On similarity invariants of matrix commutators and Jordan products

Susana Furtado a, Enide Andrade Martins b,*, Fernando C. Silva c

a Faculdade de Economia, Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal
b Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal
c Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, 1749-016 Lisboa, Portugal

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Abstract

Denote by \([X, Y]\) the additive commutator \(XY - YX\) of two square matrices \(X, Y\) over a field \(F\). In a previous paper, the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of \([\ldots [[A, X_1], X_2], \ldots, X_k] \ldots\), when \(A\) is a fixed matrix and \(X_1, \ldots, X_k\) vary, were studied. Moreover given any expression \(g(X_1, \ldots, X_k)\), obtained from distinct noncommuting variables \(X_1, \ldots, X_k\) by applying recursively the Lie product \([\cdot, \cdot]\) and without using the same variable twice, the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of \(g(X_1, \ldots, X_k)\) when one of the variables \(X_1, \ldots, X_k\) takes a fixed value in \(F^{n \times n}\) and the others vary, were studied.

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* Corresponding author.

E-mail addresses: sbf@fep.up.pt (S. Furtado), enide@mat.ua.pt (E.A. Martins), fcsilva@fc.ul.pt (F.C. Silva).

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The purpose of the present paper is to show that analogous results can be obtained when additive commutators are replaced with multiplicative commutators or Jordan products.

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### 1. Introduction

Some properties of the commutator \([A, X] = AX -XA\), when \(A\) is a fixed matrix and \(X\) varies, have been studied. Suppose that \(F\) is a division ring and \(A \in F^{n \times n}\). The rank of \([A, X]\), when \(X\) runs over \(F^{n \times n}\), was studied in [1]. The same problem, when \(X\) runs over the group of the nonsingular matrices of \(F^{n \times n}\), \(GL_n(F)\), was studied in [9]. The possible numbers of nonconstant invariant polynomials of \([A, X]\), when \(X\) runs over \(F^{n \times n}\) and also when \(X\) runs over \(GL_n(F)\), assuming that \(F\) is an arbitrary field, were studied in [4]. The possible numbers of nonconstant invariant polynomials of \([A, X]\), when \(X\) runs over \(F^{n \times n}\) and also when \(X\) runs over \(GL_n(F)\), where \(F\) is an arbitrary field, were studied in [5].

In [2], using the referred results and assuming that \(F\) is a field where all the irreducible polynomials in \(F[x]\) have degree \(\leq 2\), we have described the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of \([A, X]\), when \(A\) is fixed and \(X_1, \ldots, X_k\) vary; moreover, given any expression \(g(X_1, \ldots, X_k)\), obtained from distinct noncommuting variables \(X_1, \ldots, X_k\) by applying recursively the Lie product \([\cdot, \cdot]\) and without using the same variable twice, we have described the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of \(g(X_1, \ldots, X_k)\) when one of the variables \(X_1, \ldots, X_k\) takes a fixed value in \(F^{n \times n}\) and the others vary.

In this paper, we shall study the corresponding problems that are obtained when additive commutators are replaced with multiplicative commutators or Jordan products. Some results are already known when \(k = 1\) and will be referred later. Most of these results were proved over algebraically closed fields. In this paper, we shall also work over algebraically closed fields.

Let \(F\) be a field and \(A \in F^{n \times n}\). Let \(f_1(x) \cdots f_r(x)\) be the nonconstant invariant polynomials of \(A\) and denote the number \(r\) by \(i(A)\). We shall assume that invariant polynomials and elementary divisors are always monic. In [8], it was proved that

\[
i(A) = n - R_F(A),
\]

where \(F\) is an algebraic closure of \(F\) and

\[
R_F(A) = \min_{\lambda\in F} \text{rank } (A - \lambda I_n).
\]

It is well-known that \(C(f_1) \oplus \cdots \oplus C(f_r)\), where \(C(f_i)\) is the companion matrix of \(f_i\), \(i \in \{1, \ldots, r\}\), and \(\oplus\) denotes direct sum, is similar to \(A\). Recall also that \(A\) is said to be nonderogatory if the minimum polynomial of \(A\), \(f_r\), and the characteristic...
polynomial of $A$, $f_1 \cdots f_r$, coincide. Therefore $A$ is nonderogatory if and only if $i(A) = 1$.

2. Similarity invariants of multiplicative commutators

Throughout this section, $F$ is an algebraically closed field, $A \in \text{GL}_n(F)$ and $f_1(x) | \cdots | f_r(x)$ are the nonconstant invariant polynomials of $A$. If $X \in \text{GL}_n(F)$, $\langle A, X \rangle$ denotes the multiplicative commutator $AXA^{-1}X^{-1}$.

The main purpose of this section is to study the possible eigenvalues and numbers of nonconstant invariant polynomials of

$$B = \langle \cdots \langle \langle A, X_1 \rangle, X_2 \rangle, \ldots, X_k \rangle,$$

when $X_1, \ldots, X_k$ vary.

Suppose that $A, A' \in \text{GL}_n(F)$ are similar and $A' = P^{-1}AP$, with $P \in \text{GL}_n(F)$. Then a matrix of the form (1), where $X_1, \ldots, X_k \in \text{GL}_n(F)$, is similar to $\langle \cdots \langle A', X'_1 \rangle, X'_2 \rangle, \ldots, X'_k \rangle$, where $X'_i = P^{-1}X_iP, i \in \{1, \ldots, k\}$. Therefore, when studying possible properties, invariant under similarity, of (1), when $X_1, \ldots, X_k$ vary, the matrix $A$ can be replaced, without loss of generality, by any similar matrix.

Let $\lambda_1, \ldots, \lambda_n \in F$ be the eigenvalues of $A$, without repetitions. Suppose that

$$f_i(x) = \prod_{j=1}^{a_i} (x - \lambda_j)^{v_{ij}}, \text{ for every } i \in \{1, \ldots, r\}.$$

Then the nonconstant invariant polynomials of $A^{-1}$ are $\tilde{f}_1(x) | \cdots | \tilde{f}_r(x)$, where

$$\tilde{f}_j(x) = \prod_{i=1}^{a_i} (x - \lambda_j^{-1})^{v_{ij}}, \text{ for every } i \in \{1, \ldots, r\}.$$

Bearing in mind this remark, when $k = 1$, the following Theorem 1 is a particular case of [12, Theorem 3] and Theorem 3 is a particular case of [11, Theorem 1].

**Theorem 1.** Let $t \in \{1, \ldots, n\}$. There exist $X_1, \ldots, X_k \in \text{GL}_n(F)$ such that

$$i(\cdots \langle \langle A, X_1 \rangle, X_2 \rangle, \ldots, X_k \rangle) = t$$

if and only if

$$2^k i(A) \leq t + (2^k - 1)n.$$  \hfill (3)

**Proof.** The proof is by induction on $k$. The case $k = 1$ is covered by [12, Theorem 3]. Suppose that $k \geq 2$.

**Necessity.** According to Theorem 1, $2i(A) \leq i(A, X_1) + n$. According to the induction assumption,

$$2^{k-1} i(A, X_1) \leq t + (2^{k-1} - 1)n.$$  \hfill (4)

Then (3) follows trivially.
Sufficiency. Let \( s = \max\{1, 2i(A) - n\} \). According to [12, Theorem 3], there exists \( X_1 \in \text{GL}_n(F) \) such that \( i(A, X_1) = s \). It is easy to see that (4) is satisfied. According to the induction assumption, there exist \( X_2, \ldots, X_k \in \text{GL}_n(F) \) such that (2) is satisfied. \( \square \)

**Lemma 2.** Suppose that \( 2i(A) \leq n \). Then there exists \( X \in \text{GL}_n(F) \) such that \( i(A, X) = 1 \) and all the eigenvalues of \( (A, X) \) are different from 1.

**Proof.** The condition \( 2r = 2i(A) \leq n \) implies that \( \deg f_r \geq 2 \).

**Case 1.** Suppose that \( \deg f_r 
\geq 3 \). Let \( c_1, \ldots, c_{n-2} \) be pairwise distinct elements of \( F \setminus \{0, 1\} \) such that \( c_1, \ldots, c_{n-2} \neq 1 \). Let \( H = \{c_1, \ldots, c_{n-2}, 1, 0\} \). Let \( c_{n-1} \) be an element of \( (H \cup \{c_1, \ldots, c_{n-2}\})^{-1} : h \in H \setminus \{0\} \) such that \( c_{n-1}^{-2} \neq (c_1, \ldots, c_{n-2})^{-1} \). Let \( c_n = (c_1, \ldots, c_{n-1})^{-1} \). According to [11, Theorem 1], there exists \( X \in \text{GL}_n(F) \) such that \( (A, X) \) has eigenvalues \( c_1, \ldots, c_n \). Clearly the elements \( 1, c_1, \ldots, c_n \) are pairwise distinct. It follows that \( i(A, X) = 1 \).

**Case 2.** Suppose that \( \deg f_r = 2 \). The condition \( 2r = 2i(A) \leq n \) implies that \( 2r = n \), all the nonconstant invariant polynomials of \( A \) have degree 2 and \( A \) is similar to

\[
\bigoplus_{i=1}^{n/2} A_i.
\]

where \( A_0 \) is the companion matrix of \( f_r \). Without loss of generality, suppose that \( A \) has the form (5). Let \( c_1, \ldots, c_{n/2} \in F \setminus \{0, 1\} \) be pairwise distinct and such that

\[
\{c_1, \ldots, c_{n/2}\} \cap \{c_1^{-1}, \ldots, c_{n/2}^{-1}\} = \emptyset.
\]

According to [11, Theorem 1], for every \( i \in \{1, \ldots, n/2\} \), there exists \( X_i \in \text{GL}_2(F) \), such that \( (A_0, X_i) \) has eigenvalues \( c_i, c_i^{-1} \). Let \( X = X_1 \oplus \cdots \oplus X_{n/2} \). Then \( (A, X) \) has eigenvalues \( c_1, \ldots, c_{n/2}, c_1^{-1}, \ldots, c_{n/2}^{-1} \) and, therefore, \( i(A, X) = 1 \) and 1 is not an eigenvalue of \( (A, X) \). \( \square \)

**Theorem 3.** Let \( c_1, \ldots, c_n \in F \). Then there exist \( X_1, X_2, \ldots, X_k \in \text{GL}_n(F), k \geq 1 \), such that (1) has eigenvalues \( c_1, \ldots, c_n \) if and only if the following conditions are satisfied:

1. \( i(A, X_1) = \max\{1, 2i(A) - n\} \).
2. \( \#\{i \in \{1, \ldots, n\} : c_i = 1\} \geq 2^k i(A) - (2^k - 1)n \).
3. If \( k = 1 \), then, either \( \deg f_r \neq 2 \) or \( \deg f_r = 2 \) and there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that \( c_{\pi(j-1)} c_{\pi(j)} = 1 \) for \( 1 \leq j \leq n - i(A) \), \( c_{\pi(i)} = 1 \), for \( 2(n - i(A)) < i \leq n \).

**Proof.** The proof is by induction on \( k \). The case \( k = 1 \) follows from [11, Theorem 1]. Suppose that \( k \geq 2 \).
Necessity. Suppose that there exist $X_1$, $X_2$, ..., $X_k \in \text{GL}_n(F)$ such that (1) has eigenvalues $c_1$, ..., $c_n$. As multiplicative commutators have determinant 1, $(i_3)$ is trivial. According to the induction assumption,

$$\#\{i \in \{1, \ldots, n\} : c_i = 1\} \geq 2^{k-1} - i(A, X_1) - (2^{k-1} - 1)n.$$  \hspace{1cm} (6)

According to Theorem 1, $i(A, X_1) \geq 2i(A) - n$. Then $(ii_3)$ follows easily.

Sufficiency. Suppose that $(i_3)$ and $(ii_3)$ hold. If $A$ is scalar, then $(ii_3)$ implies that $c_1 = \cdots = c_n = 1$; clearly, for any $X_1, X_2, \ldots, X_k \in \text{GL}_n(F)$, (1) is equal to $I_n$ and has eigenvalues $c_1, \ldots, c_n$. Now suppose that $A$ is nonscalar.

Case 1. Suppose that $k = 2$.

Subcase 1.1. Suppose $2i(A) \leq n$. It follows from Theorem 1 that there exists $X_1 \in \text{GL}_n(F)$ such that $i(A, X_1) = 1$. According to the induction assumption, there exists $X_2 \in \text{GL}_n(F)$ such that $\langle (A, X_1), X_2 \rangle$ has eigenvalues $c_1, \ldots, c_n$. Note that the minimum polynomial of $(A, X_1)$ has degree $n$ and, if $n = 2$, then $c_1c_2 = 1$.

Subcase 1.2. Suppose that $2i(A) > n$. Then $A$ has, at least, $2i(A) - n$ invariant polynomials of degree 1 and is similar to a matrix of the form

$$A = aI_{2i(A) - n} \oplus A_0,$$

where $a \in F \setminus \{0\}$, $A_0 \in F^{2(\text{deg}(A)) \times 2(\text{deg}(A))}$ and $2i(A_0) \leq n - (2i(A) - n) = 2(n - i(A))$. Without loss of generality, suppose that $A$ has this form. According to Lemma 2, there exists $Y \in \text{GL}_{2(n-i(A))}(F)$ such that $i(A_0, Y) = 1$ and the eigenvalues of $\langle A_0, Y \rangle$ are distinct from 1. Let $X_1 = I_{2i(A) - n} \oplus Y$. Then $i(A, X_1) = 2i(A) - n$ and the minimum polynomial of $(A, X_1)$ has degree greater than 2. Note that $4i(A) - 3n = 2i(A, X_1) - n$. Thus, according to the induction assumption, there exists $X_2 \in \text{GL}_n(F)$ such that $(\langle A, X_1 \rangle, X_2)$ has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that $k > 2$. According to Theorem 1, there exists $X_1 \in \text{GL}_n(F)$ such that $i(A, X_1) = \max\{1, 2i(A) - n\}$. Note that the right hand side of (6) is equal to the right hand side of $(ii_3)$, when $i(A, X_1) = 2i(A) - n$; is equal to 1, when $n = 1$; and is less than 1, otherwise. In any case, (6) is satisfied. According to the induction assumption, there exist $X_2, \ldots, X_k \in \text{GL}_n(F)$ such that (1) has eigenvalues $c_1, \ldots, c_n$. \hfill \qed

Let $Z_1, Z_2, \ldots$ be pairwise distinct letters. Let $\mathcal{G}$ be the free group on the letters $Z_1, Z_2, \ldots$. An element $g \in \mathcal{G}$ will be denoted by $g(Z_{i_1}, \ldots, Z_{i_r})$ if there are no letters outside the set $\{Z_{i_1}, \ldots, Z_{i_r}\}$ that occur in $g$. Let $\mathcal{F}$ be the subset of $\mathcal{G}$ that contains all the letters $Z_1, Z_2, \ldots$, their inverses and all the elements $g(Z_{i_1}, \ldots, Z_{i_r})$ obtained by applying recursively the multiplicative commutator $\langle \cdot, \cdot \rangle$ without using the same variable twice.

Let $g(Z_{i_1}, \ldots, Z_{i_r}) \in \mathcal{F}$. For every $X_{i_1}, \ldots, X_{i_r} \in \text{GL}_n(F)$, $g(X_{i_1}, \ldots, X_{i_r})$ will denote the matrix obtained from $g(Z_{i_1}, \ldots, Z_{i_r})$ when the letters $Z_{i_1}, \ldots, Z_{i_r}$ are replaced by $X_{i_1}, \ldots, X_{i_r}$, respectively. Using the first part of this section, it is possible to study the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of $g(X_{i_1}, \ldots, X_{i_r})$ when one of the variables $X_1, \ldots, X_k$ takes a fixed value in $\text{GL}_n(F)$ and the others vary.
The following two theorems are analogous to theorems presented in [2] for additive commutators.

A matrix $C \in \text{GL}_n(F)$ can be written as a multiplicative commutator $(X,Y)$ if and only if $\det C = 1$; moreover $X$ and $Y$ can be chosen with determinant equal to 1. (cf. [3, Section 4.5].) The next theorem follows from this fact by induction.

**Theorem 4.** Let $C \in \text{GL}_n(F)$ be a matrix of determinant 1. Let $h(Z_1, \ldots, Z_t) \in \mathcal{F}$. Then there exist $X_1, \ldots, X_t \in \text{GL}_n(F)$ such that $C = h(X_1, \ldots, X_t)$.

For every $i \in \{1, 2, \ldots\}$, let $\mathcal{F}_i$ be the set of all the elements of $\mathcal{F}$ where $Z_i$ occurs. We define recursively the depth of $Z_i$ in an element $h$ of $\mathcal{F}_i$, denoted $A_i(h)$, as follows: $A_i(Z_i) = A_i(Z_i^{-1}) = 0$; if $h = (f,g)$, then $A_i(h) = A_i(f) + 1$, when $Z_i$ occurs in $f$, and $A_i(h) = A_i(g) + 1$, when $Z_i$ occurs in $g$.

The next theorem reduces the problems of studying the possible eigenvalues and numbers of nonconstant invariant polynomials of $h(X_1, \ldots, X_t)$, when one of the matrices $X_1, \ldots, X_t$ is fixed and the others vary, where $h(Z_1, \ldots, Z_t) \in \mathcal{F} \setminus \{Z_1, Z_2, \ldots\}$, to the problems studied previously.

**Theorem 5.** Let $h(Z_1, \ldots, Z_t) \in \mathcal{F}$, with $\delta = A_1(h) \geq 1$, and $C, A_1 \in \text{GL}_n(F)$. Then the following conditions are equivalent:

(a) There exist $X_2, \ldots, X_t \in \text{GL}_n(F)$ such that $C$ is similar to $h(A_1, X_2, \ldots, X_t)$.

(b) There exist $Y_1, \ldots, Y_\delta \in \text{GL}_n(F)$ such that $C$ is similar to $\langle \cdots \langle \langle A_1, Y_1, Y_2, \ldots, Y_\delta -1 \rangle \cdots \rangle \rangle$.

**Proof.** By induction on $\delta$. Firstly notice that a commutator $(Y,W)$ is similar to $(Y^{-1}, W^{-1})$ and to $(W, Y^{-1})$. Suppose that $h = (f,g)$, $f, g \in \mathcal{F}$. Also suppose that $Z_1$ occurs in $f$. If $Z_1$ occurs in $g$, the argument is analogous. Without loss of generality, $f = f(Z_1, \ldots, Z_t)$, $g = g(Z_{t+1}, \ldots, Z_t)$.

(a) implies (b). If $\delta = 1$, then $C$ is similar either to $\langle A_1, g(X_{t+1}, \ldots, X_t) \rangle$ or to $\langle A_1^{-1}, g(X_{t+1}, \ldots, X_t) \rangle$. Now suppose that $\delta \geq 2$. According to the induction assumption, there exist $P, Y_1, \ldots, Y_{\delta-1} \in \text{GL}_n(F)$ such that

$$f(A_1, X_2, \ldots, X_t) = P^{-1} \langle \cdots \langle \langle A_1, Y_1, Y_2, \ldots, Y_{\delta-1} \rangle \cdots \rangle \rangle P. \quad (7)$$

Then $C$ is similar to

$$Ph(A_1, X_2, \ldots, X_t)P^{-1} = \langle \cdots \langle \langle A_1, Y_1, Y_2, \ldots, Y_{\delta-1} \rangle \cdots \rangle, P \delta(X_{t+1}, \ldots, X_t)P^{-1} \rangle.$$ 

(b) implies (a). Let $d \in F$ be a $n$th root of $\det Y_\delta$.

Suppose that $\delta = 1$. If $f = Z_1^{-1}$, then, bearing in mind Theorem 4, choose $X_{t+1}, \ldots, X_t \in \text{GL}_n(F)$ such that $dY_\delta^{-1} = g(X_{t+1}, \ldots, X_t)$. If $f = Z_1$, choose $X_{t+1}, \ldots, X_t \in \text{GL}_n(F)$ such that $(1/d) Y_\delta = g(X_{t+1}, \ldots, X_t)$. Then $C$ is similar to $h(A_1, X_2, \ldots, X_t)$. 


Suppose that $\delta \geq 2$. Choose $W_{r+1}, \ldots, W_t \in \text{GL}_n(F)$ such that $(1/d)Y_3 = g(W_{r+1}, \ldots, W_t)$. Let $X_k = P^{-1}W_kP$, $k \in \{r+1, \ldots, t\}$. According to the induction assumption, there exist $P, X_2, \ldots, X_r \in \text{GL}_n(F)$ such that (7) holds. Then $C$ is similar to
\[ (Pf(A_1, X_2, \ldots, X_r)P^{-1}, Y_3^t) = Ph(A_1, X_2, \ldots, X_t)P^{-1}. \]

\[ \square \]

3. Similarity invariants of Jordan products

Throughout this section, $F$ is an algebraically closed field of characteristic different from 2, $A \in F^{n \times n}$ and $f_1(x) \cdots f_r(x)$ are the nonconstant invariant polynomials of $A$. If $X \in \text{GL}_n(F)$, $(A, X)$ denotes the Jordan product $AX +XA$.

The main purpose of this section is to study the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of
\[ B = ((A, X_1), X_2, \ldots, X_k), \]

when $X_1, \ldots, X_k$ vary. This problem was already considered in [2] when $F$ has characteristic 2. As in the previous section, when studying this problem, $A$ can be replaced without loss of generality with any similar matrix.

The possible eigenvalues of $(A, X)$, when $X$ varies, were studied in [6]; note that the solutions for the cases $n = 2$ and $n \geq 3$ are different. The possible numbers of nonconstant invariant polynomials of $(A, X)$, when $X$ varies, were studied in [7].

Given $A, B \in F^{n \times n}$, the possible values of rank$(X^{-1}AX - B)$, when $X$ runs over $\text{GL}_n(F)$, were described in [10]. As a particular case, a description of the possible values of rank $(A, X)$, when $X$ runs over $\text{GL}_n(F)$, can be obtained, cf. [7, Lemma 3]. The possible values of rank$(A, X)$, when $X$ runs over $F^{n \times n}$, were described in [7, Remark, p. 175], as a consequence of arguments used with other purposes; in Theorem 7, we shall give a shorter and simpler proof.

Our first step is to characterize the matrices that can be written as a Jordan product of two nonsingular matrices. Note that $A$ is always the Jordan product of two matrices. If $\lambda \in F$ and $k$ is a positive integer, we shall denote by $J_k(\lambda)$ the following Jordan block:
\[ J_k(\lambda) = \lambda I_k + \begin{bmatrix} 0 & I_{k-1} \\ 0 & 0 \end{bmatrix} \in F^{k \times k}. \]

**Theorem 6.** There exist $X, Y \in \text{GL}_n(F)$ such that $A = (X, Y)$ if and only if either $n$ is even or $A \neq 0$.

**Proof.** Necessary condition. In order to get a contradiction, suppose that $n$ is odd and there exist $X, Y \in \text{GL}_n(F)$ such that $0 = (X, Y)$. Then $-X = Y^{-1}XY$ and, therefore, $-X$ and $X$ have the same eigenvalues. As $n$ is odd and $F$ has characteristic different from 2, this is impossible.
Sufficient condition. Note that, without loss of generality, $A$ may be replaced by any similar matrix.

Case 1. Suppose that $n = 2k$, where $k$ is a positive integer, and $A = 0$. Then

$$A = \left( \bigoplus_{i=1}^{k} D_{2}, \bigoplus_{i=1}^{k} E_{2} \right),$$

where $D_{2} = \text{diag}(-1, 1)$, $E_{2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in F^{2 \times 2}$.

Case 2. Suppose that $n = 2$ and $A$ is similar to $J_{2}(0)$. Then

$$J_{2}(0) = ((J_{2}(1))^t, K_{2}), \quad \text{where} \quad K_{2} = \begin{bmatrix} -1/4 & 1/2 \\ 1/4 & -1/4 \end{bmatrix} \in F^{2 \times 2}.$$

Case 3. Suppose that $n = 3$ and $A$ is similar to $J_{3}(0)$. Then

$$J_{3}(0) = ((J_{3}(1))^t, K_{3}), \quad \text{where} \quad K_{3} = \begin{bmatrix} -1/4 & 1/2 & 0 \\ 3/8 & -1/2 & 1/2 \\ -3/8 & 3/8 & -1/4 \end{bmatrix} \in F^{3 \times 3}.$$

Case 4. Suppose that $n = 2k$, where $k$ is an integer greater than 1, and $A$ is similar to $J_{2k}(0)$. Then

$$J_{2k}(0) = \left( \bigoplus_{i=1}^{k} (J_{2}(1))^t, K_{2k} \right),$$

where

$$K_{2k} = \begin{bmatrix} K_{2} & L_{2} & 0 & \cdots & 0 \\ 0 & K_{2} & L_{2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & K_{2} & L_{2} \\ 0 & \cdots & 0 & 0 & K_{2} \end{bmatrix} \in F^{2k \times 2k},$$

$$L_{2} = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} \in F^{2 \times 2},$$

and $K_{2}$ is defined in Case 2.

Case 5. Suppose that $n = 2k + 1$, where $k$ is an integer greater than 1, and $A$ is similar to $J_{2k+1}(0)$. Then

$$J_{2k+1}(0) = \left( (J_{3}(1))^t \oplus \bigoplus_{i=1}^{k-1} (J_{2}(1))^t \right), K_{2k+1} \right),$$

where

$$K_{2k+1} = \begin{bmatrix} K_{3} & L & 0 \\ 0 & K_{2(k-1)} \end{bmatrix} \in F^{(2k+1) \times (2k+1)},$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1/2 & 0 & \cdots & 0 \end{bmatrix} \in F^{3 \times (n-3)},$$

$K_{3}$ and $K_{2(k-1)}$ are defined above.
Case 6. Suppose that \( A \) is similar to \( J_n(\lambda) \), for some \( \lambda \in F \setminus \{0\} \). Then 
\[
J_n(\lambda) = (I_n, (1/2)J_n(\lambda)).
\]

Case 7. Suppose that \( n = 2 \) and \( A \) is similar to \( \text{diag}(0, \lambda) \), for some \( \lambda \in F \). Then 
\[
\text{diag}(0, \lambda) = \begin{pmatrix}
-1 & 0 \\
0 & X_0
\end{pmatrix}, \begin{pmatrix}
0 & a \\
b & Y_0
\end{pmatrix},
\]
where 
\[
a = [1 \ 0 \ \cdots \ 0], \quad b = [0 \ \cdots \ 0 \ 1].
\]

Case 8. Suppose that \( n \geq 3 \) and \( A \) is similar to \( \begin{bmatrix} 0 \end{bmatrix} \oplus J_n-1(\lambda) \), for some \( \lambda \in F \). According to the previous cases, there exist \( X_0, Y_0 \in \text{GL}_{n-1}(F) \) such that \( J_{n-1}(\lambda) = (X_0, Y_0) \) and \( X_0 \) is lower triangular with its main elements equal to 1. Then 
\[
\begin{bmatrix} 0 \end{bmatrix} \oplus J_{n-1}(\lambda) = \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 \\
0 & X_0 & 0 & \cdots & 0 \\
0 & 0 & X_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & X_0
\end{pmatrix}, \begin{pmatrix}
0 & a \\
b & Y_0
\end{pmatrix},
\]
where 
\[
a = [1 \ 0 \ \cdots \ 0], \quad b = [0 \ \cdots \ 0 \ 1].
\]

Case 9. The general case: suppose that either \( n \) is even or \( A/\text{diag}(0) \neq 0 \). Then the Jordan blocks in a Jordan canonical form of \( A \) can be permuted and associated so that one obtains a matrix 
\[
A' = A_1 \oplus \cdots \oplus A_s,
\]
where each block \( A_i \) has one of the forms:

- \( 0 \in F^{n_i \times n_i} \), for some even integer \( n_i \geq 2 \),
- \( J_{n_i}(0) \), for some integer \( n_i \geq 2 \),
- \( J_{n_i}(\lambda) \), for some integer \( n_i \geq 1 \) and some \( \lambda \in F \setminus \{0\} \),
- \( [0] \oplus J_{n_i-1}(\lambda) \), for some integer \( n_i \geq 2 \) and some \( \lambda \in F \).

For every \( i \in \{1, \ldots, s\} \), according to the previous cases, there exist \( X_i, Y_i \in \text{GL}_{n_i}(F) \) such that \( A_i = (X_i, Y_i) \). Then \( A' = (X, Y) \), where \( X = X_1 \oplus \cdots \oplus X_s \) and \( Y = Y_1 \oplus \cdots \oplus Y_s \). \( \square \)

The nonconstant invariant polynomials of \( -A \) are \( \tilde{f}_1(x) | \cdots | \tilde{f}_r(x) \), where 
\[
\tilde{f}_i(x) = (-1)^{d_i} f_i(-x), \quad d_i = \deg f_i, \quad \text{for every} \quad i \in \{1, \ldots, r\}.
\]

For each positive integer \( p \), let \( \tau_p(A) \) be the number of elementary divisors of \( A \) of the form \( x^p \). Let 
\[
\tau(A) := \max \{ \tau_p(A) : p \text{ is odd} \}, \quad \sigma(A) := \min \{ s : \tilde{f}_i | \tilde{f}_{i+s}, \quad \text{for every} \quad i \in \{1, \ldots, r-s\} \}.
\]

**Theorem 7** [7]. Let \( \rho \in \{0, \ldots, n\} \). There exists \( X \in F^{n \times n} \) such that \( \text{rank}(A, X) = \rho \) if and only if \( \rho \leq 2 \text{rank } A \).

**Proof.** The necessity is obvious. Now we shall prove the sufficiency. If \( \rho = 0 \), take \( X = 0 \). Suppose that \( \rho \geq 1 \). Choose the smallest \( s \in \{1, \ldots, r\} \) such that
\[ d = \text{deg}(f_{r-s+1} \cdots f_r) \geq \rho. \]
Without loss of generality, suppose that \( A = A_1 \oplus A_2 \), where
\[ A_1 = C(f_1) \oplus \cdots \oplus C(f_{r-s}), \quad A_2 = C(f_{r-s+1}) \oplus \cdots \oplus C(f_r). \]

As \( A_2 \) has exactly \( s \) nonconstant invariant polynomials, it follows from the definition that \( \sigma(A_2) \leq s \). If \( s = 1 \), then \( s \leq \rho \); if \( s \geq 2 \), then \( s - 1 \leq \text{deg}(f_{r-s+2} \cdots f_r) < \rho \). In any case, \( \sigma(A_2) \leq s \leq \rho \).

If \( f_{r-s+1} = x \), then \( C(f_1) = \cdots = C(f_{r-s}) = x \) and \( \rho \leq 2 \text{ rank } A = 2 \text{ rank } A_2 \).

If \( f_{r-s+1} \neq x \), then \( \rho \leq d \leq 2 \text{ rank } A_2 \).

As \( \sigma(A_2) \leq \rho \leq 2 \text{ rank } A_2 \), it follows from [7, Lemma 3] that there exists \( Y \in \text{GL}_d(F) \) such that \( \text{rank } (A_2, Y) = \rho \). Take \( X = 0_{n-d} \oplus Y \in F^{n \times n} \). Then \( \text{rank } (A, X) = \rho \). \( \square \)

**Lemma 8.** There exist \( X_1, \ldots, X_k \in \text{GL}_n(F) \) such that all the nonconstant invariant polynomials of (8), except at most the minimum polynomial, are equal to \( x \).

\[
i(\cdots((A, X_1), X_2), \ldots, X_k) = \max\{1, n - 2^k \text{ rank } A\}, \quad (9)
\]
\[
\text{rank } (\cdots((A, X_1), X_2), \ldots, X_k) = \min\{n, 2^k \text{ rank } A\}, \quad (10)
\]
\[
\sigma(\cdots((A, X_1), X_2), \ldots, X_k) \leq 1. \quad (11)
\]

**Proof.** By induction on \( k \). The result is trivial when \( A = 0 \). Suppose that \( A \neq 0 \).

**Case 1.** Suppose that \( k = 1 \). If \( A = \lambda I_n, \lambda \in F \setminus \{0\} \), choose a nonderogatory matrix \( X_1 \in \text{GL}_n(F) \). Then (9)-(11) are satisfied. Now suppose that \( A \) is nonscalar.

**Subcase 1.1.** Suppose that \( n \leq 2 \text{ rank } A \) and the minimum polynomial of \( A \) is different from \( x^2 \). Choose pairwise distinct elements \( c_1, \ldots, c_n \) of \( F \setminus \{0\} \). According to [6], there exists \( X_1 \in \text{GL}_n(F) \) such that \( (A, X_1) \) has eigenvalues \( c_1, \ldots, c_n \). As \( c_1, \ldots, c_n, 0 \) are pairwise distinct, \( i(A, X_1) = 1 \) and \( \text{rank } (A, X_1) = n \). As \( i(A, X_1) = 1 \), it follows that \( \sigma(A, X_1) \leq 1 \).

**Subcase 1.2.** Suppose that \( n \leq 2 \text{ rank } A \) and the minimum polynomial of \( A \) is \( x^2 \). Then all the nonconstant invariant polynomials of \( A \) are equal to \( x^2 \) and \( A \) is similar to
\[
A' = \bigoplus_{i=1}^{n/2} A_0, \quad \text{where } A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]
Without loss of generality, suppose that \( A = A' \). Choose pairwise distinct elements \( c_1, \ldots, c_{n/2} \) of \( F \setminus \{0\} \). For every \( i \in \{1, \ldots, n/2\} \),
\[
A_0 Y_i + Y_i A_0 = \begin{bmatrix} c_i & 2 \\ 0 & c_i \end{bmatrix}, \quad \text{where } Y_i = \begin{bmatrix} 1 & 0 \\ c_i & 1 \end{bmatrix}.
\]
Let \( X_1 = Y_1 \oplus \cdots \oplus Y_{n/2} \). Then \( i(A, X_1) = 1 \) and \( \text{rank } (A, X_1) = n \). As \( i(A, X_1) = 1 \), it follows that \( \sigma(A, X_1) \leq 1 \).

**Subcase 1.3.** Suppose that \( n > 2 \text{ rank } A \). Then \( A \) has at least \( n - 2 \text{ rank } A \) invariant polynomials equal to \( x \) and \( A \) is similar to \( A' = 0_{n-2 \text{ rank } A} \oplus A_0 \), where \( A_0 \in \text{GL}_n(F) \).
Without loss of generality, suppose that $A = A'$. According to the previous subcases, there exists $Y \in \text{GL}_{2 \text{rank} A}(F)$ such that $i(A_0, Y) = 1$ and rank$(A_0, Y) = 2 \text{rank} A_0$. Take $X_1 = I_{n-2 \text{rank} A} \oplus Y$. Then $i(A, X_1) = n - 2 \text{rank} A$ and rank$(A, X_1) = 2 \text{rank} A$. Note that $(A, X_1)$ has $n - 2 \text{rank} A - 1$ nonconstant invariant polynomials equal to $x$. It follows that $\sigma(A, X_1) \leq 1$.

**Case 2.** Suppose that $k \geq 2$. According to the induction assumption, there exist $X_1, \ldots, X_{k-1} \in \text{GL}_n(F)$ such that the matrix

$$C = (\cdots ((A, X_1), X_2), \ldots, X_{k-1})$$

satisfies

$$\text{rank } C = \min \{ n, 2^{k-1} \text{rank } A \}. \quad (13)$$

According to Case 1, there exists $X_k \in \text{GL}_n(F)$ such that all the nonconstant invariant polynomials of $(C, X_k)$, except at most the minimum polynomial, are equal to $x$,

$$i(C, X_k) = \max \{ 1, n - 2 \text{rank } C \}, \quad (14)$$

$$\text{rank } (C, X_k) = \min \{ n, 2 \text{rank } C \}, \quad (15)$$

$$\sigma(C, X_k) \leq 1. \quad (16)$$

From (13)–(15), (9) and (10) follow easily. □

**Lemma 9.** Suppose that the following exceptional case is not satisfied:

$$(E_9) \quad A = \lambda I_n, \lambda \in F \setminus \{ 0 \}, \text{ and } n \text{ is odd.}$$

Then there exists $X \in \text{GL}_n(F)$ such that $\sigma(A, X) = 0$.

**Proof.** Firstly, we show that, if the exceptional case $(E_9)$ is satisfied, then there exists no $X \in \text{GL}_n(F)$ such that $\sigma(A, X) = 0$. In order to get a contradiction, suppose that $(E_9)$ holds and there exists $X \in \text{GL}_n(F)$ such that $\sigma(A, X) = 0$. According to [7, Lemma 3], there exists $X_2 \in \text{GL}_n(F)$ such that $(A, X, X_2) = 0$. This contradicts Theorem 6, as $(A, X) = 2 \lambda X$ and $X_2$ are both nonsingular. From now on, suppose that $(E_9)$ is not satisfied.

The conclusion is trivial when $A = 0$. Suppose that $A \neq 0$.

**Case 1.** Suppose that $A = \lambda I_n, \lambda \in F \setminus \{ 0 \}$, and $n$ is even. Then

$$(A, X) = \bigoplus_{i=1}^{n/2} \text{diag}(-2\lambda, 2\lambda) \quad \text{where} \quad X = \bigoplus_{i=1}^{n/2} \text{diag}(-1, 1).$$

Clearly $\sigma(A, X) = 0$.

**Case 2.** Suppose that $A$ is nonscalar.

- If $n = 2$, $A$ is singular and the characteristic polynomial of $A$ is different from $x^2$, then, according to [6], there exists $X \in \text{GL}_n(F)$ such that $C = (A, X)$ has eigenvalues $-1, 1$. 

• Otherwise, according to [6], there exists \( X \in \text{GL}_n(F) \) such that \( C = (A, X) \) has all its eigenvalues equal to 0.

In any case, \( \sigma(A, X) = 0 \). □

**Theorem 10.** Let \( \rho \in \{0, \ldots, n\} \). Suppose that \( n \geq 2, k \geq 2 \) and that the following exceptional case is not satisfied:

\( (E_{10}) \ A = \lambda I_n, \lambda \in F \setminus \{0\}, \rho = 0, k = 2 \) and \( n \) is odd.

Then the following conditions are equivalent:

(a10) There exist \( X_1, \ldots, X_k \in F^{n \times n} \) such that

\[
\text{rank}((A, X_1), X_2, \ldots, X_k) = \rho.
\]

(b10) There exist \( X_1, \ldots, X_k \in \text{GL}_n(F) \) such that (17) holds.

(c10) \( \rho \leq 2^k \text{rank}A \).

**Proof.** Firstly, we show that, if the exceptional case \( (E_{10}) \) is satisfied, then (b10) is impossible while (c10) is trivial. In order to get a contradiction, suppose that \( (E_{10}) \) and (b10) hold. Then \( 0 = ((A, X_1), X_2) = (2\lambda X_1, X_2), \) with \( 2\lambda X_1, X_2 \in \text{GL}_n(F) \), what contradicts Theorem 6. From now on, suppose that \( (E_{10}) \) is not satisfied.

(a10) implies (c10). This implication is valid for \( k = 1 \), according to Theorem 7. It is easy to complete the proof with an induction argument on \( k \).

(c10) implies (b10). **Case 1.** Suppose that \( \rho \geq 1 \). According to Lemma 8, there exist \( X_1, \ldots, X_{k-1} \in \text{GL}_n(F) \) such that the matrix (12) satisfies \( \sigma(C) \leq 1 \) and (13). Then

\[
\sigma(C) \leq 1 \leq \rho \leq \min[n, 2^k \text{rank}A] = \min[n, 2 \text{rank}C].
\]

According to [7, Lemma 3], there exists \( X_k \in \text{GL}_n(F) \) such that \( \text{rank}(C, X_k) = \rho \).

**Case 2.** Suppose that \( \rho = 0 \). The implication is trivial when \( A = 0 \). Suppose that \( A \neq 0 \).

• If \( k = 2 \), let \( D = A \).

• If \( k \geq 3 \) and \( A = \lambda I_n, \lambda \in F \setminus \{0\}, i \in \{1, \ldots, k-3\} \), let \( X_i = I_n \), then \( X_{k-2} \in \text{GL}_n(F) \) be a nonscalar matrix and let

\[
D = (\cdots ((A, X_1), X_2), \ldots, X_{k-2}) = 2^{k-2}\lambda X_{k-2}.
\]

• If \( k \geq 3 \) and \( A \) is nonscalar, let \( X_i = I_n, i \in \{1, \ldots, k - 2\} \), and let

\[
D = (\cdots ((A, X_1), X_2), \ldots, X_{k-2}) = 2^{k-2}A.
\]

In any case, according to Lemma 9, there exists \( X_{k-1} \in \text{GL}_n(F) \) such that \( \sigma(D, X_{k-1}) = 0 \). According to [7, Lemma 3], there exists \( X_k \in \text{GL}_n(F) \) such that \( \text{rank}((D, X_{k-1}), X_k) = 0 \). □

**Theorem 11.** Suppose that \( n \geq 2, k \geq 2 \). Let \( c_1, \ldots, c_n \in F \). Then the following statements are equivalent:
(a_{11}) There exist $X_1, \ldots, X_k \in F^{n \times n}$ such that (8) has eigenvalues $c_1, \ldots, c_n$.
(b_{11}) There exist $X_1, \ldots, X_k \in \text{GL}_n(F)$ such that (8) has eigenvalues $c_1, \ldots, c_n$.
(c_{11}) There exist at least $\max\{0, n - 2^k \text{ rank } A\}$ indices $i \in \{1, \ldots, n\}$ such that $c_i = 0$.

**Proof.** (a_{11}) implies (c_{11}). If $n \geq 3$ and (12) is nonscalar, then, according to [6], there exist at least $\nu = \max\{0, n - 2 \text{ rank } C\}$ indices $i \in \{1, \ldots, n\}$ such that $c_i = 0$. Note that the last statement is trivial when either $n = 2$ or (12) is scalar. According to Theorem 10, $\text{rank } C \leq 2^{k-1} \text{ rank } A$. Thus, $\nu \geq \max\{0, n - 2^k \text{ rank } A\}$.

(c_{11}) implies (b_{11}). If $A = 0$, this implication is trivial. Suppose that $A \neq 0$. According to Lemma 8, there exist $X_1, \ldots, X_{k-1} \in \text{GL}_n(F)$ such that all the non-constant invariant polynomials of (12), except at most the minimum polynomial, are equal to $x$, $\sigma(C) \leq 1$ and (13) is satisfied. From (13) it follows that $\text{rank } C \geq 2$. Therefore the minimum polynomial of $C$ has to be different from $x^2$. Note that $\max\{0, n - 2^k \text{ rank } A\} = \max\{0, n - 2 \text{ rank } C\}$.

According to [6], there exists $X_k \in \text{GL}_n(F)$ such that $(C, X_k)$ has eigenvalues $c_1, \ldots, c_n$. □

**Theorem 12.** Suppose that $n \geq 2$ and $k \geq 2$. Let $t \in \{1, \ldots, n\}$. Then the following statements are equivalent:

(a_{12}) There exist $X_1, \ldots, X_k \in F^{n \times n}$ such that
$$i(\cdots (A, X_1), X_2), \ldots, X_k) = t.$$  \hspace{1cm} (18)

(b_{12}) There exist $X_1, \ldots, X_k \in \text{GL}_n(F)$ such that (18) holds.

(c_{12}) $t \geq n - 2^k \text{ rank } A$.

**Proof.** (a_{12}) implies (c_{12}). According to [7, Theorem 2], the matrix (12) satisfies $i(C, X_k) \geq n - 2 \text{ rank } C$. According to either [7, Lemma 3] or Theorem 10, $\text{rank } C \leq 2^{k-1} \text{ rank } A$. Therefore, $i(C, X_k) \geq n - 2^k \text{ rank } A$.

(c_{12}) implies (b_{12}). Case 1. Suppose that $t < n$. According to Lemma 8, there exist $X_1, \ldots, X_{k-1} \in \text{GL}_n(F)$ such that the matrix (12) satisfies $\sigma(C) \leq 1$ and (13). Then
$$n - 2 \text{ rank } C = \max\{-n, n - 2^k \text{ rank } A\} \leq t \leq n - \sigma(C)$$
$$\leq n - \min(t(C), \sigma(C)),$$
According to [7, Theorem 1], there exists $X_k \in \text{GL}_n(F)$ such that $i(C, X_k) = t$.

Case 2. Suppose that $t = n$. If $A$ is scalar, then (18) is satisfied with $X_1 = \cdots = X_k = I_n$.

Suppose that $A$ is nonscalar. If $k = 2$, let $D = A$; if $k \geq 3$, let $X_1 = \cdots = X_{k-2} = I_n$ and $D = (\cdots ((A, X_1), X_2), \ldots, X_{k-2})$. According to Lemma 9, there exists
$X_{k-1} \in \text{GL}_n(F)$ such that $\sigma(D, X_{k-1}) = 0$. According to [7, Theorem 1], there exists $X_k \in \text{GL}_n(F)$ such that $i((D, X_{k-1}), X_k) = n$. □

As in the previous section and as in [2], given any expression $g(X_1, \ldots, X_k)$, obtained from distinct noncommuting variables $X_1, \ldots, X_k$ by applying recursively the Jordan product $(\cdot, \cdot)$ and without using the same variable twice, we could easily describe the possible eigenvalues, ranks and numbers of nonconstant invariant polynomials of $g(X_1, \ldots, X_k)$ when one of the variables $X_1, \ldots, X_k$ takes a fixed value in $F^n \times n$ (respectively, $\text{GL}_n(F)$) and the others vary.

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References