Symmetry properties of subdivision graphs

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\textbf{ARTICLE INFO}

\textbf{Article history:}
Available online 27 April 2011

\textbf{Keywords:}
Subdivision graph
Locally distance-transitive graph
Locally \textit{s}-arc-transitive graph
Line graph

\textbf{ABSTRACT}

The subdivision graph $S(\Sigma)$ of a graph $\Sigma$ is obtained from $\Sigma$ by ‘adding a vertex’ in the middle of every edge of $\Sigma$. Various symmetry properties of $S(\Sigma)$ are studied. We prove that, for a connected graph $\Sigma$, $S(\Sigma)$ is locally \textit{s}-arc transitive if and only if $\Sigma$ is $\lceil \frac{s+1}{2} \rceil$-arc transitive. The diameter of $S(\Sigma)$ is $2d + \delta$, where $\delta$ has diameter $d$ and $0 \leq \delta \leq 2$, and local $s$-distance transitivity of $S(\Sigma)$ is defined for $1 \leq s \leq 2d + \delta$. In the general case where $s \leq 2d - 1$ we prove that $S(\Sigma)$ is locally $s$-distance transitive if and only if $\Sigma$ is $\lceil \frac{s+1}{2} \rceil$-arc transitive. For the remaining values of $s$, namely $2d \leq s \leq 2d + \delta$, we classify the graphs $\Sigma$ for which $S(\Sigma)$ is locally $s$-distance transitive in the cases $s \leq 5$ and $s \geq 15 + \delta$. The cases $\max \{2d, 6\} \leq s < \min \{2d + \delta, 14 + \delta\}$ remain open.

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1. Introduction

All graphs in this paper are simple and undirected, that is to say, a graph $\Sigma$ consists of a vertex set $V(\Sigma)$ and a subset $E(\Sigma)$ of unordered pairs from $V(\Sigma)$, called edges. The subdivision graph $S(\Sigma)$ of a graph $\Sigma$ is defined as the graph with vertex set $V(\Sigma) \cup E(\Sigma)$ and edge set $\{x, e\} | x \in V(\Sigma), e \in E(\Sigma), x \in e\}$. Informally, $S(\Sigma)$ is the graph obtained from $\Sigma$ by ‘adding a vertex’ in the middle of each edge of $\Sigma$. The purpose of this paper is to elucidate connections between various symmetry properties of $\Sigma$ and of its subdivision graph $S(\Sigma)$, in particular local $s$-arc-transitivity, and local $s$-distance transitivity (defined in Section 2). These properties are generalisations and natural analogues of graph properties extensively studied in the literature, namely $s$-arc transitivity introduced by Tutte [12,13] and distance transitivity introduced by Higman [8], and studied further by Biggs and many others (see [2,3,11,14,15]). Moreover, the families of locally $s$-arc-transitive graphs and locally $s$-distance transitive graphs were analysed in [7] and [5], respectively.

In Section 2, we give basic graph theoretic concepts and notation, including definitions of the transitivity properties we study. Some basic properties of $S(\Sigma)$ are given in Section 3. First, $S(\Sigma)$ is bipartite and is connected if $\Sigma$ is connected, and the graph $\Sigma$ can be reconstructed from $S(\Sigma)$ (in the exceptional case where $\Sigma$ is a cycle, reconstruction of $\Sigma$ from $S(\Sigma)$ is up to isomorphism only). Cycles arise as exceptions in other ways also. The automorphism group $\text{Aut}(\Sigma)$ acts on $V(\Sigma)$ and $E(\Sigma)$ and preserves the incidence of $S(\Sigma)$ giving a natural embedding $\text{Aut}(\Sigma) \leq \text{Aut}(S(\Sigma))$. With the exception of cycles, this embedding is an isomorphism. For cycles $\Sigma = C_n$, $\text{S}(\Sigma) = C_{2n}$ and $\text{Aut}(S(\Sigma)) = D_{4n} = \text{Aut}(\Sigma)$. Note that the subdivision graphs of cycles are vertex transitive, while for all other graphs $\Sigma$, $\text{Aut}(S(\Sigma)) = \text{Aut}(\Sigma)$ fixes $V(\Sigma)$ and $E(\Sigma)$ setwise.

In Section 4, we prove a decisive relationship between the levels of arc transitivity of $\Sigma$ and $S(\Sigma)$.

\textbf{Theorem 1.1.} Let $\Sigma$ be a connected graph, $s$ a positive integer, and $G \leq \text{Aut}(\Sigma)$. Then $S(\Sigma)$ is locally $(G, s)$-arc transitive if and only if $\Sigma$ is $(G, \lceil \frac{s+1}{2} \rceil)$-arc transitive.

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doi:10.1016/j.disc.2011.03.031
Table 1

| \(\Sigma\) and \(G\) for \(s = 2, 3, 4, 5\). |
|---|---|---|---|---|
| \(K_2\) | 2 | 1 | 0 | \(S_2\) |
| \(K_3\) | 2, 3 | 1 | 1 | \(S_3\) |
| \(K_n (n \geq 4)\) | 2, 3 | 1 | 2 | \(S_n\) |
| \(\Sigma\) (\(n \geq 2\)) | 4 | 2 | 0 | \(S_n \times S_2\) |
| \(C_3\) | 4, 5 | 2 | 1 | \(D_{10}\) |
| \(P\) | 4, 5 | 2 | 2 | \(S_5\) |
| HoSi | 4, 5 | 2 | 2 | PSU(3, 5).2 |

This generalises Theorem 3.10 of [7], which proves the result for odd \(s\). The concepts of \((G, 1)\)-arc transitivity and \((G, 1)\)-distance transitivity are equivalent, as are their local variants. Thus Theorem 1.1 implies the equivalence of \((G, 1)\)-arc transitivity of \(\Sigma\) and local \((G, 1)\)-distance transitivity of \(S(\Sigma)\). Moreover, if \(s = 2\) or 3, then the local \((G, s)\)-distance transitivity of \(S(\Sigma)\) is equivalent to the \((G, 2)\)-arc transitivity of \(\Sigma\), see Corollary 4.2. A similar relationship holds for larger values of \(s\) provided \(s \leq 2 \text{diam}(\Sigma) - 1\) (proved in Section 4).

**Theorem 1.2.** Let \(\Sigma\) be a connected graph and \(s\) a positive integer such that \(s \leq 2 \text{diam}(\Sigma) - 1\) (so in particular \(\text{diam}(S(\Sigma)) > s\)). Then \(S(\Sigma)\) is locally \((G, s)\)-distance transitive if and only if \(\Sigma\) is \((G, \lceil \frac{s + 1}{2} \rceil)\)-arc transitive.

We denote the diameter of \(\Sigma\) by \(d\). The diameter of \(S(\Sigma)\) is \(2d + d\) with \(d \in \{0, 1, 2\}\) (see Remark 3.1). Thus the \(s\)-values for which Theorem 1.2 gives no information are those satisfying \(2d \leq s \leq 2d + d\). For very large and very small values of \(s\) in this range we can determine explicitly the pairs \(\Sigma, G\) for which \(S(\Sigma)\) is locally \((G, s)\)-distance transitive. In this result, \(P\) and HoSi denote the Petersen graph and the Hoffman–Singleton graph, respectively.

**Theorem 1.3.** Let \(\Sigma\) be a connected graph with \(|V(\Sigma)| \geq 2\) and diameter \(d\), let \(G \leq \text{Aut}(\Sigma)\), and let \(s\) be a positive integer such that \(2d \leq s \leq 2d + d = \text{diam}(S(\Sigma))\).

(a) If \(s \geq 15 + d\), then \(S(\Sigma)\) is locally \((G, s)\)-distance transitive if and only if \(\Sigma = C_n\) and \(G = D_{2n}\), either with \(n = s\), or with \(n = s + 1\) odd.

(b) If \(s \leq 5\), then \(S(\Sigma)\) is locally \((G, s)\)-distance transitive if and only if \(\Sigma = G\) are as in Table 1.

The first part of Theorem 1.3 is an application of the deep theorem of Weiss [15] that the only 8-arc transitive graphs are the cycles, while the second part uses the classification by Ivanov [10] of 3-arc transitive graphs of girth 5. As R. Weiss’s result relies on the finite simple group classification so also does Theorem 1.3(a). The local distance transitivity properties claimed for the graphs \(S(\Sigma)\) in Table 1 are established in Section 5.

We prove in fact that, for each of the graphs \(\Sigma\) in Table 1, \(S(\Sigma)\) is locally distance transitive. This enables us to classify all graphs \(\Sigma\) of diameter at most 2 for which \(S(\Sigma)\) is locally distance transitive.

**Corollary 4.4.** Let \(\Sigma\) be a connected graph with \(|V(\Sigma)| \geq 2\) and \(\text{diam}(\Sigma) \leq 2\), and let \(G \leq \text{Aut}(\Sigma)\). Then \(S(\Sigma)\) is locally \(G\)-distance transitive if and only if \(\Sigma, G\) are as in Table 1 (for the maximum value of \(s\)).

The first two authors are considering the unresolved cases of Theorem 1.3, attacking in particular the problem:

**Problem 1.5.** Classify graphs \(\Sigma\) for which \(S(\Sigma)\) is locally distance transitive.

2. Concepts and symmetry for graphs

For a positive integer \(s\), an \(s\)-arc of a graph \(\Sigma\) is an \((s + 1)\)-tuple \((v_0, v_1, \ldots, v_s)\) of vertices such that \(\{v_{i-1}, v_i\} \in E(\Sigma)\) for \(1 \leq i \leq s\), and \(v_{i-1} \neq v_{i+1}\) for \(1 \leq j \leq s - 1\). The integer \(s\) is called the length of the \(s\)-arc. A 1-arc is often called an arc.

The distance between two vertices \(v_1\) and \(v_2\), denoted by \(d_{\Sigma}(v_1, v_2)\), is the minimum number \(s\) such that there exists an \(s\)-arc from \(v_1\) to \(v_2\). For a connected graph \(\Sigma\), the diameter of \(\Sigma\), denoted \(\text{diam}(\Sigma)\), is the maximum distance between two vertices of \(\Sigma\). For \(0 \leq i \leq \text{diam}(\Sigma)\) define

\[\Sigma_i := \{(v, w) \in V(\Sigma) \times V(\Sigma) \mid d_{\Sigma}(v, w) = i\}\]

and, for \(v \in V(\Sigma)\), define \(\Sigma_i(v) := \{w \in V(\Sigma) \mid d_{\Sigma}(v, w) = i\}\). We often write \(\Sigma(v)\) for \(\Sigma_1(v)\). A graph \(\Sigma\) is bipartite if its vertex set can be partitioned into two non-empty sets called biparts such that every edge has one vertex in each bipart.

We denote a complete graph and a cycle on \(n\) vertices by \(K_n\) and \(C_n\), respectively; \(K_{m, n}\) denotes the complete bipartite graph with biparts of sizes \(m\) and \(n\). A graph is regular if its vertices have a constant valency, that is, lie in a constant number of edges. If \(\Sigma\) contains a cycle then the girth of \(\Sigma\) is the length of its shortest cycle.
Table 2
Properties $\mathcal{P}$ for $G$-action on a connected graph $\Sigma$.

| Property $\mathcal{P}$ | $\mathcal{P}(\Sigma) = \{\Delta_i | 1 \leq i \leq s\}$, and $\Delta_i \neq \emptyset$ |
|------------------------|------------------------------------------|
| G-arc transitivity      | $s = 1$ and $\Delta_1 = \Sigma_1$        |
| $(G, s)$-arc transitivity | $\Delta_1$ is the set of $i$-arcs of $\Sigma$ |
| $(G, s)$-distance transitivity | $\Delta_1 = \Sigma_1$                     |
| G-distance transitivity  | $s = \text{diam}(\Sigma)$ and $\Delta_1 = \Sigma_1$ |

Table 3
Local properties $\mathcal{P}$ for $G$-action on a connected graph $\Sigma$.

| Local property $\mathcal{P}$ | $\mathcal{P}(\Sigma, v) = \{\Delta_i(v) | 1 \leq i \leq s\}$, and $\Delta_i(v) \neq \emptyset$ for some $v$ |
|------------------------------|-------------------------------------------------|
| Local G-arc transitivity     | $s = 1$ and $\Delta_1(v) = \Sigma_1(v)$         |
| Local $(G, s)$-arc transitivity | $\Delta_1(v)$ is the set of $i$-arcs of $\Sigma$ with initial vertex $v$ |
| Local $(G, s)$-distance transitivity | $\Delta_1(v) = \Sigma_1(v)$                      |
| Local G-distance transitivity | $s = \text{diam}(\Sigma)$ and $\Delta_1(v) = \Sigma_1(v)$ |

Let $G \leq \text{Aut}(\Sigma)$. The properties we study are defined relative to the $G$-action on $\Sigma$. Each property $\mathcal{P}$ is defined by the requirement that $G$ be transitive on each set in some collection $\mathcal{P}(\Sigma)$ of sets, as in Table 2. Moreover, each property has a ‘local variant’ that is defined by requiring that, for each $v \in V(\Sigma)$, the stabiliser $G_v$ be transitive on each set in a related collection $\mathcal{P}(\Sigma, v)$ of sets, as in Table 3. Notice that our definition of local $(G, s)$-arc transitivity is slightly stronger than the definition in [7] (actually it is equivalent to our definition as long as $\Sigma$ has no vertex with valency 1).

These concepts are sometimes used without reference to a particular group $G$, especially when $G = \text{Aut}(\Sigma)$. It follows from the definitions that, for $s \geq 2$, (local) $(G, s)$-arc transitivity implies (local) $(G, s - 1)$-arc transitivity, and (local) $(G, s)$-distance transitivity implies (local) $(G, s - 1)$-distance transitivity.

Remark 2.1. For $s = 1$, several of the concepts coincide. Indeed, for a connected graph $\Sigma$ and $G \leq \text{Aut}(\Sigma)$, the properties of local G-arc transitivity, local $(G, 1)$-arc transitivity and local $(G, 1)$-distance transitivity are equivalent.

Lemma 2.2. Let $\Sigma$ be a connected graph and let $G \leq \text{Aut}(\Sigma)$.

(a) If $G$ is intransitive on $V(\Sigma)$, then each of the equivalent conditions of Remark 2.1 holds if and only if $G$ is transitive on $E(\Sigma)$.

Moreover, in this case, $\Sigma$ is a bipartite graph and the $G$-orbits in $V(\Sigma)$ are the two biparts.

(b) $\Sigma$ is locally $(G, 2)$-arc transitive if and only if, for all $v \in V(\Sigma)$, $G_v$ is 2-transitive on $\Sigma_1(v)$.

This Lemma is very easy to prove. See for instance [7, Lemma 3.1, Lemma 3.2] for details, noticing that with our definition of local $(G, s)$-arc transitivity, the hypothesis that no vertex has valency 1 can be dropped.

Each of the ‘global’ properties $\mathcal{P}$ for a graph $\Sigma$ implies the local variant, but the converse is not true in general. For example if $\Sigma = K_m, n$ and $G = S_m \times S_n$, then all the local properties, but none of the global properties, hold with $s = 2$. However, for each of the local properties, the corresponding global property holds if and only if $G$ is transitive on $V(\Sigma)$ (and in this case, in particular, all vertices have the same valency).

3. Basic properties of subdivision graphs

We give some further definitions and results related to graphs that will be used in the next sections. The line graph $L(\Sigma)$ of a graph $\Sigma$ is defined as the graph with vertex set $E(\Sigma)$ and edges $\{e_1, e_2\}$, for $e_1, e_2 \in E(\Sigma)$ such that $e_1 \cap e_2 \neq \emptyset$. The distance 2 graph of $\Sigma$ is the graph $\Sigma^{[2]}$ with the same vertex set as $\Sigma$ but with the edge set replaced by the set of all vertex pairs $\{u, v\}$ such that $d_\Sigma(u, v) = 2$. If $\Sigma$ is connected, then all vertices at even $\Sigma$-distance from $v$ lie in the same connected component of $\Sigma^{[2]}$, as do all vertices at odd $\Sigma$-distance from $v$, and so $\Sigma^{[2]}$ has at most two connected components. Moreover, if $\Sigma$ is connected and bipartite, then $\Sigma^{[2]}$ has exactly two components.

Clearly, if $\Sigma$ is connected, then the subdivision graph $S(\Sigma)$ is connected and bipartite with biparts $V(\Sigma)$ and $E(\Sigma)$. Note that, in $S(\Sigma)$, each vertex in $E(\Sigma)$ has valency 2 while the valency of each vertex in $V(\Sigma)$ is equal to its valency in $\Sigma$.

The graph $S(\Sigma)$ is closely related to the line graph $L(\Sigma)$ of $\Sigma$, the link arising via the distance 2 graph $S(\Sigma^{[2]})$ of $S(\Sigma)$. As mentioned in the introduction, for connected graphs $\Sigma$, we can reconstruct $\Sigma$ from its subdivision graph $S(\Sigma)$. Indeed, for a connected graph $\Sigma$ (not $K_1$), $S(\Sigma)^{[2]}$ has two connected components, namely $\Sigma$ and $L(\Sigma)$. Moreover, either $\Sigma \cong L(\Sigma) \cong C_n$ for some $n \geq 3$, or $\Sigma$ is the unique connected component of $S(\Sigma)^{[2]}$ containing vertices of valency different from 2 in $S(\Sigma)$.

The diameter of $\Sigma$ and $S(\Sigma)$ are linked in the following way.

Remark 3.1. Suppose that $\Sigma$ is a connected graph with $|V(\Sigma)| \geq 2$. Then
Theorem 1.1 and Suppose that $\Sigma$ is a connected graph of diameter $d$ with $|V(\Sigma)| \geq 2$, such that $\Sigma(\Sigma)$ has diameter $2d + \delta$, for some $\delta \in \{0, 1, 2\}$, and $G \leq \text{Aut}(\Sigma)$.

We study relationships between various symmetry properties of $\Sigma$ and $\Sigma(\Sigma)$. In particular we prove Theorems 1.1 and 1.2. First we consider the effect of local transitivity conditions on $\Sigma(\Sigma)$ related to edges of $\Sigma$. Note that the assumption $\Sigma \neq K_2$, under Hypothesis 1, is equivalent to the condition that $\Sigma$ contains at least one 2-arc.

Lemma 4.1. Suppose that Hypothesis 1 holds, and in part (b) suppose that $\Sigma \neq K_2$. Set $\Gamma := \Sigma(\Sigma)$.

(a) If $G_\alpha$ is transitive on $\Gamma_1(e)$ for all $e \in E(\Sigma)$, then $\Sigma$ is $G$-vertex transitive.

(b) $G_\alpha$ is transitive on $\Gamma_2(e)$ for all $e \in E(\Sigma)$ if and only if either $\Sigma$ is $(G, 2)$-arc transitive, or $\Sigma = C_n$ with $n$ even and $G \cong D_n$ has two orbits in $E(\Sigma)$.

Proof. (a) Let $u, v \in V(\Sigma)$. There is a path $x_1, x_2, x_3, \ldots, x_n$ in $\Sigma$ such that $x_1 = u, x_n = v$, since $\Sigma$ is connected. By assumption, for each $i$ there exists $g_i \in G_{[x_i, x_{i+1}]}$ such that $x_i = x_{i+1}$. The element $g_1g_2 \cdots g_{n-1}$ maps $x_1 = u$ to $x_n = v$.

(b) Let $e = \{u, v\} \in E(\Sigma)$. Then $\Gamma_1(e) = \{u, v\}$ and $\Gamma_2(e) = (\Gamma_1(u) \cup \Gamma_1(v)) \setminus \{e\}$. Moreover, $G_\alpha$ is transitive on $\Gamma_2(e)$ if and only if $G_\alpha$ is transitive on $\Gamma_1(e)$ and the stabiliser $G_{u,v}$ is transitive on $\Gamma_1(u) \setminus \{e\}$ (or equivalently $G_{u,v}$ is transitive on $\Sigma(u) \setminus \{v\}$).

Hypothesis 1 holds, and in part (b) suppose that $\Sigma \neq K_2$. Set $\Gamma := \Sigma(\Sigma)$.

4.1. Proof of Theorem 1.1

Let $\Delta = \Sigma(\Sigma)$, and note that $2t - 1 = s$ if $s$ is odd, and $2t - 2 = s$ if $s$ is even. Suppose that $\Sigma$ is $(G, t)$-arc transitive. We prove first that $\Gamma$ is locally $(G, 2t - 1)$-arc transitive. Let $\alpha = (u_0, e_0, u_1, e_1, \ldots, u_{t-1}, e_{t-1})$ and $\alpha' = (u_0, e_0, u_1', e_1', \ldots, u_{t-1}', e_{t-1}')$ be $(2t - 1)$-arcs in $\Gamma$ with initial vertex $u_0 \in V(\Sigma)$. Thus $\Delta' := (u_0, u_1, \ldots, u_{t-1}, u_t)$ and $\Delta'' := (u_0, u_1', \ldots, u_{t-1}', u_t')$ are $t$-arcs in $\Sigma$ where $e_{t-1} = \{u_{t-1}, u_t\}, e_{t-1}' = \{u_{t-1}', u_t'\}$. By assumption there exists $g \in G_\alpha$ such that $\tilde{\alpha}^g = \alpha'$, hence $\alpha^g = \alpha'$. Thus $G_\alpha$ acts transitively on $(2t - 1)$-arcs in $\Gamma$ starting with $u_0$. Now consider $(2t - 1)$-arcs in $\Delta$ of the form $\beta = (e_0, u_1, e_1, \ldots, u_{t-1}, e_{t-1}, u_t)$ and $\beta' = (e_0, u_1', e_1, \ldots, u_{t-1}', e_{t-1}', u_t')$ with initial vertex $e_0 \in E(\Sigma)$. Now $e_0 = \{u_0, u_1\} = \{u_0, u_1'\}$ and $\beta$ correspond to $t$-arcs $\tilde{\beta}' := (u_0, u_1, \ldots, u_{t-1}, u_t)$ and $\tilde{\beta}'' := (u_0, u_1', \ldots, u_{t-1}', u_t')$ in $\Sigma$. Hence, there exists $g \in G$ such that $\tilde{\beta}^g = \tilde{\beta}'$. The element $g$ fixes $e_0$ setwise and therefore satisfies $\tilde{\beta}^g = \tilde{\beta}'$. Thus $\Gamma$ is locally $(G, 2t - 1)$-arc transitive. Since $2t - 1 \geq s$, we have that $\Gamma$ is locally $(G, s)$-arc transitive.

Conversely, suppose $\Gamma$ is locally $(G, s)$-arc transitive. We prove that $\Sigma$ is $(G, t)$-arc transitive. Consider two $t$-arcs $\alpha = (u_0, u_1, \ldots, u_t)$ and $\alpha' = (u_0, u_1', \ldots, u_t') \in \Sigma$ with initial vertex $u_0$. The corresponding $(2t - 1)$-arcs in $\Gamma$ are $\tilde{\alpha} := (u_0, e_0, u_1, e_1, \ldots, u_{t-1}, e_{t-1})$ and $\tilde{\alpha}' := (u_0, e_0, u_1', e_1', \ldots, u_{t-1}', e_{t-1}')$, where for $i < t$, $e_i = \{u_i, u_{i+1}\}$ and $e_i' = \{u_i', u_{i+1}'\}$. Suppose first that $s$ is odd (so that $s = 2t - 1$). Then $\Gamma$ is locally $(G, 2t - 1)$-arc transitive so
there exists $g \in G_0$ such that $\omega^g = \omega'$. Thus $e^g_{i-1} = e'_{i-1}$ and $u^g_{i-1} = u'_{i-1}$, and hence $u^g = u'$ and $\omega^g = \omega'$. Therefore $\Sigma$ is locally $(G, t)$-arc transitive. By Lemma 4.1, $G$ is vertex transitive on $\Sigma$, and hence $\Sigma$ is $(G, t)$-arc transitive.

Finally suppose that $s$ is even, so that $s = 2t$ and $\Gamma$ is locally $(G, 2t)$-arc transitive. Note that $t \geq 2$ in this case. Then the $(2t-1)$-arcs of $\Gamma$ in the previous paragraph are of the form $\omega = (\beta, e_{t-1})$ and $\omega' = (\beta', e'_{t-1})$ with $\beta, \beta'$ both $(2t-2)$-arcs in $\Gamma$ with initial vertex $u_0$. Since $\Gamma$ is locally $(G, 2t)$-arc transitive, there exists $g \in G_0$ such that $\beta^g = \beta'$. Thus $\omega^g = (\beta', e^g_{t-1})$ and $e^g_{t-1} = (u^g_{t-1}, u^g_0)$. Now we also have two $(2t-2)$-arcs $\gamma := (e'_0, u'_1, e'_1, \ldots, u'_t, e'_0)$ and $\gamma' := (e'_0, u'_1, e'_1, \ldots, u'_t, e'_0)$ in $\Gamma$ with initial vertex $e'_0$, so there exists $h \in G_0$ such that $h \gamma^h = \gamma'$. Since $u_0, u'_1 = e'_0 = e_0^h = (u_0^h, u_0)$, we have $u_0^h = u_0$ and hence $\omega^gh = \omega'$, with $gh \in G_0$. Therefore $\Sigma$ is locally $(G, t)$-arc transitive, and as in the previous paragraph, $\Sigma$ is $(G, t)$-arc transitive. □

4.2. Some consequences of Theorem 1.1

We show for the cases $s = 2, 3$, how to link local $(G, s)$-distance transitivity of $S(\Sigma)$ with symmetry properties of $\Sigma$.

Corollary 4.2. Suppose that Hypothesis 1 holds, and $\Sigma \neq \Sigma_2$. Then the following four conditions are equivalent.

(a) $S(\Sigma)$ is locally $(G, 2)$-distance transitive.
(b) $\Sigma$ is locally $(G, 2)$-arc transitive.
(c) $S(\Sigma)$ is locally $(G, 3)$-arc transitive.
(d) $S(\Sigma)$ is locally $(G, 3)$-distance transitive.

Since $\Sigma \neq \Sigma_2$, $\Sigma$ contains a 2-arc, so (b) is well defined, and $\text{diam}(S(\Sigma)) \geq 3$ by Remark 3.1, so (a), (c), and (d) are well defined.

Proof. By Theorem 1.1 for $s = 3$, conditions (b) and (c) are equivalent. It follows easily from the definition of local $(G, 3)$-distance transitivity that condition (c) implies condition (d). Also, by definition, condition (a) follows from condition (d). Thus it is sufficient to prove that condition (a) implies condition (b).

Let $\Gamma = S(\Sigma)$, and suppose that $\Gamma$ is locally $(G, 2)$-distance transitive. In particular then, for all $e \in E(\Sigma)$, $G_e$ is transitive on $\Gamma(e)$ and $\Gamma(2e)$, so by Lemma 4.1(b), $\Sigma$ is $(G, 2)$-arc transitive, or $\Sigma = \Sigma_3$ and $G = D_3$ with two edge orbits in $E(\Sigma)$. However in the latter case, condition (a) does not hold since $G_{u_0} = 1$ for $u \in V(\Sigma)$. □

Remark 4.3. Typically, a graph $\Sigma$ for which one of the equivalent conditions of Corollary 4.2 holds will have girth at least 4. Otherwise $\text{girth}(\Sigma) = 3$, and since $G$ is transitive on the 2-arcs of $\Sigma$, all 2-arcs form a 3-cycle, and it follows that $\Sigma = \Sigma_n$ for some $n$. As $\Sigma \neq \Sigma_2$, $n \geq 3$ and $G$ is a 3-transitive subgroup of $S_n$.

We finish this subsection with a result about local $(G, 4)$-distance transitivity when the girth of $\Sigma$ is greater than 4.

Proposition 4.4. Suppose that Hypothesis 1 holds, and $\text{girth}(\Sigma) \geq 5$. Then $\text{diam}(S(\Sigma)) \geq 5$, and if $S(\Sigma)$ is locally $(G, 4)$-distance transitive, then $\Sigma$ is $(G, 3)$-arc transitive.

Proof. Let $\Gamma = S(\Sigma)$. Since $G = G_0$ such that $G_0$ contains a minimal cycle $(v_0, v_1, \ldots, v_{n-1})$, and so $d_{\Gamma}(v_0, v_2, v_3) = 5$, thus $d_{\Gamma}(v_0, v_2) = 5$. Let $(u_0, u_1, u_2, v_0)$ and $(u'_0, u'_1, u'_2, v'_0)$ be two 3-arcs in $\Sigma$. Since $\Gamma$ is locally $(G, 4)$-distance transitive it follows from Corollary 4.2 that $\Sigma$ is $(G, 2)$-arc transitive. Thus there exists $a \in G$ such that $(u_0, u_1, u_2)^a = (u'_0, u'_1, u'_2)$. Now we have two 4-arcs $\beta = (e_0, u'_1, e'_1, f)$ and $\beta' = (e'_0, u'_1, e'_1, f')$ in $S(\Sigma)$, where $e'_i = \{u'_0, u'_1\}$ for $0 \leq i \leq 2$ and $f = \{u'_2, u'_3\}$. Thus $d_{\Gamma}(e'_0, f) = 4$ and similarly $d_{\Gamma}(e'_0, f') = 4$. By the local $(G, 4)$-distance transitivity of $\Gamma$, there exists $b \in G_0$ such that $\beta^b = \beta'_b$. If $b \in G_0 \setminus G_0$, then $u^b_0 = u'_0$, $u^b_1 = u'_1$, $u^b_2 = u'_2$, and so the edge $(u'_0, u'_1)$ is mapped by $b$ onto $(u_0, u'_1)$ or $(u'_0, u'_1)$ respectively, however those cannot be edges, as $s \geq 5$. Thus $b \in G_0 \setminus G_0$, and $f^b = e'_2$. If $u^b_2 = u'_2$, then $\{u'_0, u'_1\}, \{u'_0, u'_2\}$ would be an edge, again a contradiction. Hence $u^b_2 = u'_2$, $(u^b_0, u'_0) = (u_0, u'_0)$, and we have that $(u_0, u_1, u_2, v_0)^{ab} = (u'_0, u'_1, u'_2, v'_0)$. Thus $\Sigma$ is $(G, 3)$-arc transitive. □

4.3. Proof of Theorem 1.2

The following Lemma 4.5 is a critical ingredient in the proof of Theorem 1.2.

Lemma 4.5. Suppose the Hypothesis 1 holds, and that $s$ is an even positive integer satisfying $s \leq 2d - 1$. If $S(\Sigma)$ is locally $(G, s)$-distance transitive, then $\text{girth}(\Sigma) \geq s + 2$.

Proof. Let $\Gamma = S(\Sigma)$, $s = 2t$, and suppose that $\Gamma$ is locally $(G, 2t)$-distance transitive. By assumption $d \geq t + 1 \geq 2$. Note that, by Lemma 4.1, $G$ is transitive on $V(\Sigma)$. We prove the lemma by induction on $t$. If $t = 1$, then $\Sigma$ is $(G, 2)$-arc transitive by Corollary 4.2, and since $d \geq 2$, some 2-arc does not lie in a 3-cycle. Hence no 2-arcs lie in a 3-cycle and $\text{girth}(\Sigma) \geq 4$. 

Table 4

| (G, s)-distance transitivity of S(Σ) for Σ = Kₙ and n ≥ 4. |
|---|---|
| s | Conditions on G |
| 1 | 2-transitive on V(Σ) |
| 2, 3 | 3-transitive on V(Σ) |
| 4 | 4-transitive on V(Σ), or n = 9 and G = PΓL(2, 8) |

by Corollary 4.2, $\Gamma_2(v) = \emptyset$ for $v \in V(\Sigma)$ and $G_\delta$ is transitive on both $\Gamma_2(e)$ and $\Gamma_4(e)$ by [4, p. 6]; it follows that $\Gamma'$ is locally $(G, 4)$-distance transitive. □

5.2. $\Sigma = K_{n,n}$ with $n \geq 2$

We have $\text{diam}(\Sigma) = 2$, $\text{diam}(S(\Sigma)) = 4$ and $\text{Aut}(\Sigma) = S_n \times S_2$ acting imprimitively on the vertices with the two biparts forming an invariant vertex partition. If a subgroup $G$ is vertex transitive then, by the Embedding Theorem [1, Theorem 8.5] for imprimitive groups, we may assume that $G \leq H \wr S_2$ with $G \cap (H \times H)$ projecting onto $H$ in each component. In this case the group $H \trianglelefteq S_n$ is called the component of $G$.

**Proposition 5.2.** Let $\Delta_1$ and $\Delta_2$ be biparts of $\Sigma = K_{n,n}$ with $n \geq 2$. Then the graph $S(\Sigma)$ is locally $(G, 4)$-distance transitive if and only if (i) $G$ is transitive on $V(\Sigma)$ with $G \leq H \wr S_2$ having component $H$, (ii) $H$ is 2-transitive on each $\Delta_i$ and, for $u_1 \in \Delta_1$, $G_{u_1}$ is transitive on $(\Delta_1 \setminus \{u_1\}) \times \Delta_2$, and (iii) for $u_1 \in \Delta_1$ and $u_2 \in \Delta_2$, the stabiliser $G_{\{u_1,u_2\}}$ interchanges $u_1$ and $u_2$, and is transitive on $\{(v_1, v_2) \mid v_i \in \Delta_i \setminus \{u_i\}\}$. In particular, $S(\Sigma)$ is locally $G$-distance transitive if and only if $G$ satisfies these three conditions.

**Proof.** Let $\Gamma' = S(\Sigma)$. It is not difficult to show that conditions (i)–(iii) together ensure that $\Gamma'$ is locally $(G, 4)$-distance transitive. So suppose conversely that $\Sigma$ is such that $\Gamma'$ is locally $(G, 4)$-distance transitive. By Lemma 4.1(a), $G$ is transitive on $V(\Sigma)$, so we may assume that $G \leq H \wr S_2$ with component $H$ and $H$ acts transitively on each $\Delta_i$. Let $u_1 \in \Delta_1$ and $u_2 \in \Delta_2$, and $e = (u_1, u_2)$. Since $G_{u_1}$ is transitive on $\Gamma_2(u_1) = \Delta_2$ and $\Gamma_4(u_1) = \Delta_1 \setminus \{u_1\}$, and these sets have coprime sizes, it follows that $G_{u_1}$ is transitive on $(\Delta_1 \setminus \{u_1\}) \times \Delta_2$. In particular $H$ is 2-transitive on each $\Delta_i$. Also, since $G_e$ is transitive on $\Gamma_1(e)$ and $\Gamma_4(e)$, it follows that $G_{\{u_1,u_2\}}$ interchanges $u_1$ and $u_2$, and is transitive on $\{(v_1, v_2) \mid v_i \in \Delta_i \setminus \{u_i\}\}$. The final assertion holds since $\text{diam}(\Gamma') = 4$. □

5.3. $\Sigma = C_n$ for $n \geq 3$

Here we consider $C_n$ in general instead of $C_5$ (which is in Table 1) because we will need it in the proof of Theorem 1.3(a). We have $\text{diam}(\Sigma) = \lceil \frac{n}{2} \rceil$, $\text{diam}(S(\Sigma)) = n = \text{girth}(\Sigma)$ (so $\delta = 0$ if $n$ is even and $\delta = 1$ if $n$ is odd), and $\text{Aut}(\Sigma) = D_{2n}$.

**Proposition 5.3.** The graph $S(\Sigma)$ is locally $(G, s)$-distance transitive for some $s \leq n$ if and only if $G = D_{2n}$ and if and only if $S(\Sigma)$ is locally $G$-distance transitive.

5.4. $\Sigma = P$, the Petersen graph

Here $\text{diam}(\Sigma) = 2$, $\text{diam}(S(\Sigma)) = 6$ and $\text{Aut}(\Sigma) = S_5$.

**Proposition 5.4.** The graph $S(\Sigma)$ is locally $(G, s)$-distance transitive, for $s = 4$ or for $s = 5$, if and only if $G = S_5$. Moreover, $S(\Sigma)$ is locally $S_5$-distance transitive.

**Proof.** Let $\Gamma' = S(\Sigma)$ and suppose that $\Gamma'$ is locally $(G, 4)$-distance transitive. Then by Lemma 4.1(a), $G$ is transitive on $V(\Sigma)$. Since $|V(\Sigma)| = 10$, $|\Gamma_4(v)| = 4$ and $|\Gamma_4(e)| = 8$, for $v \in V(\Sigma)$ and $e \in E(\Sigma)$, we have that $120 | |G|$, and so $G = S_5$. Conversely, let $G = S_5$. Then we have $G_e = S_2 \times S_2, G_e = D_8$ and we can easily check that $S(\Sigma)$ is locally $G$-distance transitive. □

5.5. $\Sigma = \text{HoSi}$, the Hoffman–Singleton graph

The Hoffman–Singleton graph is a regular graph of valency $k = 7$ and diameter $2$ with the largest possible number $k^2 + 1$ of vertices, see [9]. Here, $\text{diam}(\Sigma) = 2$, $\text{diam}(S(\Sigma)) = 6$ and $\text{Aut}(\Sigma) = H.2$, where $H = \text{PSU}(3, 5)$.

**Proposition 5.5.** The graph $S(\Sigma)$ is locally $(G, s)$-distance transitive, for $s = 4$ or for $s = 5$, if and only if $G = H$ or $H.2$. Moreover, $S(\Sigma)$ is locally $H$-distance transitive and locally $H.2$-distance transitive.

**Proof.** Let $\Gamma' = S(\Sigma)$ and suppose that $\Gamma'$ is $(G, 4)$-distance transitive. Then by Lemma 4.1(a), $G$ is transitive on $V(\Sigma)$. Since $|V(\Sigma)| = 50, |\Gamma_1(v)| = 7$ and $|\Gamma_4(e)| = 72, for v \in V(\Sigma)$ and $e \in E(\Sigma)$, we have that $2^3 \cdot 3^2 \cdot 5^2 \cdot 7 = |H|/10$ divides $|G|$, and hence $G = H$ or $H.2$ by [4, p. 34]. On the other hand, for $G = H$, it follows from [4, p. 34] that $G_e = A_7$ and $G_e = M_{10}$, and so we can easily prove that $\Gamma'$ is locally $X$-distance transitive for $X = H$ or $H.2$. □

6. Proofs of Theorem 1.3 and Corollary 1.4

**Proof of Theorem 1.3(a).** Suppose that $2d \leq s \leq 2d + \delta$ and that $s \geq 15 + \delta$. Then $d \geq 8$ and so $s \geq \max\{16, 15 + \delta\}$. Let $\Gamma' = S(\Sigma)$. Suppose first that $\Gamma' = S(\Sigma)$ is locally $(G, s)$-distance transitive. Set $s' := s - \delta - 1$. Then $s' \leq 2d - 1$ and $\Gamma' = S(\Sigma)$ is locally $(G, s')$-distance transitive. By Theorem 1.2, $\Sigma$ is $(G, t)$-arc transitive, where $t = \lceil \frac{s' + 1}{2} \rceil$. If $s = 16$ then
16 \geq 15 + \delta \text{ so } \delta \leq 1 \text{ and } s' = s - \delta - 1 \geq 14. \text{ On the other hand if } s \geq 17, \text{ then } s' = s - \delta - 1 \geq 14. \text{ Thus in both cases } s' \geq 14 \text{ and hence } t \geq 8. \text{ By Weiss' Theorem [15], it follows that } \Sigma = C_n \text{ for some } n. \text{ Since } S(\Sigma) \text{ is locally } (G, s)\text{-distance transitive, } G = D_{2n} \text{ by Proposition 5.3. If } n \text{ is even, then } \delta = 0, \text{ and so } s = 2d = n, \text{ whence } \Sigma = C_n. \text{ If } n \text{ is odd, then } \delta = 1, \text{ and so } s = 2d = n - 1 \text{ or } s = 2d + 1 = n, \text{ giving } \Sigma = C_{n+1} \text{ or } C_n \text{ respectively. Notice that in the former case, } s \text{ is even.}

Conversely, suppose that } \Sigma = C_n \text{ and } G = D_{2s} \text{ or } s \text{ is even, } \Sigma = C_{s+1} \text{ and } G = D_{2s+2}. \text{ Then } S(\Sigma) \text{ is locally } G\text{-distance transitive by Proposition 5.3, and since } \text{diam}(S(\Sigma)) \geq s, \text{ we have that } S(\Sigma) \text{ is locally } (G, s)\text{-distance transitive. Notice that } s \geq 16, \text{ as } \delta = 1 \text{ for } \Sigma = C_s \text{ with } s \text{ odd.} \square

**Proof of Theorem 1.3(b).** Suppose that Hypothesis 1 holds and that } 2d \geq s \leq 2d + \delta \text{ and } s \leq 5. \text{ In particular } d \leq 2. \text{ Suppose first that } d = 1. \text{ Then } \Sigma = K_n \text{ for some } n \geq 2. \text{ If } n = 2 \text{ or } n = 3 \text{ then the assertions of Theorem 1.3 and the entries in Table 1 hold, see Section 5.1. If } n \geq 4, \text{ then these hold by Proposition 5.1.}

Suppose now that } d = 2, \text{ so } 4 \leq s \leq 4 + \delta \text{ and by assumption } s \leq 5. \text{ Let } G' = S(\Sigma). \text{ Suppose first that } G' \text{ is locally } (G, s)\text{-distance transitive. Then in particular } G' \text{ is locally } (G, 3)\text{-distance transitive, and so by Theorem 1.2, } \Sigma \text{ is } (G, 2)\text{-arc transitive. Note that the girth of } \Sigma \text{ is at most 5 (since } d = 2) \text{ and at least 4 (since } \Sigma \text{ is } (G, 2)\text{-arc transitive and } d = 1, \text{ see Remark 4.3).}

We now prove that if girth(\Sigma) = 4 \text{ then } \Sigma = K_{n,n}, \text{ for some } n \geq 2. \text{ Let } (u_0, u_1, u_2, u_3) \text{ be a 4-cycle in } \Sigma \text{ and set } e_1 = \{u_0, u_1, u_2, u_3\} \text{ for } 0 \leq i \leq 2. \text{ Then } \Sigma = \{e_0, e_1, ... e_3\} \text{ is a 4-arc in } G', \text{ and } d_{G'}(e_0, e_2) = 4 \text{ since } e_0 \cap e_2 = \emptyset. \text{ Thus } I'_{\Sigma}(e_0) = \emptyset. \text{ For every edge } e'_i = \{u_1, u_2\} \text{ containing } u_1, d_{\Sigma}(u_1, u'_2) = 2 \text{ since girth}(\Sigma) > 3. \text{ For the same reason, for every edge } e'_2 = \{u'_1, u'_3\} \text{ containing } u'_2, e'_2 \neq e'_1, \text{ we have } d_{\Sigma}(u'_2, u'_1) = 2 \text{ and } u'_2 \neq u_0, \text{ so } e'_2 \neq e'_1 \text{ and } \Sigma = \{u'_1, u'_2\} = \emptyset. \text{ Thus } e'_2, e'_2 \in I'_{\Sigma}(e'_0) \text{ and so } e'_3 = e'_3 \text{ for some } a = \emptyset. \text{ This implies that } u'_1 \in \Sigma_1(u'_0) \cap \Sigma_2(u'_1) \cap e'_3. \text{ If } u_0 \text{ does not fix } u_1 \text{ then we conclude that } u'_0 = u'_2 \in \Sigma_2(u_1) \text{ and hence } u'_0 \in \Sigma_1(u_0). \text{ On the other hand, if } a \text{ does not fix } u_0 \text{ then it must interchange } u_0 \text{ and } u_1. \text{ Thus } u'_0 \text{ is adjacent to } u'_1 \text{ in } \Sigma. \text{ Thus } u'_1 \in \Sigma_1(u'_0) \cap \Sigma_2(u'_1) \text{ for each } u'_0 \in \Sigma_2(u_0). \text{ It follows that } \Sigma_1(u'_0) = \Sigma_1(u_0) \text{ since } u_0 \text{ and } u'_0 \text{ have the same valency, and hence that } \Sigma \text{ is a complete bipartite graph } K_{n,m} \text{ for some } n, m \geq 2. \text{ Since } G \text{ is transitive on } V(\Sigma), n = m. \text{ It now follows, from Propositions 5.3–5.5, that the assertions of Theorem 1.3 and the entries in Table 1 hold in the case where girth(\Sigma) = 5. This completes the proof.} \square

**Proof of Corollary 1.4.** Suppose that } \Sigma \text{ is a connected graph of diameter } d \leq 2 \text{ such that } |V(\Sigma)| \geq 2 \text{ and } G \leq \text{Aut}(\Sigma). \text{ Thus } \text{diam}(S(\Sigma)) = 2d + \delta \leq 6. \text{ Suppose that } S(\Sigma) \text{ is locally } G\text{-distance transitive. If } 2d + \delta \leq 5 \text{ then, applying Theorem 1.3(b) with } s = 2d + \delta, \text{ we find that } \Sigma, G \text{ are as in Table 1. Thus we may assume that } \text{diam}(S(\Sigma)) = 2d + \delta = 6, \text{ so } (d, \delta) = (2, 2). \text{ Then Theorem 1.3(b) applies with } s = 5, \text{ again yielding the graphs in Table 1. The local } G\text{-distance transitivity of the graphs in Table 1 follows from the properties proved in Section 5 about these graphs.} \square

**Acknowledgements**

The first author thanks Bu-Ali Sina University and the University of Western Australia for support during her Sabbatical leave. The authors thank the anonymous referees for helpful suggestions with the exposition. The third author is supported by the Australian Research Council Federation Fellowship FF0776186. This paper forms part of the work from the Research Council Federation Fellowship project.

**References**


