ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 338 (2008) 299–316
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On interval-valued nonlinear programming problems

Hsien-Chung Wu

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

Received 5 February 2007
Available online 18 May 2007
Submitted by Goong Chen

Abstract

The Wolfe's duality theorems in interval-valued optimization problems are derived in this paper. Four kinds of interval-valued optimization problems are formulated. The Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problems are derived for the purpose of proving the strong duality theorems. The concept of having no duality gap in weak and strong sense are also introduced, and the strong duality theorems in weak and strong sense are then derived naturally.

Keywords: Karush–Kuhn–Tucker optimality conditions; Wolfe's primal and dual problems; Duality theorems

1. Introduction

The methodology for solving optimization problems has widely applied to many research fields. If the coefficients of optimization problems are taken as real numbers, they are categorized as the deterministic optimization problems. However, the coefficients can be taken as uncertain quantities. If the coefficients of optimization problem are assumed as random variables with known distributions, they are categorized as the stochastic optimization problems. The books written by Birge and Louveaux [3], Kall [5], Prékopa [11], Stancu-Minasian [15] and Vajda [17] give the main stream of this topic, and also give many useful techniques for solving the stochastic optimization problems. On the other hand, if the coefficients are taken as closed intervals, they will be categorized as interval-valued optimization problems.

As we have known in the stochastic optimization problems, the coefficients are assumed as random variables with known distributions in most of cases. However, the specifications of the distributions are very subjective. For example, many researchers invoke the Gaussian (normal) distributions with different parameters in the stochastic optimization problems. These specifications may not perfectly match the real problems. Therefore, interval-valued optimization problems may provide an alternative choice for considering the uncertainty into the optimization problems. That is to say, the coefficients in the interval-valued optimization problems are assumed as closed intervals. Although the specifications of closed intervals may still be judged as subjective viewpoint, we might argue that the bounds of uncertain data (i.e., determining the closed intervals to bound the possible observed data) are easier to be handled than specifying the Gaussian distributions in stochastic optimization problems.

E-mail address: hcwu@nknucc.nknu.edu.tw.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.05.023

The duality theory for inexact linear programming problem was proposed by Soyster [12–14] and Thuente [16]. Falk [4] provided some properties on this problem. However, Pomerol [10] pointed out some drawbacks of Soyster's results and also provided some mild conditions to improve Soyster's results. The interval-valued optimization problem proposed in this paper is closely related with the inexact linear programming problem. The main difference between those two problems are the solutions concepts imposed upon the objective functions. The solution concept in the inexact linear programming problem used the conventional solution concept in the scalar linear programming problems. The solution concept of interval-valued optimization problems proposed in this paper follows from the nondominated solution concept employed in multiobjective programming problems. Moreover, the interval-valued objective function and interval-valued constraint functions are considered as nonlinear-type rather than linear-type in this paper.

Four kinds of interval-valued optimization problems are formulated in this paper. The solution concept for primal and dual problems is proposed by following the nondominated solution concept employed in multiobjective programming problem to interpret the optimality of primal and dual problems. In order to prove the strong duality theorems, we derive the Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problems. We also introduce the concept of having no duality gap in weak and strong sense. Finally, we derive the strong duality theorems in weak and strong sense.

In Section 2, we introduce some basic properties for closed intervals. In Section 3, four kinds of interval-valued optimization problems are formulated. In Section 4, we formulate the Wolfe's primal and dual pair problems. In Section 5, we derive the Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problems which will be used to prove the strong duality theorems. In Section 6, we discuss the solvability for Wolfe's primal and dual problems. In Section 7, the duality theorems in weak and strong sense are derived.

2. Interval analysis

Let us denote by $\mathcal{I}$ the class of all closed intervals in $\mathbb{R}$. If C is a closed interval, we also adopt the notation $C = [c^L, c^U]$, where c^L and c^U mean the lower and upper bounds of C, respectively. Let $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be in $\mathcal{I}$. Then, by definition, we have

(i) $C + D = \{c + d\colon\ c \in C \text{ and } d \in D\} = [c^L + d^L, c^U + d^U]$;
(ii) $-C = \{-c\colon\ c \in C\} = [-c^U, -c^L]$.

Therefore, we see that $C - D = C + (-D) = [c^L - d^U, c^U - d^L]$. We also see that

$$kC = \{kc\colon\ c \in C\} = \begin{cases} [kc^L, kc^U] & \text{if } k \geqslant 0, \\ [kc^U, kc^L] & \text{if } k < 0, \end{cases}$$

where k is a real number.

Let $C = [c^L, c^U]$ be a closed interval. We also write $C_L = c^L$ and $C_U = c^U$. We see that if $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be closed intervals, then

$$(C + D)_L = c^L + d^L = C_L + D_L \quad \text{and} \quad (C + D)_U = c^U + d^U = C_U + D_U. \tag{1}$$

The function $F : \mathbb{R}^n \to \mathcal{I}$ defined on the Euclidean space $\mathbb{R}^n$ is called an interval-valued function, i.e., $F(\mathbf{x}) = F(x_1, \ldots, x_n)$ is a closed interval in $\mathbb{R}$ for each $\mathbf{x} \in \mathbb{R}^n$. The interval-valued function F can also be written as $F(\mathbf{x}) = [F_L(\mathbf{x}), F_U(\mathbf{x})]$, where F_L and F_U are real-valued functions defined on $\mathbb{R}^n$ and satisfy $F_L(\mathbf{x}) \leqslant F_U(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$. We also see that $(F(\mathbf{x}))_L = F_L(\mathbf{x})$ and $(F(\mathbf{x}))_U = F_U(\mathbf{x})$. We say that the interval-valued function F is *differentiable* at $\mathbf{x}_0 \in \mathbb{R}^n$ if and only if the real-valued functions F_L and F_U are differentiable at $\mathbf{x}_0$. For more details on the topic of interval analysis, we refer to Moore [6,7] and Alefeld and Herzberger [1].

3. Formulation of interval-valued optimization problems

Let $\{K_j\}_{j=1}^n$ and K be $n + 1$ nonempty convex subsets of $\mathbb{R}^m$. We consider the following set:

$$X = \big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon\ x_1K_1 + \cdots + x_nK_n \subseteq K\big\}.$$

Soyster [12] has shown that X is also a convex subset of $\mathbb{R}^n$. Let $\mathbf{b} \in \mathbb{R}^m$ and

$$K(\mathbf{b}) = \{\mathbf{k} \in \mathbb{R}^m: \mathbf{k} \leqslant \mathbf{b}\},$$

where $\mathbf{k} \leqslant \mathbf{b}$ means that $k_i \leqslant b_i$ for all $i = 1, \ldots, m$. Then $K(\mathbf{b})$ is also a convex subset of $\mathbb{R}^m$. Therefore, we can consider the following set:

$$X(\mathbf{b}) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n: x_1 K_1 + \cdots + x_n K_n \subseteq K(\mathbf{b})\}. \tag{2}$$

The support functional $s_K : \mathbb{R}^m \to \mathbb{R}$ of a convex set K is defined by, for $\mathbf{y} \in \mathbb{R}^m$,

$$s_K(\mathbf{y}) = \sup_{\mathbf{k} \in K} \mathbf{y}^T \mathbf{k},$$

where $\mathbf{y}^T$ means the transpose of $\mathbf{y}$. Let $\{K_j\}_{j=1}^n$ be n nonempty convex subsets of $\mathbb{R}^m$. We consider the column vector $\bar{\mathbf{a}}_j$ with the ith component defined by $\bar{a}_{ij} = s_{K_j}(\mathbf{e}_i)$, where $\mathbf{e}_i$ is the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position. Then we see that

$$\bar{a}_{ij} = s_{K_j}(\mathbf{e}_i) = \sup_{\mathbf{k}_j \in K_j} \mathbf{e}_i^T \mathbf{k}_j = \sup_{\mathbf{k}_j \in K_j} k_{ij},$$

where $\mathbf{k}_j = (k_{1j}, \ldots, k_{ij}, \ldots, k_{mj})$. We further assume that $\{K_j\}_{j=1}^n$ are compact subsets in $\mathbb{R}^m$ (i.e., closed and bounded in $\mathbb{R}^m$). Then $\bar{a}_{ij} = \sup_{\mathbf{k}_j \in K_j} \mathbf{e}_i^T \mathbf{k}_j < \infty$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $\bar{A}$ be an $m \times n$ matrix comprises the column vectors $\bar{\mathbf{a}}_j$ for $j = 1, \ldots, n$, i.e., $\bar{A} = (\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_n)$. Now we consider the following set:

$$\bar{X}(\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n: \bar{A}\mathbf{x} \leqslant \mathbf{b} \text{ and } \mathbf{x} \geqslant \mathbf{0}\}. \tag{3}$$

Proposition 3.1. *(See Soyster [12].) The sets $X(\mathbf{b})$ and $\overline{X}(\mathbf{b})$ described in* (2) *and* (3)*, respectively, are identical.*

Now we consider the following interval-valued optimization problem:

$$\text{(IVP1)} \quad \min \quad F(\mathbf{x}) = [F_L(\mathbf{x}), F_U(\mathbf{x})]$$
$$\text{subject to} \quad x_1 K_1 + \cdots + x_n K_n \subseteq K(\mathbf{b}) \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}.$$

Then $\mathbf{x} = (x_1, \ldots, x_n)$ is called a feasible solution of problem (IVP1) if and only if $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \in K(\mathbf{b})$ for all possible $\mathbf{a}_j \in K_j$, $j = 1, \ldots, n$. We also consider the auxiliary interval-valued optimization problem of (IVP1) as follows:

$$\text{(IVP2)} \quad \min \quad F(\mathbf{x}) = [F_L(\mathbf{x}), F_U(\mathbf{x})]$$
$$\text{subject to} \quad \bar{A}\mathbf{x} \leqslant \mathbf{b} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}.$$

Then we have the following useful result.

Proposition 3.2. *Suppose that problems* (IVP1) *and* (IVP2) *use the same solution concept. Then* (IVP1) *and* (IVP2) *have the same optimal solutions.*

Proof. We see that the feasible sets of problems (IVP1) and (IVP2) are $X(\mathbf{b})$ and $\overline{X}(\mathbf{b})$, respectively. Since problems (IVP1) and (IVP2) have the same objective functions, the result follows from Proposition 3.1 immediately. □

The above proposition shows that the solution of problem (IVP1) can be obtained from the solution of an easier problem (IVP2), since the feasible set of problem (IVP2) is easier to be handled than that of problem (IVP1).

Now let us consider the following minimization problem with interval-valued coefficients:

$$\text{(IVP3)} \quad \min \quad F(\mathbf{x}) = [F_L(\mathbf{x}), F_U(\mathbf{x})]$$
$$\text{subject to} \quad x_1[k_{11}^L, k_{11}^U] + x_2[k_{12}^L, k_{12}^U] + \cdots + x_n[k_{1n}^L, k_{1n}^U] \subseteq [b_1^L, b_1^U],$$
$$x_1[k_{21}^L, k_{21}^U] + x_2[k_{22}^L, k_{22}^U] + \cdots + x_n[k_{2n}^L, k_{2n}^U] \subseteq [b_2^L, b_2^U],$$

$$\vdots$$

$$x_1\big[k_{m1}^L, k_{m1}^U\big] + x_2\big[k_{m2}^L, k_{m2}^U\big] + \cdots + x_n\big[k_{mn}^L, k_{mn}^U\big] \subseteq \big[b_m^L, b_m^U\big],$$

$$x_j \geqslant 0, \quad j = 1, \ldots, n.$$

Then we say that $\mathbf{x} = (x_1, \ldots, x_n)$ is a feasible solution of problem (IVP3) if and only if $x_1k_{i1} + \cdots + x_jk_{ij} + \cdots + x_nk_{in} \in [b_i^L, b_i^U]$ for all possible $k_{ij} \in [k_{ij}^L, k_{ij}^U]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. In other words, $\mathbf{x} = (x_1, \ldots, x_n)$ is a feasible solution of problem (IVP3) if and only if

$$b_i^L \leqslant \sum_{j=1}^n x_jk_{ij} \leqslant b_i^U \tag{4}$$

for all possible $k_{ij} \in [k_{ij}^L, k_{ij}^U]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We adopt the notations $\mathbf{b}^L = (b_1^L, \ldots, b_m^L)$ and $\mathbf{b}^U = (b_1^L, \ldots, b_m^U)$. Let

$$K\big(\mathbf{b}^L, \mathbf{b}^U\big) = \big\{\mathbf{k} = (k_1, \ldots, k_m)\colon\ k_i \in \big[b_i^L, b_i^U\big],\ i = 1, \ldots, m\big\},$$

and, for $j = 1, \ldots, n$,

$$K_j = \big\{\mathbf{k}_j = (k_{1j}, \ldots, k_{ij}, \ldots, k_{mj})\colon\ k_{ij} \in \big[k_{ij}^L, k_{ij}^U\big],\ i = 1, \ldots, m\big\}.$$

Then it is easy to see that $\{K_j\}_{j=1}^n$ and $K(\mathbf{b}^L, \mathbf{b}^U)$ are compact and convex subsets of $\mathbb{R}^m$. Now let A^L and A^U be two $m \times n$ matrices comprise the column vectors $\mathbf{a}_j^L$ and $\mathbf{a}_j^U$ for $j = 1, \ldots, n$, respectively, where $\mathbf{a}_j^L$ and $\mathbf{a}_j^U$ are given by

$$a_{ij}^L = \inf_{\mathbf{k}_j \in K_j} \mathbf{e}_i^T\mathbf{k}_j = \inf_{\mathbf{k}_j \in K_j} k_{ij} = k_{ij}^L \quad \text{and} \quad a_{ij}^U = \sup_{\mathbf{k}_j \in K_j} \mathbf{e}_i^T\mathbf{k}_j = \sup_{\mathbf{k}_j \in K_j} k_{ij} = k_{ij}^U. \tag{5}$$

We consider the augmented matrix $\hat{A} = [A^U, -A^L]$ and vector $\hat{\mathbf{b}} = [\mathbf{b}^U, -\mathbf{b}^L]$. Then we consider the following auxiliary interval-valued optimization problem of (IVP3):

$$\begin{array}{lll}\text{(IVP4)} & \min & F(\mathbf{x}) = \big[F_L(\mathbf{x}), F_U(\mathbf{x})\big] \\ & \text{subject to} & \hat{A}\mathbf{x} \leqslant \hat{\mathbf{b}} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}.\end{array}$$

Proposition 3.3. *The feasible set of problem* (IVP4) *can be rewritten as*

$$\big\{\mathbf{x}\colon\ A^U\mathbf{x} \leqslant \mathbf{b}^U,\ -A^L\mathbf{x} \leqslant -\mathbf{b}^L \text{ and } \mathbf{x} \geqslant \mathbf{0}\big\} = \big\{\mathbf{x}\colon\ A^U\mathbf{x} \leqslant \mathbf{b}^U,\ A^L\mathbf{x} \geqslant \mathbf{b}^L \text{ and } \mathbf{x} \geqslant \mathbf{0}\big\}.$$

Next we are going to show that the feasible sets of problems (IVP3) and (IVP4) are identical with each other.

Proposition 3.4. *Let X_1 and X_2 be the feasible sets of problems* (IVP3) *and* (IVP4)*, respectively. Then $X_1 = X_2$.*

Proof. Let $\mathcal{M}$ be the set of matrices given by $\mathcal{M} = \{A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)\colon\ \mathbf{a}_j \in K_j,\ j = 1, \ldots, n\}$. Then it is obvious that $A^L \leqslant A \leqslant A^U$ for all $A \in \mathcal{M}$ from (5). Let $\bar{\mathbf{x}} \in X_2$, i.e., $\bar{\mathbf{x}}$ is a feasible solution of problem (IVP4). Then, from Proposition 3.3, we see that $A^U\bar{\mathbf{x}} \leqslant \mathbf{b}^U$ and $A^L\bar{\mathbf{x}} \geqslant \mathbf{b}^L$. Since $A^L \leqslant A \leqslant A^U$ for all $A \in \mathcal{M}$ and $\bar{\mathbf{x}} \geqslant \mathbf{0}$, we have $\mathbf{b}^L \leqslant A^L\bar{\mathbf{x}} \leqslant A\bar{\mathbf{x}} \leqslant A^U\bar{\mathbf{x}} \leqslant \mathbf{b}^U$ for all $A \in \mathcal{M}$, which shows that $\bar{\mathbf{x}}$ is a feasible solution of problem (IVP3) from expression (4), i.e., $X_2 \subseteq X_1$. Conversely, if $\bar{\mathbf{x}}$ is a feasible solution of problem (IVP3), then

$$b_i^L \leqslant \sum_{j=1}^n \bar{x}_j \cdot \inf_{\mathbf{k}_j \in K_j} k_{ij} \quad \text{and} \quad \sum_{j=1}^n \bar{x}_j \cdot \sup_{\mathbf{k}_j \in K_j} k_{ij} \leqslant b_i^U$$

since (4) is satisfied for all possible $k_{ij} \in [k_{ij}^L, k_{ij}^U]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Equivalently, from (5), we have

$$b_i^L \leqslant \sum_{j=1}^n \bar{x}_ja_{ij}^L \quad \text{and} \quad \sum_{j=1}^n \bar{x}_ja_{ij}^U \leqslant b_i^U,$$

which shows that $\bar{\mathbf{x}}$ is a feasible solution of problem (IVP4) from Proposition 3.3. This completes the proof. □

The above proposition also shows the following useful result.

Proposition 3.5. *Suppose that problems* (IVP3) *and* (IVP4) *use the same solution concept. Then* (IVP3) *and* (IVP4) *have the same optimal solutions.*

Let us say that the constraints discussed in the above problems are linear-type constraints with interval-valued coefficients. Next we are going to discuss the nonlinear-type constraints with interval-valued coefficients. Suppose that we consider the following simple problem:

$$\begin{aligned} \text{(IVP5)} \quad \min \quad & F(x_1,x_2,x_3)=\big[F_L(x_1,x_2,x_3), F_U(x_1,x_2,x_3)\big] \\ \text{subject to} \quad & x_1^2\big[k_{11}^L,k_{11}^U\big]+x_1x_2\big[k_{12}^L,k_{12}^U\big]+x_2^3x_3^2\big[k_{13}^L,k_{13}^U\big]\subseteq\big[b_1^L,b_1^U\big], \\ & x_1x_3\big[k_{21}^L,k_{21}^U\big]+x_2x_3\big[k_{22}^L,k_{22}^U\big]+x_1x_3\big[k_{23}^L,k_{23}^U\big]\subseteq\big[b_2^L,b_2^U\big], \\ & x_1x_2x_3\big[k_{31}^L,k_{31}^U\big]+x_2\big[k_{32}^L,k_{32}^U\big]+x_3\big[k_{33}^L,k_{33}^U\big]\subseteq\big[b_3^L,b_3^U\big], \\ & x_1,x_2,x_3\geqslant 0. \end{aligned}$$

Let $\mathbf{k}_1=(k_{11},k_{12},k_{13})$ for some $k_{11}\in[k_{11}^L,k_{11}^U]$, $k_{12}\in[k_{12}^L,k_{12}^U]$ and $k_{13}\in[k_{13}^L,k_{13}^U]$. For the first constraint, we can form a constraint function

$$G_{\mathbf{k}_1}(x_1,x_2,x_3)=k_{11}\cdot x_1^2+k_{12}\cdot x_1x_2+k_{13}\cdot x_2^3x_3^2. \tag{6}$$

Similarly, let $\mathbf{k}_2=(k_{21},k_{22},k_{23})$ and $\mathbf{k}_3=(k_{31},k_{32},k_{33})$. For the second and third constraints, we can also form two corresponding constraint functions $G_{\mathbf{k}_2}$ and $G_{\mathbf{k}_3}$. Then $\mathbf{x}=(x_1,x_2,x_3)$ is the feasible solution of problem (IVP5) if and only if $G_{\mathbf{k}_i}(x_1,x_2,x_3)\in[b_i^L,b_i^U]$ (where $\mathbf{k}_i=(k_{i1},k_{i2},k_{i3})$) for all possible $k_{ij}\in[k_{ij}^L,k_{ij}^U]$, $i=1,\dots,3$ and $j=1,\dots,3$. Also, the first constraint can be written as

$$\begin{aligned} G_1(x_1,x_2,x_3)&=x_1^2\big[k_{11}^L,k_{11}^U\big]+x_1x_2\big[k_{12}^L,k_{12}^U\big]+x_2^3x_3^2\big[k_{13}^L,k_{13}^U\big] \\ &=\big[k_{11}^L\cdot x_1^2+k_{12}^L\cdot x_1x_2+k_{13}^L\cdot x_2^3x_3^2, k_{11}^U\cdot x_1^2+k_{12}^U\cdot x_1x_2+k_{13}^U\cdot x_2^3x_3^2\big] \\ &\subseteq\big[b_1^L,b_1^U\big], \end{aligned}$$

i.e., $G_1(x_1,x_2,x_3)=[G_1^L(x_1,x_2,x_3), G_1^U(x_1,x_2,x_3)]$ is an interval-valued constraint function with

$$\begin{aligned} G_1^L(x_1,x_2,x_3)&=k_{11}^L\cdot x_1^2+k_{12}^L\cdot x_1x_2+k_{13}^L\cdot x_2^3x_3^2=G_{\mathbf{k}_1^L}(x_1,x_2,x_3), \\ G_1^U(x_1,x_2,x_3)&=k_{11}^U\cdot x_1^2+k_{12}^U\cdot x_1x_2+k_{13}^U\cdot x_2^3x_3^2=G_{\mathbf{k}_1^U}(x_1,x_2,x_3), \end{aligned}$$

where $\mathbf{k}_1^L=(k_{11}^L,k_{12}^L,k_{13}^L)$ and $\mathbf{k}_1^U=(k_{11}^U,k_{12}^U,k_{13}^U)$ by referring to (6). Therefore, the three original constraints in problem (IVP5) can be written as $G_i(x_1,x_2,x_3)\subseteq[b_i^L,b_i^U]$ for $i=1,2,3$, where $G_i(x_1,x_2,x_3)=[G_i^L(x_1,x_2,x_3), G_i^U(x_1,x_2,x_3)]$ are interval-valued constraint functions for $i=1,2,3$. Let $\mathbf{k}_i^L=(k_{i1}^L,k_{i2}^L,k_{i3}^L)$ and $\mathbf{k}_i^U=(k_{i1}^U,k_{i2}^U,k_{i3}^U)$ for $i=1,2,3$. Then it is not hard to see that $G_i^L(x_1,x_2,x_3)=G_{\mathbf{k}_i^L}(x_1,x_2,x_3)$ and $G_i^U(x_1,x_2,x_3)=G_{\mathbf{k}_i^U}(x_1,x_2,x_3)$ for $i=1,2,3$, since $x_1,x_2,x_3\geqslant 0$.

According to the above convenient notations, we consider the following general problem:

$$\begin{aligned} \text{(IVP6)} \quad \min \quad & F(\mathbf{x})=\big[F_L(\mathbf{x}), F_U(\mathbf{x})\big] \\ \text{subject to} \quad & G_i(\mathbf{x})\subseteq\big[b_i^L,b_i^U\big], \quad i=1,\dots,m, \\ & \mathbf{x}\geqslant\mathbf{0}, \end{aligned}$$

where $G_i(\mathbf{x})=[G_i^L(\mathbf{x}), G_i^U(\mathbf{x})]$ are interval-valued constraint functions for $i=1,\dots,m$. Then $\mathbf{x}=(x_1,\dots,x_n)$ is the feasible solution of problem (IVP6) if and only if $G_{\mathbf{k}_i}(x_1,\dots,x_n)\in[b_i^L,b_i^U]$ (where $\mathbf{k}_i=(k_{i1},k_{i2},\dots,k_{ir_i})$) for all possible $k_{ij}\in[k_{ij}^L,k_{ij}^U]$, $i=1,\dots,m$ and $j=1,\dots,r_i$. Let $\mathbf{k}_i^L=(k_{i1}^L,k_{i2}^L,\dots,k_{ir_i}^L)$ and $\mathbf{k}_i^U=(k_{i1}^U,k_{i2}^U,\dots,k_{ir_i}^U)$ for $i=1,\dots,m$. Then we can consider its auxiliary interval-valued optimization problem described as follows:

$$
\begin{aligned}
\text{(IVP7)}\quad & \min && F(\mathbf{x}) = \big[F_L(\mathbf{x}), F_U(\mathbf{x})\big] \\
& \text{subject to} && G_i^L(\mathbf{x}) = G_{\mathbf{k}_i^L}(\mathbf{x}) \geqslant b_i^L, \quad i = 1, \ldots, m, \\
& && G_i^U(\mathbf{x}) = G_{\mathbf{k}_i^U}(\mathbf{x}) \leqslant b_i^U, \quad i = 1, \ldots, m, \\
& && \mathbf{x} \geqslant \mathbf{0}.
\end{aligned}
$$

Since $\mathbf{x} \geqslant \mathbf{0}$, we see that

$$
b_i^L \leqslant G_i^L(\mathbf{x}) = G_{\mathbf{k}_i^L}(\mathbf{x}) \leqslant G_{\mathbf{k}_i}(\mathbf{x}) \leqslant G_{\mathbf{k}_i^U}(\mathbf{x}) = G_i^U(\mathbf{x}) \leqslant b_i^U
$$

for all possible $k_{ij} \in [k_{ij}^L, k_{ij}^U]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Using the similar arguments of Proposition 3.4, we have the following results.

Proposition 3.6.

(i) *The feasible sets of problems* (IVP6) *and* (IVP7) *are identical with each other.*
(ii) *Suppose that problems* (IVP6) *and* (IVP7) *use the same solution concepts. Then* (IVP6) *and* (IVP7) *have the same optimal solutions.*

Let $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be two closed intervals in $\mathbb{R}$. We write

$$
C \preccurlyeq D \quad \text{if and only if} \quad c^L \leqslant d^L \text{ and } c^U \leqslant d^U.
$$

It means that C is inferior to D, or D is superior to C. It is easy to see that "$\preccurlyeq$" is a partial ordering on $\mathcal{I}$. Now we consider another problem by using the ordering relation "$\preccurlyeq$" for the constraints:

$$
\begin{aligned}
\text{(IVP8)}\quad & \min && F(\mathbf{x}) = \big[F_L(\mathbf{x}), F_U(\mathbf{x})\big] \\
& \text{subject to} && G_i(\mathbf{x}) \preccurlyeq \big[b_i^L, b_i^U\big], \quad i = 1, \ldots, m, \\
& && \mathbf{x} \geqslant \mathbf{0},
\end{aligned}
$$

where $G_i(\mathbf{x}) = [G_i^L(\mathbf{x}), G_i^U(\mathbf{x})]$ are interval-valued constraints functions for $i = 1, \ldots, m$. Then $\mathbf{x} = (x_1, \ldots, x_n)$ is a feasible solution of problem (IVP8) if $G_i(\mathbf{x}) \preccurlyeq [b_i^L, b_i^U]$ for all $i = 1, \ldots, m$; or, equivalently, $G_i^L(\mathbf{x}) = G_{\mathbf{k}_i^L}(\mathbf{x}) \leqslant b_i^L$ and $G_i^U(\mathbf{x}) = G_{\mathbf{k}_i^U}(\mathbf{x}) \leqslant b_i^U$ for all $i = 1, \ldots, m$. Then the auxiliary interval-valued optimization problem of (IVP8) can be taken as follows:

$$
\begin{aligned}
\text{(IVP9)}\quad & \min && F(\mathbf{x}) = \big[F_L(\mathbf{x}), F_U(\mathbf{x})\big] \\
& \text{subject to} && G_i^L(\mathbf{x}) = G_{\mathbf{k}_i^L}(\mathbf{x}) \leqslant b_i^L, \quad i = 1, \ldots, m, \\
& && G_i^U(\mathbf{x}) = G_{\mathbf{k}_i^U}(\mathbf{x}) \leqslant b_i^U, \quad i = 1, \ldots, m, \\
& && \mathbf{x} \geqslant \mathbf{0}.
\end{aligned}
$$

It is obvious that the feasible sets of problems (IVP8) and (IVP9) are identical with each other. Then the following proposition is also obvious.

Proposition 3.7. *Suppose that problems* (IVP8) *and* (IVP9) *use the same solution concept. Then* (IVP8) *and* (IVP9) *have the same optimal solutions.*

In the sequel, we are going to propose the dual problems of the above interval-valued optimization problems (IVP1), (IVP3), (IVP6) and (IVP8), respectively, and derive the duality theorems.

4. The Wolfe's primal and dual problems

First of all, we present the Wolfe's primal and dual pair problems for conventional nonlinear programming problem by referring to Wolfe [18]. Let f and g_i, $i = 1, \ldots, m$, be real-valued functions defined on $\mathbb{R}^n$. Then we consider the following primal minimization problem:

$$\text{(P1)} \quad \min \quad f(\mathbf{x}) = f(x_1, \ldots, x_n)$$
$$\text{subject to} \quad g_i(\mathbf{x}) \leqslant 0, \quad i = 1, \ldots, m,$$
$$\mathbf{x} \geqslant \mathbf{0}.$$

Suppose that f and g_i, $i = 1, \ldots, m$, are differentiable on $\mathbb{R}^n_+$. The dual problem is formulated as follows:

$$\text{(D1)} \quad \max \quad f(\mathbf{x}) + \sum_{i=1}^{m} u_i \cdot g_i(\mathbf{x})$$
$$\text{subject to} \quad \nabla f(\mathbf{x}) + \sum_{i=1}^{m} u_i \cdot \nabla g_i(\mathbf{x}) = \mathbf{0},$$
$$\mathbf{x} \geqslant \mathbf{0},$$
$$\mathbf{u} = (u_1, \ldots, u_m) \geqslant \mathbf{0}.$$

From Propositions 3.2, 3.5–3.7, we see that, in order to propose the dual problems of problems (IVP1), (IVP3), (IVP6) and (IVP8), respectively, it will be enough to propose the dual problems of problems (IVP2), (IVP4), (IVP7) and (IVP9), respectively. We see that the interval-valued optimization problems (IVP2), (IVP4), (IVP7) and (IVP9) have the common form as shown below:

$$\text{(IVP)} \quad \min \quad F(\mathbf{x})$$
$$\text{subject to} \quad g_i(\mathbf{x}) \leqslant 0, \quad i = 1, \ldots, m,$$
$$h_i(\mathbf{x}) \leqslant 0, \quad i = 1, \ldots, m,$$
$$\mathbf{x} \geqslant \mathbf{0},$$

where $F : \mathbb{R}^n \to \mathcal{I}$ is an interval-valued function, and $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are real-valued functions.

Example 4.1. We consider the following problem:

$$\min \quad F(x_1, x_2) = [1, 1]x_1^2 + [1, 1]x_2^2 + [1, 2] = \left[x_1^2 + x_2^2 + 1, x_1^2 + x_2^2 + 2\right]$$
$$\text{subject to} \quad [1, 6]x_1 + [1, 2]x_2 \succcurlyeq [1, 12],$$
$$x_1, x_2 \geqslant 0.$$

Therefore, we have

$$F_L(x_1, x_2) = x_1^2 + x_2^2 + 1 \quad \text{and} \quad F_U(x_1, x_2) = x_1^2 + x_2^2 + 2$$

and

$$g_1(x_1, x_2) = -x_1 - x_2 + 1 \quad \text{and} \quad h_1(x_1, x_2) = -6x_1 - 2x_1 + 12.$$

We denote by

$$X = \left\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{x} \geqslant \mathbf{0},\ g_i(\mathbf{x}) \leqslant 0 \text{ and } h_i(\mathbf{x}) \leqslant 0,\ i = 1, \ldots, m\right\}$$

the feasible set of primal problem (IVP). We also denote by

$$\mathrm{Obj}_P(F, X) = \left\{F(\mathbf{x}) \colon \mathbf{x} \in X\right\}$$

the set of all objective values of primal problem (IVP).

Let $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be two closed intervals in $\mathbb{R}$. Let us recall that $C \preccurlyeq D$ if and only if $c^L \leqslant d^L$ and $c^U \leqslant d^U$. Now we write $C \prec D$ if and only if $C \preccurlyeq D$ and $C \neq D$. Equivalently, $C \prec D$ if and only if

$$\begin{cases} c^L < d^L, \\ c^U \leqslant d^U \end{cases} \quad \text{or} \quad \begin{cases} c^L \leqslant d^L, \\ c^U < d^U \end{cases} \quad \text{or} \quad \begin{cases} c^L < d^L, \\ c^U < d^U. \end{cases} \tag{7}$$

We need to interpret the meaning of minimization in problem (IVP). Since "$\preccurlyeq$" is a partial ordering, not a total ordering, on $\mathcal{I}$, we may follow the similar solution concept (the nondominated solution) used in multiobjective programming problem to interpret the meaning of minimization in primal problem (IVP).

In the minimization problem (IVP), we say that the feasible solution $\bar{\mathbf{x}}$ is better than (dominates) the feasible solution $\mathbf{x}^*$ if $F(\bar{\mathbf{x}}) \prec F(\mathbf{x}^*)$. Therefore, we propose the following definition.

Definition 4.1. Let $\mathbf{x}^*$ be a feasible solution of primal problem (IVP). We say that $\mathbf{x}^*$ is a *nondominated solution* of problem (IVP) if there exists no $\bar{\mathbf{x}} \in X$ such that $F(\bar{\mathbf{x}}) \prec F(\mathbf{x}^*)$. In this case, $F(\mathbf{x}^*)$ is called the *nondominated objective value* of F.

We denote by $\mathrm{Min}(F, X)$ the set of all nondominated objective values of problem (IVP). More precisely, we write

$$\mathrm{Min}(F, X) = \{F(\mathbf{x}^*)\colon\ \mathbf{x}^* \text{ is a nondominated solution of (IVP)}\}.$$

Let c be a real number. Then we can regard the real number c as an interval $[c, c]$. Let $C = [c^L, c^U]$ be a closed interval. If we write $C + c$, then we shall mean that $C + [c, c] = [c^L + c, c^U + c]$.

Now we assume that the interval-valued function F and the real-valued functions g_i and h_i, $i = 1, \ldots, m$, are differentiable on $\mathbb{R}^n_+$. The dual problem of (IVP) is formulated as follows:

$$\begin{aligned} \text{(DIVP)} \quad \max \quad & F(\mathbf{x}) + \sum_{i=1}^{m} \mu_i \cdot g_i(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \cdot h_i(\mathbf{x}) \\ \text{subject to} \quad & \nabla F_L(\mathbf{x}) + \nabla F_U(\mathbf{x}) + \sum_{i=1}^{m} \mu_i \cdot \nabla g_i(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \cdot \nabla h_i(\mathbf{x}) = \mathbf{0}, \\ & \boldsymbol{\mu} = (\mu_1, \ldots, \mu_m) \geqslant \mathbf{0}, \\ & \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \geqslant \mathbf{0}, \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

We denote by Y the feasible set of dual problem (DIVP) consisting of elements $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+$. We write

$$H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = F(\mathbf{x}) + \sum_{i=1}^{m} \mu_i \cdot g_i(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \cdot h_i(\mathbf{x})$$

and denote by

$$\mathrm{Obj}_D(H, Y) = \{H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})\colon\ (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in Y\}$$

the set of all objective values of dual problem (DIVP). We also see that H is an interval-valued function with

$$H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \equiv \big(H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})\big)_L = F_L(\mathbf{x}) + \sum_{i=1}^{m} \mu_i g_i(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) \tag{8}$$

and

$$H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \equiv \big(H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})\big)_U = F_U(\mathbf{x}) + \sum_{i=1}^{m} \mu_i g_i(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}). \tag{9}$$

Example 4.2. Continued from Example 10, the dual problem is formulated as follows:

$$\begin{array}{ll} \max & \left[x_1^2+x_2^2+1, x_1^2+x_2^2+2\right]+\mu\cdot(-x_1-x_2+1)+\lambda\cdot(-6x_1-2x_1+12) \\ \text{subject to} & \begin{bmatrix}2x_1\\2x_2\end{bmatrix}+\begin{bmatrix}2x_1\\2x_2\end{bmatrix}+\mu\begin{bmatrix}-1\\-1\end{bmatrix}+\lambda\begin{bmatrix}-6\\-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \\ & \mu,\lambda,x_1,x_2\geqslant 0. \end{array}$$

We also have

$$H_L(x_1,x_2,\mu,\lambda)=x_1^2+x_2^2+1+\mu\cdot(-x_1-x_2+1)+\lambda\cdot(-6x_1-2x_1+12)$$

and

$$H_U(x_1,x_2,\mu,\lambda)=x_1^2+x_2^2+2+\mu\cdot(-x_1-x_2+1)+\lambda\cdot(-6x_1-2x_1+12).$$

Definition 4.2. Let $(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)$ be a feasible solution of dual problem (DIVP). We say that $(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)$ is a *nondominated solution* of dual problem (DIVP) if there exists no $(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$ such that $H(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)\prec H(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$. In this case, $H(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)$ is called the *nondominated objective value* of problem (DIVP).

We denote by $\operatorname{Max}(H,Y)$ the set of all nondominated objective values of problem (DIVP). More precisely, we write

$$\operatorname{Max}(H,Y)=\left\{H\left(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*\right):\ \left(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*\right)\text{ is a nondominated solution of (DIVP)}\right\}.$$

5. Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problems

Now we consider the following optimization problem:

$$\begin{array}{lll} \text{(P2)} & \min & f(\mathbf{x})=F_L(\mathbf{x})+F_U(\mathbf{x}) \\ & \text{subject to} & g_i(\mathbf{x})\leqslant 0,\quad i=1,\ldots,m, \\ & & h_i(\mathbf{x})\leqslant 0,\quad i=1,\ldots,m, \\ & & \mathbf{x}\geqslant \mathbf{0}. \end{array}$$

Then we have the following observation.

Proposition 5.1. *If* $\mathbf{x}^*$ *is an optimal solution of problem* (P2)*, then* $\mathbf{x}^*$ *is a nondominated solution of problem* (IVP).

Proof. We see that problems (P2) and (IVP) have the identical feasible sets. Suppose that $\mathbf{x}^*$ is not a nondominated solution. Then there exists a feasible solution $\mathbf{x}$ such that $F(\mathbf{x})\prec F(\mathbf{x}^*)$. From (7), it means that

$$\begin{cases} F_L(\mathbf{x})<F_L\left(\mathbf{x}^*\right), \\ F_U(\mathbf{x})\leqslant F_U\left(\mathbf{x}^*\right) \end{cases} \text{or} \quad \begin{cases} F_L(\mathbf{x})\leqslant F_L\left(\mathbf{x}^*\right), \\ F_U(\mathbf{x})<F_U\left(\mathbf{x}^*\right) \end{cases} \text{or} \quad \begin{cases} F_L(\mathbf{x})<F_L\left(\mathbf{x}^*\right), \\ F_U(\mathbf{x})<F_U\left(\mathbf{x}^*\right). \end{cases}$$

It also shows that $f(\mathbf{x})<f(\mathbf{x}^*)$, which contradicts the fact that $\mathbf{x}^*$ is an optimal solution of problem (P2). We complete the proof. □

Example 5.1. Continued from Example 4.1, we are going to obtain the nondominated solution of primal problem by applying Proposition 5.1. It means that we shall minimize the objective function $f(x_1,x_2)=2x_1^2+2x_2^2+3$ subject to $(x_1,x_2)\in X$. The optimal solution is $(x_1,x_2)=(9/5,3/5)$, which is also a nondominated solution by applying Proposition 5.1.

Now we consider the following optimization problem:

$$\begin{array}{lll} \text{(D2)} & \max & f(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})=H_L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})+H_U(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda}) \\ & \text{subject to} & \nabla F_L(\mathbf{x})+\nabla F_U(\mathbf{x})+\sum_{i=1}^m \mu_i\cdot\nabla g_i(\mathbf{x})+\sum_{i=1}^m \lambda_i\cdot\nabla h_i(\mathbf{x})=\mathbf{0}, \end{array}$$

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m) \geqslant \mathbf{0},$$
$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \geqslant \mathbf{0},$$
$$\mathbf{x} \geqslant \mathbf{0}.$$

Then we also have the following observation.

Proposition 5.2. *If* $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ *is an optimal solution of problem* (D2)*, then* $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ *is a nondominated solution of problem* (DIVP).

Proof. We see that problems (D2) and (DIVP) have the identical feasible sets. Suppose that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is not a nondominated solution. Then there exists a feasible solution $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ such that $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \prec H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$. From (7), it means that

$$\begin{cases} H_L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) < H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ H_U(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \leqslant H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \end{cases} \quad \text{or} \quad \begin{cases} H_L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \leqslant H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ H_U(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) < H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \end{cases}$$
$$\text{or} \quad \begin{cases} H_L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) < H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ H_U(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) < H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}). \end{cases}$$

It also shows that $f(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) < f(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$, which contradicts the fact that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is an optimal solution of problem (D2). We complete the proof. □

Example 5.2. Continued from Example 4.2, we are going to obtain the nondominated solution of dual problem by applying Proposition 5.2. Therefore, we shall solve the following problem:

$$\begin{aligned} \max \quad & 2x_1^2 + 2x_2^2 + 3 + \mu \cdot (-x_1 - x_2 + 1) + \lambda \cdot (-6x_1 - 2x_1 + 12) \\ \text{subject to} \quad & 4x_1 - \mu - 6\lambda = 0, \\ & 4x_2 - \mu - 2\lambda = 0, \\ & \mu, \lambda, x_1, x_2 \geqslant 0. \end{aligned}$$

From the constraints, we can obtain

$$x_1 = \frac{1}{4}(\mu + 6\lambda) \quad \text{and} \quad x_2 = \frac{1}{4}(\mu + 2\lambda),$$

which can be substituted into the objective function to obtain a function of μ and λ. After some algebraic calculations, we obtain $\mu = 0$ and $\lambda = 6/5$, which also implies $x_1 = 9/5$ and $x_2 = 3/5$. Therefore, the optimal solution is $(x_1, x_2, \mu, \lambda) = (9/5, 3/5, 0, 6/5)$, which is also a nondominated solution of dual problem by applying Proposition 5.2.

Let us rename the constraint functions h_i for $i = 1, \ldots, m$ as $g_{m+i} = h_i$ for $i = 1, \ldots, m$. Let $J(\mathbf{x}^*)$ be the index set defined by

$$J(\mathbf{x}^*) = \{i\colon\, g_i(\mathbf{x}^*) = 0 \text{ for } i = 1, \ldots, 2m\}.$$

We say that the constraint functions g_i and h_i, $i = 1, \ldots, m$, satisfy the *Kuhn–Tucker constraint qualification* at $\mathbf{x}^*$ if and only if the real-valued functions g_i, $i = 1, \ldots, 2m$, satisfy the Kuhn–Tucker constraint qualification at $\mathbf{x}^*$; that is to say, if $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leqslant 0$ for all $i \in J(\mathbf{x}^*)$, where $\mathbf{d} \in \mathbb{R}^n$, then there exists an n-dimensional vector function $\mathbf{a} : [0, 1] \to \mathbb{R}^n$ defined on $[0, 1]$ such that $\mathbf{a}$ is right-differentiable at 0, $\mathbf{a}(0) = \mathbf{x}^*$, $\mathbf{a}(t) \in X$ for all $t \in [0, 1]$, and there exists a real number $\alpha > 0$ with $\mathbf{a}'_+(0) = \alpha\mathbf{d}$.

For the further discussions, we need the Motzkin's theorem of the alternative. It states that, given matrices $A \neq \mathbf{0}$ and C, exactly one of the following system has a solution:

System I: $A\mathbf{x} < \mathbf{0}$, $C\mathbf{x} \leqslant \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$;
System II: $A^T\boldsymbol{\lambda} + C^T\boldsymbol{\mu} = \mathbf{0}$ for some $\boldsymbol{\mu} \geqslant \mathbf{0}$ and $\boldsymbol{\lambda} \geqslant \mathbf{0}$ with $\boldsymbol{\lambda} \neq \mathbf{0}$.

Theorem 5.1 *(KKT Conditions). Suppose that* $\mathbf{x}^*$ *is a nondominated solution of primal problem* (IVP)*, and* F*,* g_i *and* h_i*,* $i = 1, \ldots, m$*, are differentiable at* $\mathbf{x}^*$*. We also assume that the constraint functions* g_i *and* h_i*,* $i = 1, \ldots, m$*, satisfy the Kuhn–Tucker constraint qualification at* $\mathbf{x}^*$*. Then there exist multipliers* $\mathbf{0} \leqslant \boldsymbol{\zeta} = (\zeta_L, \zeta_U) \in \mathbb{R}^2$ *with* $\boldsymbol{\zeta} \neq \mathbf{0}$ *and* $0 \leqslant \mu_i, \lambda_i \in \mathbb{R}$*,* $i = 1, \ldots, m$*, such that*

$$\zeta_L \cdot \nabla F_L(\mathbf{x}^*) + \zeta_U \cdot \nabla F_U(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \cdot \nabla g_i(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \cdot \nabla h_i(\mathbf{x}^*) = \mathbf{0};$$
$$\mu_i \cdot g_i(\mathbf{x}^*) = 0 = \lambda_i \cdot h_i(\mathbf{x}^*) \quad \text{for all } i = 1, \ldots, m.$$

Proof. Since F is differentiable at $\mathbf{x}^*$, we see that F_L and F_U are differentiable at $\mathbf{x}^*$ by definition. Suppose that there exists $\mathbf{d} \in \mathbb{R}^n$ such that

$$\begin{aligned} &\nabla F_L(\mathbf{x}^*)^T \mathbf{d} < 0, \\ &\nabla F_U(\mathbf{x}^*)^T \mathbf{d} < 0, \\ &\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leqslant 0 \quad \text{for all } i \in J(\mathbf{x}^*). \end{aligned} \tag{10}$$

Since g_i, $i = 1, \ldots, 2m$, satisfy the Kuhn–Tucker constraint qualification at $\mathbf{x}^*$ and F_L is differentiable at $\mathbf{x}^*$, we have

$$\begin{aligned} F_L(\mathbf{a}(t)) &= F_L(\mathbf{x}^*) + \nabla F_L(\mathbf{x}^*)^T (\mathbf{a}(t) - \mathbf{x}^*) + \|\mathbf{a}(t) - \mathbf{x}^*\| \cdot \varepsilon(\mathbf{a}(t), \mathbf{x}^*) \\ &= F_L(\mathbf{x}^*) + \nabla F_L(\mathbf{x}^*)^T (\mathbf{a}(t) - \mathbf{a}(0)) + \|\mathbf{a}(t) - \mathbf{a}(0)\| \cdot \varepsilon(\mathbf{a}(t), \mathbf{a}(0)) \\ &= F_L(\mathbf{x}^*) + t \cdot \nabla F_L(\mathbf{x}^*)^T \left(\frac{\mathbf{a}(0+t) - \mathbf{a}(0)}{t} \right) + \|\mathbf{a}(t) - \mathbf{a}(0)\| \cdot \varepsilon(\mathbf{a}(t), \mathbf{a}(0)), \end{aligned}$$

where $\varepsilon(\mathbf{a}(t), \mathbf{a}(0)) \to 0$ as $\|\mathbf{a}(t) - \mathbf{a}(0)\| \to 0$. Therefore, as $t \to 0+$, we see that $\|\mathbf{a}(t) - \mathbf{a}(0)\| \to 0$ and

$$\frac{\mathbf{a}(0+t) - \mathbf{a}(0)}{t} \to \mathbf{a}'_+(0) = \alpha \mathbf{d}, \quad \text{where } \alpha > 0.$$

Since $\nabla F_L(\mathbf{x}^*)^T \mathbf{d} < 0$, we obtain that $F_L(\mathbf{a}(t_1)) < F_L(\mathbf{x}^*)$ for a sufficiently small $t_1 > 0$. Similarly, since $\nabla F_U(\mathbf{x}^*)^T \mathbf{d} < 0$, we can also obtain that $F_U(\mathbf{a}(t_2)) < F_U(\mathbf{x}^*)$ for a sufficiently small $t_2 > 0$. In other words, we have that $F_L(\mathbf{a}(t)) < F_L(\mathbf{x}^*)$ and $F_U(\mathbf{a}(t)) < F_U(\mathbf{x}^*)$ for a sufficiently small $t < \min\{t_1, t_2\}$; or equivalently, $F(\mathbf{a}(t)) \prec F(\mathbf{x}^*)$ for a sufficiently small t, which contradicts that $\mathbf{x}^*$ is a nondominated solution of primal problem (IVP), since $\mathbf{a}(t)$ is a feasible solution. Therefore, we conclude that the system of inequalities presented in (10) has no solutions. Let A be the matrix whose rows are $\nabla F_L(\mathbf{x}^*)^T$, $\nabla F_U(\mathbf{x}^*)^T$ and C be the matrix whose rows are $\nabla g_i(\mathbf{x}^*)^T$ for $i \in J(\mathbf{x}^*)$. According to the Motzkin's theorem of the alternative, since system I, i.e., (10), has no solutions, there exist multipliers $\mathbf{0} \leqslant \boldsymbol{\zeta} = (\zeta_L, \zeta_U) \in \mathbb{R}^2$ with $\boldsymbol{\zeta} \neq \mathbf{0}$ and $0 \leqslant \mu_i \in \mathbb{R}$ for $i \in J(\mathbf{x}^*)$ such that

$$\zeta_L \nabla F_L(\mathbf{x}^*) + \zeta_U \nabla F_U(\mathbf{x}^*) + \sum_{i \in J(\mathbf{x}^*)} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}.$$

We set $\mu_i = 0$ for $i \in \{1, \ldots, 2m\} \setminus J(\mathbf{x}^*)$. Then we obtain that

$$\zeta_L \nabla F_L(\mathbf{x}^*) + \zeta_U \nabla F_U(\mathbf{x}^*) + \sum_{i=1}^{2m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0};$$
$$\mu_i g_i(\mathbf{x}^*) = 0 \quad \text{for all } i = 1, \ldots, 2m.$$

Let $\lambda_i = \mu_{i+m}$ for $i = 1, \ldots, m$. Then we complete the proof. □

Theorem 5.2 *(KKT Conditions). Suppose that* $\mathbf{x}^*$ *is an optimal solution of problem* (P2) (*also a nondominated solution of primal problem* (IVP) *by Proposition* 5.1)*, and* F*,* g_i *and* h_i*,* $i = 1, \ldots, m$*, are differentiable at* $\mathbf{x}^*$*. We also assume that the constraint functions* g_i *and* h_i*,* $i = 1, \ldots, m$*, satisfy the Kuhn–Tucker constraint qualification at* $\mathbf{x}^*$*. Then there exist multipliers* $0 \leqslant \mu_i, \lambda_i \in \mathbb{R}$ *for* $i = 1, \ldots, m$ *such that*

$$\nabla F_L(\mathbf{x}^*) + \nabla F_U(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \cdot \nabla g_i(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \cdot \nabla h_i(\mathbf{x}^*) = \mathbf{0};$$
$$\mu_i \cdot g_i(\mathbf{x}^*) = 0 = \lambda_i \cdot h_i(\mathbf{x}^*) \quad \text{for all } i = 1, \ldots, m.$$

Proof. Since F is differentiable at $\mathbf{x}^*$, we see that F_L and F_U are differentiable at $\mathbf{x}^*$. It also says that $f = F_L + F_U$ is differentiable at $\mathbf{x}^*$. Suppose that there exists $\mathbf{d} \in \mathbb{R}^n$ such that

$$\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0,$$
$$\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leqslant 0 \quad \text{for all } i \in J(\mathbf{x}^*). \tag{11}$$

Using the similar arguments in the proof of Theorem 5.1, we obtain $f(\mathbf{a}(t)) < f(\mathbf{x}^*)$ for a sufficiently small t, which contradicts the fact that $\mathbf{x}^*$ is an optimal solution of problem (P2). Therefore, we conclude that the system of inequalities presented in (11) has no solutions. According to the Motzkin's theorem of the alternative, there exist multipliers $0 < \zeta$ and $0 \leqslant \mu_i \in \mathbb{R}$ for $i \in J(\mathbf{x}^*)$ such that

$$\zeta \cdot \nabla f(\mathbf{x}^*) + \sum_{i \in J(\mathbf{x}^*)} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0};$$

or equivalently,

$$\nabla f(\mathbf{x}^*) + \sum_{i \in J(\mathbf{x}^*)} \bar{\mu}_i \nabla g_i(\mathbf{x}^*) = \mathbf{0},$$

where $\bar{\mu}_i = \mu_i / \zeta$. We complete the proof by renaming the multipliers. □

Example 5.3. Continued from Example 5.1, we see that

$$g_2(x_1, x_2) = h_1(x_1, x_2) = -6x_1 - 2x_1 + 12.$$

It is not hard to check that the constraint functions $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ satisfy the Kuhn–Tucker constraint qualification at $(x_1^*, x_2^*) = (9/5, 3/5)$. We also see that $J(9/5, 3/5) = \{2\}$. Now we have

$$\nabla F_L(9/5, 3/5) + \nabla F_U(9/5, 3/5) + \mu \cdot \nabla g_1(9/5, 3/5) + \lambda \cdot \nabla h_1(9/5, 3/5)$$
$$= \begin{bmatrix} 18/5 \\ 6/5 \end{bmatrix} + \begin{bmatrix} 18/5 \\ 6/5 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then we obtain $\mu = 0$ and $\lambda = 6/5$ that verify Theorem 5.2.

6. Solvability

Let f be a differentiable real-valued function defined on a nonempty open convex subset X of $\mathbb{R}^n$. Then f is convex at $\mathbf{x}^*$ if and only if

$$f(\mathbf{x}) \geqslant f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \tag{12}$$

for $\mathbf{x} \in X$ (ref. Bazarra et al. [2, Theorem 3.3.3]).

Definition 6.1. Let X be a nonempty convex subset of $\mathbb{R}^n$ and F be an interval-valued function defined on X. We say that F is convex at $\mathbf{x}^*$ if

$$F(\lambda \mathbf{x}^* + (1 - \lambda)\mathbf{x}) \preccurlyeq \lambda F(\mathbf{x}^*) + (1 - \lambda) F(\mathbf{x})$$

for each $\lambda \in (0, 1)$ and each $\mathbf{x} \in X$.

Proposition 6.1. *Let X be a nonempty convex subset of $\mathbb{R}^n$ and F be an interval-valued function defined on X. The interval-valued function F is convex at $\mathbf{x}^*$ if and only if the real-valued functions F_L and F_U are convex at $\mathbf{x}^*$.*

Proof. By definition, we have

$$F_L\big(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}\big) \leqslant \big[\lambda F\big(\mathbf{x}^*\big) + (1-\lambda)F(\mathbf{x})\big]_L$$

and

$$F_U\big(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}\big) \leqslant \big[\lambda F\big(\mathbf{x}^*\big) + (1-\lambda)F(\mathbf{x})\big]_U.$$

From Eq. (1), since $\lambda > 0$ and $1-\lambda > 0$, we obtain

$$F_L\big(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}\big) \leqslant \lambda F_L\big(\mathbf{x}^*\big) + (1-\lambda)F_L(\mathbf{x})$$

and

$$F_U\big(\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}\big) \leqslant \lambda F_U\big(\mathbf{x}^*\big) + (1-\lambda)F_U(\mathbf{x}).$$

We complete the proof. □

Lemma 6.1. *Let F, g_i and h_i, $i = 1,\dots,m$, be differentiable on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}}$ is a feasible solution of primal problem* (IVP) *and $(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$ is a feasible solution of dual problem* (DIVP). *If F, g_i and h_i, $i = 1,\dots,m$, are convex at $\mathbf{x}$, then the following statements hold true.*

(i) *If $F_U(\mathbf{x}) \geqslant F_U(\bar{\mathbf{x}})$, then $F_L(\bar{\mathbf{x}}) \geqslant H_L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$.*
(ii) *If $F_U(\mathbf{x}) > F_U(\bar{\mathbf{x}})$, then $F_L(\bar{\mathbf{x}}) > H_L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$.*
(iii) *If $F_L(\mathbf{x}) \geqslant F_L(\bar{\mathbf{x}})$, then $F_U(\bar{\mathbf{x}}) \geqslant H_U(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$.*
(iv) *If $F_L(\mathbf{x}) > F_L(\bar{\mathbf{x}})$, then $F_U(\bar{\mathbf{x}}) > H_U(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda})$.*

Proof. From Proposition 6.1, we see that F_L and F_U are differentiable on $\mathbb{R}^n_+$ and convex at $\mathbf{x}$. Since $\bar{\mathbf{x}}$ is a feasible solution of primal problem, we see that

$$g_i(\bar{\mathbf{x}}) \leqslant 0 \quad \text{and} \quad h_i(\bar{\mathbf{x}}) \leqslant 0, \tag{13}$$

for all $i = 1,\dots,m$. Then we have

$$\begin{aligned}
F_L(\bar{\mathbf{x}}) &\geqslant F_L(\mathbf{x}) + \nabla F_L(\mathbf{x})^T(\bar{\mathbf{x}} - \mathbf{x}) \quad \big(\text{by Eq. (12)}\big)\\
&= F_L(\mathbf{x}) - \nabla F_U(\mathbf{x})^T(\bar{\mathbf{x}} - \mathbf{x}) - \sum_{i=1}^m \mu_i \cdot \nabla g_i(\mathbf{x})^T(\bar{\mathbf{x}} - \mathbf{x}) - \sum_{i=1}^m \lambda_i \cdot \nabla h_i(\mathbf{x})^T(\bar{\mathbf{x}} - \mathbf{x})\\
&\qquad \big(\text{since } (\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda}) \text{ is a feasible solution of dual problem (DIVP)}\big)\\
&\geqslant F_L(\mathbf{x}) + F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) + \sum_{i=1}^m \mu_i \cdot \big[g_i(\mathbf{x}) - g_i(\bar{\mathbf{x}})\big] + \sum_{i=1}^m \lambda_i \cdot \big[h_i(\mathbf{x}) - h_i(\bar{\mathbf{x}})\big]\\
&\qquad \big(\text{by } \mu_i, \lambda_i \geqslant 0 \text{ and Eq. (12)}\big)\\
&\geqslant F_L(\mathbf{x}) + F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) + \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) + \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x})\\
&\qquad \big(\text{by } \mu_i, \lambda_i \geqslant 0 \text{ and Eq. (13)}\big)\\
&\geqslant F_L(\mathbf{x}) + \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) + \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x}), \quad \text{if } F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) \geqslant 0\\
&= H_L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda}), \quad \text{if } F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) \geqslant 0 \quad \big(\text{by Eq. (8)}\big).
\end{aligned}$$

Therefore statement (i) holds true. We also see that if $F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) > 0$, then

$$F_L(\bar{\mathbf{x}}) > F_L(\mathbf{x}) + \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) + \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x}) = H_L(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda}),$$

which proves statement (ii). On the other hand, considering the function F_U, statements (iii) and (iv) can also be obtained by using the similar arguments. □

Lemma 6.2. *Let F, g_i and h_i, $i = 1, \ldots, m$, be differentiable on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}}$ is a feasible solution of primal problem* (IVP) *and $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is a feasible solution of dual problem* (DIVP). *If F, g_i and h_i, $i = 1, \ldots, m$, are convex at $\mathbf{x}$, then the following statements hold true.*

(i) *If $F_L(\mathbf{x}) \leqslant F_L(\bar{\mathbf{x}})$, then $F_L(\bar{\mathbf{x}}) \geqslant H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*
(ii) *If $F_L(\mathbf{x}) < F_L(\bar{\mathbf{x}})$, then $F_L(\bar{\mathbf{x}}) > H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*
(iii) *If $F_U(\mathbf{x}) \leqslant F_U(\bar{\mathbf{x}})$, then $F_U(\bar{\mathbf{x}}) \geqslant H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*
(iv) *If $F_U(\mathbf{x}) < F_U(\bar{\mathbf{x}})$, then $F_U(\bar{\mathbf{x}}) > H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*

Proof. We have that

$$
\begin{aligned}
&F_L(\bar{\mathbf{x}}) - H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \\
&\quad = F_L(\bar{\mathbf{x}}) - F_L(\mathbf{x}) - \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) - \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x}) \\
&\quad \geqslant \nabla F_L(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) - \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) - \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x}) \quad \big(\text{by Eq. (12)}\big) \\
&\quad = \nabla F_L(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) + \left[-\sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) + \sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) - \sum_{i=1}^m \mu_i \cdot g_i(\mathbf{x}) \right] \\
&\qquad + \left[-\sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) + \sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) - \sum_{i=1}^m \lambda_i \cdot h_i(\mathbf{x}) \right] \\
&\quad \geqslant \nabla F_L(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) + \left[-\sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) + \sum_{i=1}^m \mu_i \cdot \nabla g_i(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) \right] \\
&\qquad + \left[-\sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) + \sum_{i=1}^m \lambda_i \cdot \nabla h_i(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) \right] \\
&\qquad \big(\text{by } \mu_i, \lambda_i \geqslant 0 \text{ and Eq. (12)}\big) \\
&\quad = \left[\nabla F_L(\mathbf{x})^T + \sum_{i=1}^m \mu_i \cdot \nabla g_i(\mathbf{x})^T + \sum_{i=1}^m \lambda_i \cdot \nabla h_i(\mathbf{x})^T \right] (\bar{\mathbf{x}} - \mathbf{x}) - \sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) - \sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) \\
&\quad = -\nabla F_U(\mathbf{x})^T (\bar{\mathbf{x}} - \mathbf{x}) - \sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) - \sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) \\
&\qquad \big(\text{since } (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \text{ is a feasible solution of dual problem (DIVP)}\big) \\
&\quad \geqslant F_U(\mathbf{x}) - F_U(\bar{\mathbf{x}}) - \sum_{i=1}^m \mu_i \cdot g_i(\bar{\mathbf{x}}) - \sum_{i=1}^m \lambda_i \cdot h_i(\bar{\mathbf{x}}) \quad \big(\text{by Eq. (12)}\big) \\
&\quad = F_U(\mathbf{x}) - H_U(\bar{\mathbf{x}}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \quad \big(\text{by Eq. (9)}\big) \\
&\quad \geqslant 0, \quad \text{if } F_L(\mathbf{x}) \leqslant F_L(\bar{\mathbf{x}}) \quad \big(\text{using Lemma 6.1(iii)}\big).
\end{aligned}
$$

Therefore statement (i) holds true. We also see that if $F_L(\mathbf{x}) < F_L(\bar{\mathbf{x}})$, then statement (ii) holds true by using Lemma 6.1(iv). On the other hand, statements (iii) and (iv) can also be obtained by using the similar arguments and Lemma 6.1(i) and (ii), respectively. □

Let $C = [c^L, c^U]$ and $D = [d^L, d^U]$ be two closed intervals. We say that C and D are *comparable* if and only if $C \preccurlyeq D$ or $C \succcurlyeq D$. Therefore if C and D are not comparable, then

$$
\begin{cases} c^L \leqslant d^L, \\ c^U > d^U, \end{cases} \quad \begin{cases} c^L < d^L, \\ c^U \geqslant d^U, \end{cases} \quad \begin{cases} c^L < d^L, \\ c^U > d^U, \end{cases} \quad \begin{cases} c^L \geqslant d^L, \\ c^U < d^U, \end{cases} \quad \begin{cases} c^L > d^L, \\ c^U \leqslant d^U \end{cases} \quad \text{or} \quad \begin{cases} c^L > d^L, \\ c^U < d^U. \end{cases} \tag{14}
$$

In other words, if C and D are not comparable, then $C \neq D$, and $C \supseteq D$ or $C \subseteq D$.

Proposition 6.2. *Let F, g_i and h_i, $i = 1, \dots, m$, be differentiable on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}}$ is a feasible solution of primal problem* (IVP) *and $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is a feasible solution of dual problem* (DIVP). *If F, g_i and h_i, $i = 1, \dots, m$, are convex at $\mathbf{x}$, then the following statements hold true.*

(i) *If $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are comparable, then $F(\bar{\mathbf{x}}) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*
(ii) *If $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are not comparable, then $F_L(\bar{\mathbf{x}}) > H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ or $F_U(\bar{\mathbf{x}}) > H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$.*

Proof. (i) If $F(\mathbf{x}) \succcurlyeq F(\bar{\mathbf{x}})$, then $F(\bar{\mathbf{x}}) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ using Lemma 6.1(i) and (iii). On the other hand, if $F(\mathbf{x}) \preccurlyeq F(\bar{\mathbf{x}})$, then $F(\bar{\mathbf{x}}) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ using Lemma 6.2(i) and (iii).

(ii) Since $F(\mathbf{x}) \neq F(\bar{\mathbf{x}})$, from (14), we have that

$$\begin{cases} F_L(\mathbf{x}) \leqslant F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) > F_U(\bar{\mathbf{x}}), \end{cases} \quad \begin{cases} F_L(\mathbf{x}) < F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) \geqslant F_U(\bar{\mathbf{x}}), \end{cases} \quad \begin{cases} F_L(\mathbf{x}) < F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) > F_U(\bar{\mathbf{x}}), \end{cases}$$
$$\begin{cases} F_L(\mathbf{x}) \geqslant F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) < F_U(\bar{\mathbf{x}}), \end{cases} \quad \begin{cases} F_L(\mathbf{x}) > F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) \leqslant F_U(\bar{\mathbf{x}}) \end{cases} \quad \text{or} \quad \begin{cases} F_L(\mathbf{x}) > F_L(\bar{\mathbf{x}}), \\ F_U(\mathbf{x}) < F_U(\bar{\mathbf{x}}). \end{cases}$$

Using Lemma 6.1(ii) and (iv), and Lemma 6.2(ii) and (iv), we conclude that $F_L(\bar{\mathbf{x}}) > H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ or $F_U(\bar{\mathbf{x}}) > H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$. □

Theorem 6.1 *(Solvability). Let F, g_i and h_i, $i = 1, \dots, m$, be convex and differentiable on $\mathbb{R}^n_+$. Suppose that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is a feasible solution of dual problem* (DIVP) *and $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Obj}_P(F, X)$. Then $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ solves dual problem* (DIVP)*, i.e., $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Max}(H, Y)$.*

Proof. We are going to prove this result by contradiction. Suppose that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is not a nondominated solution of dual problem (DIVP). Then there exists a feasible solution $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ in the feasible set of dual problem (DIVP) such that $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \prec H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$. Since $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Obj}_P(F, X)$, it says that there exists a feasible solution $\bar{\mathbf{x}}$ of primal problem (IVP) such that

$$F(\bar{\mathbf{x}}) = H\big(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*\big) \prec H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}). \tag{15}$$

It also means that

$$\begin{cases} F_L(\bar{\mathbf{x}}) < H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ F_U(\bar{\mathbf{x}}) \leqslant H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \end{cases} \quad \begin{cases} F_L(\bar{\mathbf{x}}) \leqslant H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ F_U(\bar{\mathbf{x}}) < H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \end{cases} \quad \text{or} \quad \begin{cases} F_L(\bar{\mathbf{x}}) < H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}), \\ F_U(\bar{\mathbf{x}}) < H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}). \end{cases} \tag{16}$$

Suppose that $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are comparable. Then, from Proposition 6.2(i), we have that $F(\bar{\mathbf{x}}) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$, which violates expressions (15) or (16). Suppose now that $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are not comparable. Then, from Proposition 6.2(ii), we have that $F_L(\bar{\mathbf{x}}) > H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ or $F_U(\bar{\mathbf{x}}) > H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$, which also violates expression (16). This completes the proof. □

Theorem 6.2 *(Solvability). Let F, g_i and h_i, $i = 1, \dots, m$, be convex and differentiable on $\mathbb{R}^n_+$. Suppose that $\mathbf{x}^*$ is a feasible solution of primal problem* (IVP) *and $F(\mathbf{x}^*) \in \mathrm{Obj}_D(H, Y)$. Then $\mathbf{x}^*$ solves primal problem* (IVP)*, i.e., $F(\mathbf{x}^*) \in \mathrm{Min}(F, X)$.*

Proof. Suppose that $\mathbf{x}^*$ is not a nondominated solution of primal problem (IVP). Then there exists a feasible solution $\mathbf{x}$ in the feasible set of primal problem (IVP) such that $F(\mathbf{x}) \prec F(\mathbf{x}^*)$. Since $F(\mathbf{x}^*) \in \mathrm{Obj}_D(H, Y)$, it says that there exists a feasible solution $(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ of dual problem (DIVP) such that

$$H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) = F\big(\mathbf{x}^*\big) \succ F(\mathbf{x}). \tag{17}$$

It also means that

$$\begin{cases} F_L(\mathbf{x}) < H_L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}), \\ F_U(\mathbf{x}) \leqslant H_U(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}), \end{cases} \quad \begin{cases} F_L(\mathbf{x}) \leqslant H_L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}), \\ F_U(\mathbf{x}) < H_U(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \end{cases} \quad \text{or} \quad \begin{cases} F_L(\mathbf{x}) < H_L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}), \\ F_U(\mathbf{x}) < H_U(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}). \end{cases} \tag{18}$$

Suppose that $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are comparable. Then, from Proposition 6.2(i), we have that $F(\mathbf{x}) \succcurlyeq H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$, which violates expressions (17) or (18). Suppose now that $F(\mathbf{x})$ and $F(\bar{\mathbf{x}})$ are not comparable. Then, from Proposition 6.2(ii), we have that $F_L(\mathbf{x}) > H_L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ or $F_U(\mathbf{x}) > H_U(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$, which also violates expression (18). This completes the proof. □

Theorem 6.3 *(Solvability). Let F, g_i and h_i, $i = 1, \ldots, m$, be convex and differentiable on $\mathbb{R}^n_+$. Suppose that $\mathbf{x}^*$ is a feasible solution of primal problem* (IVP) *and $(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ is a feasible solution of dual problem* (DIVP). *If $H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) = F(\mathbf{x}^*)$, then $\mathbf{x}^*$ solves primal problem* (IVP) *and $(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ solves dual problem* (DIVP).

Proof. Since $F(\mathbf{x}^*) = H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \in \mathrm{Obj}_D(H, Y)$, according to the proof of Theorem 6.2, we see that $\mathbf{x}^*$ solves primal problem (IVP). On the other hand, since $H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) = F(\mathbf{x}^*) \in \mathrm{Obj}_P(F, X)$, according to the proof of Theorem 6.1, we see that $(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ solves dual problem (DIVP). The proof is complete. □

7. The duality theorems

In some sense, $\mathrm{Min}(F, X)$ and $\mathrm{Max}(H, Y)$ can be regarded as kinds of "optimal objective values" of primal problem (IVP) and dual problem (DIVP), respectively. Therefore, we are going to present the strong duality theorem by considering

$$\mathrm{Min}(F, X) \cap \mathrm{Max}(H, Y) \neq \emptyset,$$

which also means that there exist $F(\mathbf{x}^*) \in \mathrm{Min}(F, X)$ and $L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \in \mathrm{Max}(H, Y)$ such that

$$F(\mathbf{x}^*) = L(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}). \tag{19}$$

We provide the following definition.

Definition 7.1. Two kinds of concepts for having no duality gap are presented below:

(i) We say that the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense if and only if $\mathrm{Min}(F, X) \cap \mathrm{Max}(H, Y) \neq \emptyset$.
(ii) We say that the primal problem (IVP) and dual problem (DIVP) have no duality gap in strong sense if and only if there exist $F(\mathbf{x}^*) \in \mathrm{Min}(F, X)$ and $L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Max}(H, Y)$ such that $F(\mathbf{x}^*) = L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$.

We see that the primal problem (IVP) and dual problem (DIVP) having no duality gap in strong sense implies that the primal problem (IVP) and dual problem (DIVP) have no duality gap in weak sense. We also see that if $\mathbf{x}^* = \bar{\mathbf{x}}$ in Eq. (19), then the primal problem (IVP) and dual problem (DIVP) having no duality gap in weak sense implies that the primal problem (IVP) and dual problem (DIVP) have no duality gap in strong sense.

Theorem 7.1 *(Strong duality theorem in weak sense). Let F, g_i and h_i, $i = 1, \ldots, m$, be convex and differentiable on $\mathbb{R}^n_+$. Suppose that one of the following conditions is satisfied*:

(i) *there exists a feasible solution $\mathbf{x}^*$ of primal problem* (IVP) *such that $F(\mathbf{x}^*) \in \mathrm{Obj}_D(H, Y)$*;
(ii) *there exists a feasible solution $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ of dual problem* (DIVP) *such that $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Obj}_P(F, X)$.*

Then the primal problem (IVP) *and dual problem* (DIVP) *have no duality gap in weak sense.*

Proof. Suppose that condition (i) is satisfied. Then, from Theorem 6.2, it remains to show that $F(\mathbf{x}^*) \in \mathrm{Max}(H, Y)$. Since $F(\mathbf{x}^*) \in \mathrm{Obj}_D(H, Y)$, there exists a feasible solution $(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$ of dual problem (DIVP) such that $F(\mathbf{x}^*) = H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}})$. Therefore we just need to show that $H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \in \mathrm{Max}(H, Y)$. Using the similar arguments in the proof of Theorem 6.1 by looking at Eq. (15), we obtain that $H(\bar{\mathbf{x}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \in \mathrm{Max}(H, Y)$. Now suppose that condition (ii) is satisfied. Then, from Theorem 6.1, it remains to show that $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Min}(F, X)$. Since $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathrm{Obj}_P(F, X)$, there exists a feasible solution $\bar{\mathbf{x}}$ of primal problem (IVP) such that $F(\bar{\mathbf{x}}) = H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$. Using the similar arguments in the proof of Theorem 6.2 by looking at Eq. (17), we obtain that $F(\bar{\mathbf{x}}) \in \mathrm{Min}(F, X)$. This completes the proof. □

Theorem 7.2 *(Strong duality theorem in strong sense). Let F, g_i and h_i, $i = 1, \ldots, m$, be convex and differentiable on $\mathbb{R}^n_+$. Suppose that $\mathbf{x}^*$ is an optimal solution of problem* (P2) *(also a nondominated solution of primal problem* (IVP) *by Proposition* 5.1*). We also assume that the constraint functions g_i and h_i, $i = 1, \ldots, m$, satisfy the Kuhn–Tucker constraint qualification at $\mathbf{x}^*$. Then there exist $\boldsymbol{\mu}^*, \boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ solves dual problem* (DIVP) *and $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = F(\mathbf{x}^*)$; that is to say, the primal problem* (IVP) *and dual problem* (DIVP) *have no duality gap in strong sense.*

Proof. Using Theorem 5.2, there exist $\mathbf{0} \leqslant \boldsymbol{\mu}^*, \boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that

$$\nabla F_L(\mathbf{x}^*) + \nabla F_U(\mathbf{x}^*) + \sum_{i=1}^m \mu_i^* \cdot \nabla g_i(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i^* \cdot \nabla h_i(\mathbf{x}^*) = \mathbf{0};$$
$$\mu_i^* \cdot g_i(\mathbf{x}^*) = 0 = \lambda_i^* \cdot h_i(\mathbf{x}^*) \quad \text{for all } i = 1, \ldots, m.$$

It shows that $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is a feasible solution of dual problem (DIVP) and $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = F(\mathbf{x}^*)$. From Theorem 6.3, we complete the proof. □

Example 7.1. Continued from Examples 5.1 and 5.2, we see that $(x_1^*, x_2^*) = (9/5, 3/5)$ solves the primal problem and $(x_1^*, x_2^*, \mu^*, \lambda^*) = (9/5, 3/5, 0, 6/5)$ solves the dual problem. We also see that the primal and dual problems have the same objective value $[23/5, 28/5]$, which verifies Theorem 7.2.

8. Conclusions

We see that "$\preccurlyeq$" is a partial ordering. That is to say, $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ and $H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ are not comparable in general, even though $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is a nondominated solution of dual problem (DIVP). In other words, we cannot always give the relationship $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ in general for any feasible solutions $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ of problem (DIVP), even though $(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ is a nondominated solution of dual problem (DIVP). However, under the assumptions in Theorem 7.2, we can get an interesting result. From Theorem 7.2 and Proposition 6.2, we have that, for any feasible solutions $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ of problem (DIVP),

(i) $H(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = F(\mathbf{x}^*) \succcurlyeq H(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ if $F(\mathbf{x})$ and $F(\mathbf{x}^*)$ are comparable;
(ii) $H_L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = F_L(\mathbf{x}^*) > H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ or $H_U(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = F_L(\mathbf{x}^*) > H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ if $F(\mathbf{x})$ and $F(\mathbf{x}^*)$ are not comparable.

Therefore we conclude that $H_L(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \geqslant H_L(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ or $H_U(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \geqslant H_U(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ for any feasible solutions $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ of problem (DIVP).

From (12), we see that if $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$ for all $x \in X$, then $\mathbf{x}^*$ minimizes the real-valued function f. The inequality of the form $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$ is called the *variational inequality*. The details of this topic may refer to Noor [8,9]. Arising from this inspiration, it is naturally to investigate the variational inequalities for interval-valued functions. Let $F(\mathbf{x}) = [F_L(\mathbf{x}), F_U(\mathbf{x})]$ be an interval-valued function. Suppose that F is differentiable. Then we can consider the following two variational inequalities:

$$\nabla F_L(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0 \quad \text{and} \quad \nabla F_U(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$$

for all $\mathbf{x} \in X$, which also says that $\mathbf{x}^*$ minimizes F_L and F_U simultaneously. Moreover, we have

$$\nabla F_L(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \nabla F_U(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$$

for all $\mathbf{x} \in X$, which is equivalent to

$$\nabla\big(F_L(\mathbf{x}^*) + F_U(\mathbf{x}^*)\big)(\mathbf{x} - \mathbf{x}^*) = \big(\nabla F_L(\mathbf{x}^*)^T + \nabla F_U(\mathbf{x}^*)^T\big)(\mathbf{x} - \mathbf{x}^*) \geqslant 0 \tag{20}$$

for all $\mathbf{x} \in X$. By referring to problem (P2), since $f(\mathbf{x}) = F_L(\mathbf{x}) + F_U(\mathbf{x})$, Eq. (20) says that $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$ for all $x \in X$. That is to say, $\mathbf{x}^*$ is an optimal solution of problem (P2). From Proposition 5.1, we conclude that $\mathbf{x}^*$ is a nondominated solution of problem (IVP). In other words, if $\nabla F_L(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$ and $\nabla F_U(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geqslant 0$ for all $\mathbf{x} \in X$, then $\mathbf{x}^*$ is a nondominated solution. In the future research, we may study more interesting properties and results about the variational inequalities for interval-valued functions.

References

[1] G. Alefeld, J. Herzberger, Introduction to Interval Computations, Academic Press, NY, 1983.
[2] M.S. Bazarra, H.D. Sherali, C.M. Shetty, Nonlinear Programming, Wiley, NY, 1993.
[3] J.R. Birge, F. Louveaux, Introduction to Stochastic Programming, Physica-Verlag, NY, 1997.
[4] J.E. Falk, Exact solutions of inexact linear programs, Oper. Res. 24 (1976) 783–787.
[5] P. Kall, Stochastic Linear Programming, Springer-Verlag, NY, 1976.
[6] R.E. Moore, Interval Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1966.
[7] R.E. Moore, Method and Applications of Interval Analysis, SIAM, Philadelphia, 1979.
[8] M.A. Noor, Some recent advances in variational inequalities, parts I, II, New Zealand J. Math. 26 (1997) 53–80, 229–255.
[9] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199–277.
[10] J.C. Pomerol, Constraint qualification for inexact linear programs, Oper. Res. 27 (1979) 843–847.
[11] A. Prékopa, Stochastic Programming, Kluwer Academic Publishers, Boston, 1995.
[12] A.L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming, Oper. Res. 21 (1973) 1154–1157.
[13] A.L. Soyster, A duality theory for convex programming with set-inclusive constraints, Oper. Res. 22 (1974) 892–898, Oper. Res. 22 (1974) 1279–1280 (Erratum).
[14] A.L. Soyster, Inexact linear programming with generalized resource sets, European J. Oper. Res. 3 (1979) 316–321.
[15] I.M. Stancu-Minasian, Stochastic Programming with Multiple Objective Functions, Reidel Publishing Company, 1984.
[16] D.J. Thuente, Duality theory for generalized linear programs with computational methods, Oper. Res. 28 (1980) 1005–1011.
[17] S. Vajda, Probabilistic Programming, Academic Press, NY, 1972.
[18] P. Wolfe, A duality theorem for nonlinear programming, Quart. Appl. Math. 19 (1961) 239–244.