, then $T+\phi \vdash \Box^{\Pi_n}_T \neg\phi \to \neg\phi$ and hence $T \vdash \Box_T \neg\phi \to \neg\phi$. By Löb's theorem we conclude $T \vdash \neg\phi$, that is, $T+\phi$ is inconsistent. □

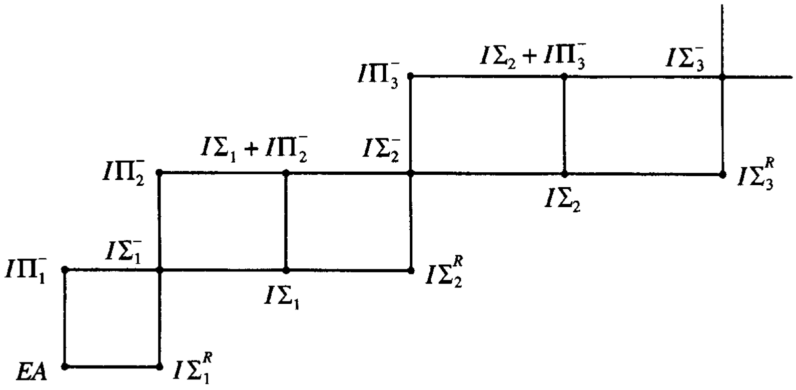
As a corollary we obtain the following result.

Theorem 7. *$I\Sigma^-_n$ is not contained in any consistent extension of EA by finitely many $\mathscr{B}(\Sigma_{n+1})$ sentences.*

Corollary 5.4. *Any consistent theory extending $I\Sigma^-_n$ by $\mathscr{B}(\Sigma_{n+1})$ sentences is not finitely axiomatizable.*

Proof. This follows from Theorem 7 and the fact that the theory $I\Sigma^-_n$ itself has a $\mathscr{B}(\Sigma_{n+1})$ axiomatization. □

Finally, we draw a diagram representing the structure of parametric and parameter free induction schemata of bounded arithmetical complexity.



Notice that $I\Pi_{n+1}^- \nsubseteq I\Sigma_{n+1}^R$ follows from Corollary 5.3. $I\Sigma_n \nsubseteq I\Pi_{n+1}^-$ follows from the fact that $I\Pi_{n+1}^-$ has a Σ_{n+2} axiomatization, whereas $I\Sigma_n$ contains $\mathrm{RFN}_{EA}(\Pi_{n+2})$ (Leivant [15]). $I\Sigma_{n+1}^R \nsubseteq I\Sigma_n + I\Pi_{n+1}^-$ follows from the fact that $I\Sigma_n + I\Pi_{n+1}^-$ is an extension of $I\Sigma_n$ by a set of Σ_{n+2} sentences, whereas $I\Sigma_{n+1}^R$ contains $\mathrm{RFN}_{\Pi_{n+2}}(I\Sigma_n)$ by Proposition 2.2. Therefore, all inclusions corresponding to the edges of the diagram are strict.

6. Parameter free induction and fast growing functions

Classes of p.t.c.f. of theories containing *EA* are often measured in terms of the *extended Grzegorczyk* (or *Fast Growing*) hierarchy.

We fix a canonical fundamental sequences assignment for limit ordinals $<\varepsilon_0$ based on Cantor normal form (see [22]). $\alpha[n]$ denotes the nth term of the fundamental sequence for an ordinal α. If the Cantor normal form of a limit ordinal α is $\alpha_0 + \omega^\beta$, then

$$\alpha[n] := \begin{cases} \alpha_0 + \omega^\gamma \cdot (n+1) & \text{if } \beta = \gamma + 1, \\ \alpha_0 + \omega^{\beta[n]} & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

For this fundamental sequences assignment, a hierarchy of functions F_α, for $\alpha < \varepsilon_0$, is defined as follows.

$$F_0(x) := x + 1,$$

$$F_{\alpha+1}(x) := F_\alpha^{(x+1)}(x),$$

$$F_\alpha(x) := F_{\alpha[x]}(x) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

As usual $F^{(n)}(x)$ denotes the n-fold iteration of a function $F(x)$, that is, the expression $F(F(\dots F(x)\dots))$ (n times).

Classes of functions $\mathscr{E}^\alpha$ for $\alpha<\varepsilon_0$ (the extended Grzegorczyk hierarchy) are defined as follows.

$$\mathscr{E}^\alpha := E(\{F_\beta \mid \beta<\alpha\}),$$

where $E(K)$ denotes the elementary closure of a class K, that is, the closure of K and all elementary functions under composition and bounded recursion.

For $3\leqslant\alpha<\omega$ the classes $\mathscr{E}^\alpha$ thus defined coincide with the classes $\mathscr{E}^\alpha$ of the familiar Grzegorczyk hierarchy. In particular, $\mathscr{E}^3$ is the class of Kalmar elementary functions, and $\mathscr{E}^\omega$ is the class of primitive recursive functions. $\mathscr{E}^{\omega^k}$ coincides with the class of k-recursive functions in the sense of Peter (see [21, 16]).

It is well-known that $\mathscr{E}^{\varepsilon_0}$ coincides with the class of p.t.c.f. of Peano Arithmetic (Kreisel–Schwichtenberg–Wainer), see [9] for a modern self-contained exposition. The results of Parsons in combination with those of Tait (see e.g. [22, 19]) sharpen this to $\mathscr{F}(I\Sigma_n)=\mathscr{E}^{\omega_n}$, for each $n\geqslant 1$, where we define

$$\omega_0(\alpha):=\alpha,$$
$$\omega_{k+1}(\alpha):=\omega^{\omega_k(\alpha)}$$

and $\omega_n:=\omega_n(1)$. From Corollary 4.2 we thus immediately infer the following result.

Theorem 8. *For* $n\geqslant 1$, $\mathscr{F}(I\,\Pi^-_{n+1})=\mathscr{E}^{\omega_n}$.

The characterization of p.t.c.f. of the theories of the form $I\Sigma_n + I\,\Pi^-_{n+1}$ is more interesting.

Theorem 9. *For* $n\geqslant 1$, $\mathscr{F}(I\Sigma_n+I\,\Pi^-_{n+1})=\mathscr{E}^{\omega_n(2)}$. *In particular*, $\mathscr{F}(I\Sigma_1+I\,\Pi^-_2)=\mathscr{E}^{\omega^2}$, *that is, coincides with the class of doubly recursive functions of Peter.*

Proof. For a proof of this theorem, in addition to the results of the previous section, we apply the machinery of transfinitely iterated reflection principles. This topic goes back to the works of Turing and Feferman. Essential ingredients for our proof are contained in the works [23, 2] and particularly [26]. Neither Schmerl, nor Sommer present all technical details in their papers, therefore the reader is also referred to their Ph.D. Theses cited therein.

First, following Sommer [26], we represent the system of ordinal notation up to ε_0 by bounded arithmetical formulas [3] in such a way that basic properties of ordinal functions and Cantor normal forms become provable in EA. Then we construct a bounded formula $F_\alpha(x)\simeq y$ of the variables α, x, y that uniformly represents the graphs of the functions in the Fast Growing hierarchy as defined above. For these formulas one can verify basic monotonicity properties and functionality property in EA. As in [26, p. 285], we then define the theories S_α, for $\alpha<\varepsilon_0$, as follows:

$$S_\alpha := EA + \{\forall x\exists y\, F_{3+\beta}(x)\simeq y \mid \beta<\alpha\}.$$

[3] In fact, a $\Delta_0(\exp)$ natural well-ordering representation will do for our present purposes.

As a corollary of Herbrand's Theorem (or Proposition 6.4 in [26]) we obtain the following statement. □

Proposition 6.1. *For all* $\alpha < \varepsilon_0$, $\mathscr{F}(S_\alpha) = \mathscr{E}^{3+\alpha}$.

Proposition 6.10 of [26] can then be reformulated as follows.

Proposition 6.2. *Provably in EA*,

$$\forall \alpha < \varepsilon_0 \quad S_\alpha \equiv EA + \{\mathrm{RFN}_{S_\beta}(\Pi_2) \mid \beta < \alpha\}.$$

Uniqueness Lemma 2.3 of [2] formulated for iterated consistency assertions holds for iterated Π_2 reflection principles with the same proof. It implies that there is only one, up to EA-provable equivalence, sequence of theories S_α satisfying the statement of the previous proposition. This means that the theories S_α coincide with the hierarchy of transfinitely iterated uniform Π_2 reflection principles built up over EA along the canonical system of ordinal notation in the sense of [23, 2].

More precisely (see [2]), for a given $\Delta_0(\exp)$ well-ordering representation, an initial theory T, and a fixed $n \geqslant 1$, there is a $\Delta_0(\exp)$ formula $Ax_T(\alpha, x)$ numerating in EA the axioms of a theory denoted by $(T)^n_\alpha$ such that, provably in EA,

$$\forall \alpha < \varepsilon_0 \quad (T)^n_\alpha \equiv T + \{\mathrm{RFN}_{(T)^n_\beta}(\Pi_n) \mid \beta < \alpha\}.$$

Actually, the equivalence above can be viewed as a fixed point equation implicitly defining $Ax_T(\alpha, x)$. By Lemma 2.3 of [2], for a fixed initial theory T and a well-ordering representation, such a sequence of theories is defined uniquely up to EA-provable equivalence. So, applying this to the canonical well-ordering representation up to ε_0 we obtain the following.

Proposition 6.3. *Provably in EA*,

$$\forall \alpha < \varepsilon_0, \quad S_\alpha \equiv (EA)^2_\alpha.$$

By the same Uniqueness lemma, the transfinite progression of iterated reflection principles over primitive recursive arithmetic, $(PRA)^{n+1}_\alpha$, coincides with the one considered in Schmerl [23], which he denotes $\binom{n}{\alpha}$. By inspection of the so-called Fine Structure theorem [23, p. 347] it is not too difficult to convince oneself that its proof works for EA, as well as for PRA, and to obtain the following statement. (A more general form of this theorem with a new proof will appear in [6].)

Proposition 6.4. *For each* $n, k \geqslant 1$, *and all ordinals* $\alpha \geqslant 1$, $((EA)^{n+k}_\alpha)^n_\beta$ *proves the same* Π_n *sentences as* $(EA)^n_{\omega_k(\alpha)\cdot(1+\beta)}$.

(In fact, the mutual Π_n conservativity above holds provably in EA^+, uniformly in α, β.) Now we are ready to complete the proof of Theorem 9. Since $I\Sigma_n$ is a finitely

Π_{n+2} axiomatizable theory, Theorem 5 implies that $I\Sigma_n + I\Pi^-_{n+1}$ is Π_{n+1} conservative over $(I\Sigma_n)^{n+1}_\omega$. But $I\Sigma_n$ is equivalent to $(EA)^{n+2}_1$, therefore

$$(I\Sigma_n)^{n+1}_\omega \equiv ((EA)^{n+2}_1)^{n+1}_\omega.$$

By Proposition 6.4 $((EA)^{n+2}_1)^{n+1}_\omega$ proves the same Π_{n+1} sentences as $(EA)^{n+1}_{\omega^2}$, and the latter theory proves the same Π_2 sentences as $(EA)^2_{\omega_{n-1}(\omega^2)} \equiv (EA)^2_{\omega_n(2)}$. Therefore $I\Sigma_n + I\Pi^-_{n+1}$ and $(EA)^2_{\omega_n(2)}$ prove the same Π_2 sentences and have the same classes of p.t.c.f.. The result follows now by Propositions 6.1 and 6.3. □

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