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Worldline deviations of charged spinning particles

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Abstract

The geodesic deviation equation is generalized to worldline deviation equations describing the relative accelerations of charged spinning particles in the framework of Dixon–Souriau equations of motion.

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Keywords: Charged spinning particles; Worldline deviation

1. Introduction

In the study of the dynamics of particles in a given space–time, one important object is the relative acceleration of particles. For nearby test particles, this is given by the well-known geodesic deviation equation. This equation gives in a frame-independent way, how much two nearby geodesics deviate from each other. For test particles which are not necessarily nearby, similar equations may be derived by keeping higher order terms in the approximation whose first order terms lead to the equation of geodesic deviation. When the particles under study possess some internal structure like spin, or if they are subject to extra interactions like the Lorentz force, their worldlines will no longer be geodesics. In such cases the geodesic deviation equation should be modified so as to accommodate the effects of those extra interactions on the relative accelerations of the particles. For the case of test particles with charge moving in an arbitrary space–time in the presence of some electromagnetic fields, a set of worldline deviation equations has been obtained in [1]. For particles with spin (but no charge), described by the Mathison–Papapetrou–Dixon equations, a set of such generalized worldline deviation equations was derived in a recent publication [2]. In [3] the relative motion of a spinning particle with respect to a nearby

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free test particle in the gravitational field of a rotating source was studied. The aim of the present Letter is to obtain worldline deviation equations for charged spinning particles moving in an arbitrary space–time in the presence of electromagnetic fields.

In general relativity, the motion of charged spinning particles is described by the so-called Dixon–Souriau (DS, for abbreviation) equations [4,5]. These equations reflect the effects of the spin-curvature, the charge-electromagnetic field, and the spin-electromagnetic field on the motion of the particle. These equations consist of seven independent equations to describe the particle's four-momentum and spin tensor supplemented by three other equations to render the equations of motion complete. The particle's trajectory is then determined by integrating its four-velocity which is obtained indirectly from the equations of motion. These equations or simplified versions of them have been used to study the motion of charged spinning particles in different space–times [6–13]. An extension of these equations to the case where torsion fields are also included was obtained in [14,15]. It has also been shown that the DS equations reduce to the well-known Bargmann–Michel–Telegdi equations in the limit of the weak and homogeneous external field [16].

In the following sections we first review the DS equations briefly and derive the worldline deviation equations along the lines of Ref. [2]. We then apply these equations to the case of motion of charged spinning particles in a gravitational wave space–time and a uniform magnetic field and show how they can be used to calculate the relative accelerations and also to generate approximate solutions to the DS equations via a known one. In the last section we present our conclusions.

2. The DS equations of motion

The motion of a charged spinning particle is described by the Dixon–Souriau equations [17]

$$\frac{Dp^{\mu}}{D\tau} = -\frac{1}{2}R^{\mu}{}_{\nu\lambda\rho}s^{\lambda\rho}\dot{x}^{\nu} + qF^{\mu}{}_{\beta}\dot{x}^{\beta} + \frac{k}{2}s^{\kappa\rho}D^{\mu}F_{\kappa\rho}, \tag{1}$$

$$\frac{Ds^{\mu\nu}}{D\tau} = p^{\mu}\dot{x}^{\nu} - p^{\nu}\dot{x}^{\mu} - k\big(s^{\mu\kappa}F_{\kappa}{}^{\nu} - s^{\nu\kappa}F_{\kappa}{}^{\mu}\big), \tag{2}$$

supplemented by the following equations

$$p_{\mu}s^{\mu\nu} = 0, \tag{3}$$

where D represents covariant differentiation, τ is an affine parameter along the worldline of the particle, $\dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}$ is the 4-velocity of the particle, p^{μ} is the components of its 4-momentum, $s^{\mu\nu}$ is the particle's spin tensor, q is the particle's charge, $F^{\mu\nu}$ is the electromagnetic tensor, and $k = \frac{qg}{2m}$ is a constant with g being the particle's gyromagnetic ratio and m its mass, and $R^{\mu}{}_{\nu\alpha\beta} = \partial_{\alpha}\Gamma^{\mu}_{\nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\alpha\kappa}\Gamma^{\kappa}_{\nu\beta} - \Gamma^{\mu}_{\beta\kappa}\Gamma^{\kappa}_{\nu\alpha}$. The particle's spin may also be described by a four-vector

$$s^{\mu} = \frac{1}{2m\sqrt{-g}}\varepsilon^{\mu}{}_{\nu\kappa\rho}p^{\nu}s^{\kappa\rho}.$$

It can be shown that these equations lead to

$$\frac{1}{2}s_{\mu\nu}s^{\mu\nu} = s^2 \tag{4}$$

in which the spin s of the particle is constant. We can also deduce the following relation from the DS equations [5]

$$p_{\mu}\dot{x}^{\mu}\frac{d}{d\tau}\big(p_{\mu}p^{\mu}\big) - p_{\mu}p^{\mu}\frac{d}{d\tau}\left(\frac{qg}{2}F_{\mu\nu}s^{\mu\nu}\right) = 0. \tag{5}$$

We fix the gauge by

$$p_{\mu}\dot{x}^{\mu} - \frac{qg}{2m}F_{\mu\nu}s^{\mu\nu} = -m, \tag{6}$$

which reduces to the Dixon's gauge introduced in [18] if we let the electromagnetic field to be absent. Now it follows from Eq. (5) that

$$p_{\mu}p^{\mu} - \frac{qg}{2}F_{\mu\nu}s^{\mu\nu} = -m^2 \tag{7}$$

is a constant of motion.

In the DS framework no direct equations of motion exist for $\dot{x}^{\mu}$, but it can be shown that in the gauge (6), the following relation results from Eqs. (1)–(3)

$$\begin{aligned}\dot{x}^{\mu} &= \frac{1}{m}p^{\mu} \\ &\quad + \frac{qg}{2m(p_{\mu}p^{\mu})}\left(\frac{1}{2}s^{\alpha\beta}D^{\mu}F_{\alpha\beta} - p^{\nu}s^{\mu\kappa}F_{\kappa\nu}\right) \\ &\quad + \frac{s^{\mu\nu}f^{\kappa}l_{\nu\kappa}}{1 + \frac{1}{2}s^{\kappa\rho}l_{\kappa\rho}},\end{aligned} \tag{8}$$

where

$$l_{\nu\kappa} = \frac{1}{p_{\mu}p^{\mu}}\left(-\frac{1}{2}R_{\nu\kappa\alpha\beta}s^{\alpha\beta} + qF_{\nu\kappa}\right).$$

The above equation reduces to the equation given in [19] in the case of $F_{\mu\nu}=0$.

3. Worldline deviations

Let us now consider a one-parameter family of worldlines $\{x^\mu(\tau,\lambda)\}$ describing the worldlines of charged spinning particles of the same spin-to-mass ratios and the same charge-to-mass ratios. In this family, worldlines are obtained from a specific "fiducial" worldline via $x^\mu \to x^\mu + \Delta\lambda n^\mu$ with $n^\mu = \frac{dx^\mu}{d\lambda}$. We now define

$$J^{\mu\nu} = \frac{Ds^{\mu\nu}}{D\lambda}, \qquad j^\mu = \frac{Dp^\mu}{D\lambda}. \tag{9}$$

Now by setting $x^\mu \to x^\mu + \Delta\lambda n^\mu$ in Eqs. (1)–(3), and keeping only linear terms of $\Delta\lambda$ in the resulting equations, and making use of

$$\frac{D\dot{x}^\mu}{D\lambda} = \frac{Dn^\mu}{D\tau},$$

we obtain the worldline deviation equations, which read

$$\begin{aligned}\frac{Dj^\mu}{D\tau} &= -R^\mu{}_{\beta\alpha\kappa}\dot{x}^\kappa n^\alpha p^\beta - \frac{1}{2}n^\kappa D_\kappa R^\mu{}_{\nu\lambda\rho}s^{\lambda\rho}\dot{x}^\nu \\ &\quad - \frac{1}{2}R^\mu{}_{\nu\lambda\rho}J^{\lambda\rho}\dot{x}^\nu - \frac{1}{2}R^\mu{}_{\nu\lambda\rho}s^{\lambda\rho}\frac{Dn^\nu}{D\tau} \\ &\quad + qF^\mu{}_\beta\frac{Dn^\beta}{D\tau} + q\frac{DF^\mu{}_\alpha}{D\tau}\dot{x}^\alpha \\ &\quad + \frac{qg}{4m}J^{\kappa\rho}D^\mu F_{\kappa\rho} + \frac{qg}{4m}s^{\kappa\rho}n^\nu D_\nu D^\mu F_{\kappa\rho},\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{DJ^{\mu\nu}}{D\tau} &= s^{\kappa[\mu}R^{\nu]}{}_{\kappa\alpha\beta}n^\alpha\dot{x}^\beta + p^{[\mu}\frac{D}{D\tau}n^{\nu]} + j^{[\mu}\dot{x}^{\nu]} \\ &\quad - \frac{qg}{2m}J^{[\mu\kappa}F_\kappa{}^{\nu]} - \frac{qg}{2m}s^{[\mu\alpha}\frac{D}{D\lambda}F_\alpha{}^{\nu]},\end{aligned} \tag{11}$$

and

$$s_{\mu\nu}j^\nu + J_{\mu\nu}p^\nu = 0, \tag{12}$$

respectively. Here, $A^{[\mu}B^{\nu]}$ means $A^\mu B^\nu - A^\nu B^\mu$. Similarly one can obtain from Eqs. (4) and (7) the following useful relations

$$s_{\mu\nu}J^{\mu\nu} = 0, \tag{13}$$

$$p_\mu j^\mu - \frac{qg}{4}F_{\mu\nu}J^{\mu\nu} - \frac{qg}{4}s^{\mu\nu}\frac{DF_{\mu\nu}}{D\lambda} = 0, \tag{14}$$

respectively. Also the relation (6) leads to

$$j_\mu\dot{x}^\mu + p_\mu\frac{Dn^\mu}{D\tau} - \frac{qg}{2m}F_{\mu\nu}J^{\mu\nu} - \frac{qg}{2m}s^{\mu\nu}\frac{DF_{\mu\nu}}{D\lambda} = 0. \tag{15}$$

To find the deviation n^μ itself, one should solve the above equations for j^μ, $J^{\mu\nu}$ and find n^μ from these indirectly. Having this in mind, a useful equation may be obtained by starting from Eq. (8) and following the same procedure described above. Thus we have

$$\frac{Dn^\mu(\tau)}{D\tau} = \frac{1}{m}j^\mu(\tau) + \frac{D}{D\lambda}\left(\frac{r^\mu}{p_\mu p^\mu}\right) \tag{16}$$

where

$$\begin{aligned}r^\mu = s^{\mu\nu}\bigg(&-\frac{1}{2}R_{\nu\kappa\alpha\beta}\dot{x}^\kappa s^{\alpha\beta} + qF_{\nu\kappa}\dot{x}^\kappa \\ &+ \frac{qg}{4m}s^{\alpha\beta}D_\nu F_{\alpha\beta}\bigg) - \frac{qg}{2m}p^\nu s^{\mu\kappa}F_{\kappa\nu}.\end{aligned}$$

4. Motion in a gravitational wave

Here, we apply our results to the case of the motion of a charged spinning particle in the space–time of a plane gravitational wave when a uniform magnetic field is present. We take this magnetic field to be in the same direction the wave propagates and the gyromagnetic ratio as $g=2$.

The space–time metric is given by

$$ds^2 = -du\,dv - K(u,x,y)\,du^2 + dx^2 + dy^2 \tag{17}$$

representing a gravitational wave propagating along the z-direction. Here, (u,v) are the light-cone coordinates given by $u=t-z$ and $v=t+z$ and $K(u,x,y)$ is given by

$$K(u,x,y) = f(u)\big(x^2 - y^2\big) \tag{18}$$

in which $f(u)$ is an arbitrary function corresponding to the linear polarization of the wave. We also take the non-vanishing components of the electromagnetic tensor $F_{\mu\nu}$ as follows

$$F_{34} = B \tag{19}$$

which corresponds to a uniform magnetic field in the z-direction. We label the coordinates u, v, x, y with $1, 2, 3, 4$, respectively.

With this metric and field, the DS equations admit the following solution

$$v^\mu = (1, 1, 0, 0),$$
$$p^\mu = (M, M, 0, 0),$$
$$s^{34} = S, \qquad s^{1\mu} = s^{2\mu} = 0, \tag{20}$$

where $M = \sqrt{m^2 - qBS}$. This describes a particle sitting in the origin of the coordinates with its spin directed along the z-direction. We take the worldline of this particle, $(\tau, \tau, 0, 0)$, as a fiducial worldline and calculate the relative acceleration of nearby charged spinning particles and also show that how approximate solutions to the DS equations may be found in the vicinity of the above fiducial worldline. Now by using the deviation equations of the previous section we reach at

$$n^1(\tau) = -n^2(\tau) = \alpha, \tag{21}$$

and consequently

$$n^z(\tau) = -\alpha, \tag{22}$$

where α is a constant. This means that the particles gain no relative velocity in the z-direction. For convenience we set $\alpha = 0$ hereafter. We also obtain

$$\frac{dj^3(\tau)}{d\tau} = f(u)\big(J^{13}(\tau) - Mn^3(\tau)\big) + qB\frac{dn^4(\tau)}{d\tau}, \tag{23}$$

$$\frac{dj^4(\tau)}{d\tau} = -f(u)\big(J^{14}(\tau) - Mn^4(\tau)\big) - qB\frac{dn^3(\tau)}{d\tau}, \tag{24}$$

$$2Sj^4(\tau) = -M\big(J^{13}(\tau) + J^{23}(\tau)\big), \tag{25}$$

$$2Sj^3(\tau) = M\big(J^{14}(\tau) + J^{24}(\tau)\big), \tag{26}$$

$$J^{12}(\tau) = 0, \tag{27}$$

$$J^{34}(\tau) = 0, \tag{28}$$

$$\frac{dJ^{13}(\tau)}{d\tau} = M\frac{dn^3(\tau)}{d\tau} - j^3(\tau) + \omega J^{14}(\tau), \tag{29}$$

$$\frac{dJ^{14}(\tau)}{d\tau} = M\frac{dn^4(\tau)}{d\tau} - j^4(\tau) - \omega J^{13}(\tau), \tag{30}$$

$$\frac{dJ^{23}(\tau)}{d\tau} = M\frac{dn^3(\tau)}{d\tau} - j^3(\tau) + \omega J^{24}(\tau) - 2Sf(u)n^4(\tau), \tag{31}$$

$$\frac{dJ^{24}(\tau)}{d\tau} = M\frac{dn^4(\tau)}{d\tau} - j^4(\tau) - \omega J^{23}(\tau) - 2Sf(u)n^3(\tau), \tag{32}$$

$$\frac{dn^3(\tau)}{d\tau} = \frac{1}{m}j^3(\tau) + \frac{S}{M^2 - qBS}f(u)J^{14}(\tau), \tag{33}$$

$$\frac{dn^4(\tau)}{d\tau} = \frac{1}{m}j^4(\tau) + \frac{S}{M^2 - qBS}f(u)J^{13}(\tau), \tag{34}$$

where $\omega = \frac{qB}{m}$. The solutions to these equations give n^μ and hence the relative accelerations of the particles near the fiducial worldline and also approximate solutions of the DS equations there. These equations are simplified in the interesting situation where $M = 0$, that is in the fine-tuned case of $\frac{s}{m} = \frac{1}{\omega}$. In this case the above equations result in

$$j^3(\tau) = 0 = j^4(\tau), \tag{35}$$

$$J^{12}(\tau) = 0 = J^{34}(\tau), \tag{36}$$

$$J^{13}(\tau) = J\sin(\omega\tau + \phi), \tag{37}$$

$$J^{14}(\tau) = J\cos(\omega\tau + \phi), \tag{38}$$

$$n^3(\tau) = -\frac{J}{qB}\int f(\tau)\cos(\omega\tau + \phi)\,d\tau, \tag{39}$$

$$n^4(\tau) = -\frac{J}{qB}\int f(\tau)\sin(\omega\tau + \phi)\,d\tau, \tag{40}$$

$$J^{23}(\tau) = -\cos(\omega\tau + \chi)\eta(\tau) - \sin(\omega\tau + \chi)\rho(\tau), \tag{41}$$

$$J^{24}(\tau) = \sin(\omega\tau + \chi)\eta(\tau) - \cos(\omega\tau + \chi)\rho(\tau) \tag{42}$$

with J, ϕ, χ being constants and

$$\eta(\tau) = 2S\int \big(n^4(\tau)\cos(\omega\tau + \chi) - n^3(\tau)\sin(\omega\tau + \chi)\big)f(\tau)\,d\tau,$$

$$\rho(\tau) = 2S\int \big(n^3(\tau)\cos(\omega\tau + \chi) + n^4(\tau)\sin(\omega\tau + \chi)\big)f(\tau)\,d\tau.$$

5. Conclusions

In this Letter we have studied worldline deviations of charged spinning particles of the same spin-to-mass and the same charge-to-mass ratios in the framework of the DS equations and determined the effects of the spin-curvature, the charge-electromagnetic field, and the spin-electromagnetic couplings on the relative acceleration of nearby particles. The equations we have found, reduce to those of [2] if the electromagnetic field is turned off.

The equation of geodesic deviation is usually considered in the literature together with an extra equation, namely $\dot{x}_\mu n^\mu = 0$. This is because $\dot{x}_\mu n^\mu$ is constant along a time-like geodesic. However, in the case of spinning particles, the relation $\dot{x}_\mu \dot{x}^\mu = -1$ is no longer guaranteed by the equations of motion and $\dot{x}_\mu n^\mu$ is not a constant along the particle's worldlines. In the framework of the DS equations, one may consider Eq. (15) having the same role the equation $\dot{x}_\mu n^\mu = 0$ plays in the case of geodesics.

It is possible to extend the above worldline deviation equations systematically to equations of higher accuracy by keeping higher order terms in $\Delta\lambda$. These extensions then may be used to calculate the relative accelerations of particles of arbitrary separations.

Another interesting application of our worldline deviation equations is to generate approximate solutions of the DS equations from a given solution. The application of these equations in the study of the dynamics of charged spinning particles in some interesting space–times would be reported in a future work.

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References

[1] A. Balakin, J.W. van Holten, R. Kerner, Class. Quantum Grav. 17 (2000) 5009.
[2] M. Mohseni, Phys. Lett. B 587 (2004) 133.
[3] C. Chicone, B. Mashhoon, B. Punsly, Phys. Lett. A 343 (2005) 1.
[4] W.G. Dixon, Nuovo Cimento 34 (1964) 317.
[5] J.M. Souriau, Ann. Inst. H. Poincaré A 20 (1974) 315.
[6] R. Hojman, S. Hojman, Phys. Rev. D 15 (1977) 2724.
[7] A.R. Prasanna, K.S. Virbhadra, Phys. Lett. A 138 (1989) 242.
[8] A.R. Prasanna, K.S. Virbhadra, Phys. Lett. A 145 (1990) 410.
[9] D. Bini, G. Gemelli, R. Ruffini, Phys. Rev. D 61 (2000) 064013.
[10] Z. Stuchlik, S. Hledik, Phys. Rev. D 64 (2001) 104016.
[11] D. Baleanu, Nuovo Cimento B 111 (1996) 973.
[12] A.R. Prasanna, K.S. Virbhadra, Pramana 32 (1989) 449.
[13] D.B. Papadopoulos, Gen. Relativ. Gravit. 36 (2004) 949.
[14] G. Cognola, S. Soldati, L. Vanzo, S. Zerbini, Phys. Rev. D 25 (1982) 3109.
[15] G. Cognola, R. Soldati, L. Vanzo, S. Zerbini, Nuovo Cimento B 76 (1983) 109.
[16] G. Cognola, L. Vanzo, S. Zerbini, Phys. Lett. B 104 (1981) 88.
[17] I. Bailey, W. Israel, Commun. Math. Phys. 42 (1975) 65.
[18] W.G. Dixon, Proc. R. Soc. London, Ser. A 314 (1970) 499.
[19] K.P. Tod, F. de Felice, M. Calvani, Nuovo Cimento B 34 (1976) 365.