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Applications of additive sequence of permutations

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ABSTRACT

Let X^1 be the m-vector $(-r, -r+1, \ldots, -1, 0, 1, \ldots, r-1, r)$, $m = 2r+1$, and $X^2, \ldots, X^n$ be permutations of X^1. Then $X^1, X^2, \ldots, X^n$ is said to be an additive sequence of permutations (ASP) of order m and length n if the vector sum of every subsequence of consecutive permutations is again a permutation of X^1. ASPs had been extensively studied and used to construct perfect difference families. In this paper, ASPs are used to construct perfect difference families and properly centered permutation matrices (which are related to radar arrays). More existence results on perfect difference families and properly centered permutation matrices are obtained.

1. Introduction

Given an additive group G of order v, a (v, k, λ) *difference family* ((v, k, λ)-DF in short) over G is a family of subsets of G (*base blocks*) having size k and such that each non-zero element of G can be represented as the difference of two elements of some base block in exactly λ ways. The interested reader may refer [11] for more details about difference families.

It is clear that the necessary conditions for the existence of a (v, k, λ)-DF are $\lambda(v-1) \equiv 0 (\mathrm{mod}\ k(k-1))$ and $v \geq k$. Much work had been done on the existence of (v, k, λ)-DFs (see [4–10]).

Let $\mathscr{D} = \{D_1, D_2, \ldots, D_h\}$, where $D_i = \{x_{i1}, x_{i2}, \ldots, x_{ik}\}$ is a collection of h subsets of $Z_g = \{0, 1, \ldots, g-1\}$ called *blocks*. If the differences $\Delta\mathscr{D} = \{x_{im} - x_{in} : i = 1, 2, \ldots, h, 1 \leq n < m \leq k\}$ cover the set $\{1, 2, \ldots, (g-1)/2\}$, then we call $\mathscr{D}$ a *perfect* $(g, k, 1)$ *difference family*, or briefly, a perfect $(g, k, 1)$-DF. A perfect $(g, k, 1)$-DF is also a *graceful labeling* of a graph with g connected components all isomorphic to the complete graph on k-vertices [18].

A perfect $(g, k, 1)$-DF is a powerful tool for constructing optimal optical orthogonal codes [2]. It is not difficult to see that the necessary conditions for the existence of a perfect $(v, k, 1)$-DF are $v - 1 \equiv 0 (\mathrm{mod}\ k(k-1))$ and $v \geq k$. The existence of perfect $(v, 3, 1)$-DFs was solved. It was also proved that there are no perfect $(v, k, 1)$-DFs for $k \geq 6$. The following is a brief summary of known results on perfect difference families.

Theorem 1.1 (*See [1,14,16]*).

(1) *If* $v \equiv 1, 7 (\mathrm{mod}\ 24)$, *then there exists a perfect* $(v, 3, 1)$-*DF.*
(2) *Let* $v = 12t + 1$. *Then perfect* $(v, 4, 1)$-*DFs exist for the following values of* $t < 50 : 1, 4-33, 36, 41$.
(3) *Suppose a perfect* $(12t+1, 4, 1)$-*DF exists. Then perfect* $(v, 4, 1)$-*DFs exist for* $v = 60t+13, 156t+13, 228t+49, 276t+61, 300t+61$ *and* $300t+73$.
(4) *Let* $v = 20t + 1$. *Then perfect* $(v, 5, 1)$-*DFs are known for* $t = 6, 8, 10$ *but for no other small values of* t.

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(5) *There are no perfect* $(v, k, 1)$*-DFs for the following values:*
  (a) $k = 3$, $v \equiv 13, 19 \pmod{24}$,
  (b) $k = 4$, $v \in \{25, 37\}$,
  (c) $k = 5$, $v \equiv 21 \pmod{40}$ *or* $v \in \{41, 81\}$,
  (d) $k \geq 6$.

In this paper, we focus our attention on perfect $(v, k, 1)$-DFs for $k = 4, 5$. The following results are obtained.

Theorem 1.2. *If there exist both a perfect* $(v, k, 1)$*-DF and a perfect* $(w, k, 1)$*-DF, then there exists a perfect* $(vw, k, 1)$*-DF, for* $k = 4, 5$.

Theorem 1.3. *Suppose a perfect* $(12t+1, 4, 1)$*-DF exists. Then perfect* $(L, 4, 1)$*-DFs exist for* $L = 324t+61, 324t+73, 348t+73, 348t+85$.

In [20], properly centered permutation matrices are introduced to construct radar arrays. A permutation matrix is a square matrix containing exactly one dot in each row and in each column. Let $\mathscr{P}_L$ be the collection of all $L \times L$ permutation matrices. As in [20], we denote an $L \times L$ permutation matrix by an $L \times 1$ vector whose elements correspond to the column positions of dots. Let $A, B \in \mathscr{P}_L$, and $A = (a_1, a_2, \ldots, a_L)^{\mathrm{T}}$, $B = (b_1, b_2, \ldots, b_L)^{\mathrm{T}}$. Subtract b_i from a_i term by term to form the ''set of differences'' of A and B, which is denoted as

$$S_{AB} = \{a_1 - b_1, a_2 - b_2, \ldots, a_L - b_L\}.$$

If $S_{AB} = \{0, \pm 1, \pm 2, \ldots, \pm \frac{L-1}{2}\}$ for some $A, B \in \mathscr{P}_L$ where L is an odd integer, then we say that S_{AB} is properly centered, and A and B are a pair of properly centered permutation matrices. A set of n pairwise properly centered $L \times L$ permutation matrices can be briefly denoted as an $L \times n$ array, where the n columns correspond to the n properly centered $L \times L$ permutation matrices.

In their conclusion section, Zhang and Tu [20] posed the following open problems:

Finding permutation matrices with pairwise properly centered sets of differences is an open problem. Except for $L = 5, 7, 13$ (and their products), we did not find any other cases. There are three interesting questions regarding these special permutation matrices.

(1) Do they exist for other values of L?
(2) Can we find more than four permutation matrices with such properties?
(3) Does there exist a systematic method (except for the direct product method used here) for constructing these permutation matrices?

Ge, Ling and Miao [12] gave positive answers to Problems (1) and (3) by constructing such properly centered permutation matrices.

In this paper, more properly centered permutation matrices are constructed, and the following results are obtained.

Theorem 1.4. (1) *There exist sets of 3 pairwise properly centered* $v \times v$ *permutation matrices for each* $v = \prod_{i=6}^{14}(2i+1)^{a_{i-5}} 5^{b_1} 7^{b_2} 45^{b_3} 121^{b_4} 161^{b_5} 201^{5_6}$*, where* $a_1, \ldots, a_9, b_1, \ldots, b_6$ *are non-negative integers, not all equal to zero;*

(2) *There exist sets of 4 pairwise properly centered* $w \times w$ *permutation matrices for each* $w = 5^{a_1} 121^{a_2} 161^{a_3} 201^{a_4}$*, where* $a_1, \ldots, a_4$ *are non-negative integers, not all equal to zero.*

From Theorem 1.4 and general constructions in Section B of [20], more radar arrays can be obtained.

2. Application to perfect difference families

Additive sequence of permutations were first considered in [13]. In [16], additive sequence of permutations were used to construct perfect difference families. Let X^1 be the m-vector $(-r, -r+1, \ldots, -1, 0, 1, \ldots, r-1, r)$, $m = 2r+1$, and $X^2, \ldots, X^n$ be permutations of X^1. Then $X^1, X^2, \ldots, X^n$ is said to be an *additive sequence of permutations* of order m and length n, or briefly ASP(m, n), if the vector sum of every subsequence of consecutive permutations is again a permutation of X^1. ASPs had been extensively studied, it also can be used to construct optimal optical orthogonal codes and optimal constant weight cyclically permutable codes. The interested readers may refer to [3,4,14,15,17], and the references therein for the details.

In [4], an $n \times k$ difference array, which is also called a difference matrix, denoted by (n, k)-DM, is used to the recursive construction of optimal constant weight cyclically permutable codes. Let n be an odd integer. An (n, k)-DM over Z_n is called *perfect* if the entries of this matrix are all lie in $\{0, 1, \ldots, (n-1)/2\}$.

Lemma 2.1. *A perfect* $(v, k+1)$*-DM is equivalent to an ASP*(v, k).

Proof. Suppose D is a perfect $(v, k+1)$-DM. Let Y^i be the i'th row of D, $1 \leq i \leq k+1$. Set $X^i = Y^{i+1} - Y^i$, $1 \leq i \leq k$, then $X^1, X^2, \ldots, X^k$ is an ASP(v, k).

Suppose $X^1, X^2, \ldots, X^k$ is an ASP(v, k). Let Y^0 be a row of v zeros, and $Y^i = Y^{i-1} + X^i$, $1 \le i \le k$. Then M is a $(v, k+1)$-DM, where the i'th row of M is Y^i, $0 \le i \le k$. For each $0 \le j \le v-1$, suppose $y_{i_j j}$ is the smallest element in column j, add $-y_{i_j j}$ to all entries in column j, the resulting matrix M' is a perfect $(v, k+1)$-DM. This completes the proof. $\square$

Remark. If we add $(v-1)/2$ to all entries of M in Lemma 2.1, and delete the 1st row, then the resulting matrix is a *PHUDM*(k, v) in Section 3.

The following result was stated in [4].

Lemma 2.2. *If there exist both a (v_1, k)-DM and a (v_2, k)-DM, then there exists a $(v_1 v_2, k)$-DM.*

Similar to the proof of Lemma 2.2, one can obtain the following result.

Lemma 2.3. *If there exist both a perfect (v_1, k)-DM and a perfect (v_2, k)-DM, then there exists a perfect $(v_1 v_2, k)$-DM.*

From Lemmas 2.1 and 2.3, one can obtain the following product construction of additive sequence of permutations in [15] (no explicit construction appeared in [15]).

Theorem 2.4. *If there exist both an ASP(v_1, k) and an ASP(v_2, k), then there exists an ASP$(v_1 v_2, k)$.*

Lemma 2.5. *For $k \in \{4, 5\}$, if there exists a perfect $(k(k-1)t+1, k, 1)$-DF, then there exist an ASP$(k(k-1)t+1, k-1)$.*

Proof. For $k = 4$, the result comes from [16]. For $k = 5$, the result from [19]. We can also prove the result as follows.

Let $v = k(k-1)t+1$, where k is a prime power. If there exists a $(v, k, 1)$-DF, then one can obtain a matrix D: (1) replace each base block $\{a_1, a_2, \ldots, a_k\}$ by the $k(k-1)$ columns of $k-1$ orthogonal Latin squares (MOLS) of order k on the symbols $\{a_1, a_2, \ldots, a_k\}$; (2) add a column of k zeros. It is clear that D is a (v, k)-DM. It is not difficult to check that if the $(v, k, 1)$-DF is perfect, then the (v, k)-DM is also perfect. The conclusion comes from Lemma 2.1. $\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Since a perfect $(w, k, 1)$-DF exists, then an ASP$(w, k-1)$ exists from Lemma 2.5, thus a perfect (w, k)-DM exists from Lemma 2.1. The conclusion comes from the recursive construction of optimal constant weight cyclically permutable codes in [4]. $\square$

In the reminder of this section, we will prove Theorem 1.3.

In order to construct new perfect difference families, one needs to find ASPs.

Lemma 2.6. *There exists an ASP$(m, 3)$ for each $m \in \{21, 27, 29\}$.*

Proof. For $m = 21$:

$$\begin{aligned}
X^1 &= (-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)\\
X^2 &= (0, 1, -1, 2, 3, -2, 4, 5, 8, 9, 10, 6, 7, -9, -8, -7, -5, -4, -3, -10, -6)\\
X^3 &= (6, 7, 1, 5, -7, 0, -9, -8, 2, -10, -5, 3, -3, 10, -1, 9, 8, -6, -4, 4, -2)
\end{aligned}$$

For $m = 27$:

$$\begin{aligned}
X^1 &= (-13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)\\
X^2 &= (0, 1, -1, 2, 3, -2, 4, -3, 5, 6, 10, 11, 12, 13, 7, 8, 9, -11, -10, -8, -6, -12, -4, -7, -5, -13, -9)\\
X^3 &= (2, 4, 8, -1, 10, 13, -9, 9, -13, -12, -10, -8, -2, -5, 5, -11, 0, 1, -3, 12, -6, 11, -7, 3, -4, 6, 7)
\end{aligned}$$

For $m = 29$:

$$\begin{aligned}
X^1 &= (-14, -13, -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1,\\
&\quad 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14)\\
X^2 &= (0, 1, -1, 2, 3, -2, 4, -3, 5, 6, 7, 11, 12, 13, 14, 8, 9, 10, -12, -11,\\
&\quad -9, -7, -13, -4, -8, -5, -14, -6, -10)\\
X^3 &= (5, 7, 11, 9, -4, -3, -8, 12, -12, 8, -13, -14, -11, -6, -1, -2,\\
&\quad 3, -10, 0, 14, 13, -7, 2, -9, 10, -5, 6, 4, 1). \quad \square
\end{aligned}$$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. For $m \in \{27, 29\}$, let X^1, X^2, X^3 be the ASP in Lemma 2.6, and

$$\alpha^l = \{\alpha_1^l, \alpha_2^l, \ldots, \alpha_m^l\} = \sum_{1 \le i \le l} X^i, \quad 1 \le l \le 3.$$

Let $D_i = \{0, a_i, b_i, c_i\}$, $1 \le i \le t$ be the perfect $(12t+1, 4, 1)$-DF,

$$B_{mi-m+j} = \{0, ma_i + \alpha_1^j, mb_i + \alpha_2^j, mc_i + \alpha_3^j\}, \quad 1 \le i \le t,\ 1 \le j \le m,$$

and $\mathscr{B}_m = \bigcup_{1\le s\le mt} B_s$. Then

$$\Delta\mathscr{B}_m = \bigcup_{1\le s\le mt} \Delta B_s = \{(m-1)/2+1, (m-1)/2+2, \dots, (m-1)/2+6mt\}.$$

For $(m, x) \in \{(27, 61), (27, 73), (29, 73), (29, 85)\}$, define

$$\begin{aligned}
\mathscr{R}^{(27,61)} &= \{\{0, 1, 6tm+18, 6tm+23\}, \{0, 2, 12, 6tm+26\}, \{0, 3, 11, 6tm+30\},\\
&\quad \{0, 4, 6tm+20, 6tm+29\}, \{0, 7, 13, 6tm+28\}\}.\\
\mathscr{R}^{(27,73)} &= \{\{0, 1, 10, 6tm+36\}, \{0, 5, 6tm+19, 6tm+25\}, \{0, 2, 6tm+24, 6tm+32\},\\
&\quad \{0, 3, 6tm+18, 6tm+31\}, \{0, 4, 6tm+21, 6tm+33\}, \{0, 7, 6tm+23, 6tm+34\}\}.\\
\mathscr{R}^{(29,73)} &= \{\{0, 1, 5, 6tm+25\}, \{0, 2, 6tm+18, 6tm+32\}, \{0, 3, 13, 6tm+34\}, \{0, 6, 6tm+23, 6tm+35\},\\
&\quad \{0, 7, 6tm+22, 6tm+33\}, \{0, 8, 6tm+27, 6tm+36\}\}.\\
\mathscr{R}^{(29,85)} &= \{\{0, 1, 6tm+16, 6tm+20\}, \{0, 2, 6tm+31, 6tm+40\}, \{0, 3, 6tm+26, 6tm+39\},\\
&\quad \{0, 5, 6tm+22, 6tm+33\}, \{0, 6, 6tm+27, 6tm+41\},\\
&\quad \{0, 7, 6tm+25, 6tm+37\}, \{0, 8, 6tm+32, 6tm+42\}\}.
\end{aligned}$$

Then it is not difficult to check that $\mathscr{B}_m \bigcup \mathscr{R}^{(m,x)}$ form a perfect $(L, 4, 1)$-DF, where $L = 12tm + x \in \{324t+61, 324t+73, 348t+73, 348t+85\}$. This completes the proof. □

Remark. From the known existence results of perfect $(v, 4, 1)$-DFs and new results in Theorem 1.3, one can obtain more existence results of perfect $(L, 4, 1)$-DFs. For example, since perfect $(v, 4, 1)$-DFs exist for $v = 61, 73, 85$, then there exist perfect $(L, 4, 1)$-DFs for $L = 732t+61, 876t+73, 1020t+85$ provided that a perfect $(12t+1, 4, 1)$-DF exists. One can also obtain more perfect $(v, 4, 1)$-DFs by the product construction in Theorem 1.2.

At the end of this section, we will use our construction to improve result (2) of perfect $(12t+1, 4, 1)$-DFs in Theorem 1.1.

Theorem 2.7. *Let $v = 12t+1$. Then perfect $(v, 4, 1)$-DFs exist for the following values of $t < 50$: $1, 4-36, 41, 46$.*

Proof. From Theorem 1.1, one needs only to prove that there exists a perfect $(12t+1, 4, 1)$-DF for $t = 34, 35$ and 46.

Let

$$\begin{aligned}
\mathscr{D} = \{&\{0, 1, 202, 204\}, \{0, 3, 178, 200\}, \{0, 4, 110, 155\}, \{0, 5, 117, 170\}, \{0, 6, 109, 191\},\\
&\{0, 7, 97, 184\}, \{0, 8, 85, 126\}, \{0, 9, 138, 159\}, \{0, 11, 76, 116\}, \{0, 12, 179, 192\},\\
&\{0, 14, 162, 187\}, \{0, 15, 95, 157\}, \{0, 16, 114, 174\}, \{0, 18, 49, 92\}, \{0, 19, 149, 172\},\\
&\{0, 20, 111, 181\}, \{0, 26, 119, 190\}, \{0, 27, 79, 96\}, \{0, 28, 143, 199\}, \{0, 29, 123, 133\},\\
&\{0, 30, 154, 196\}, \{0, 32, 121, 169\}, \{0, 35, 108, 163\}, \{0, 37, 144, 168\}, \{0, 38, 99, 183\},\\
&\{0, 46, 146, 182\}, \{0, 54, 113, 194\}, \{0, 58, 160, 193\}, \{0, 63, 141, 188\}, \{0, 64, 147, 198\},\\
&\{0, 66, 152, 186\}, \{0, 67, 139, 189\}, \{0, 68, 156, 195\}, \{0, 75, 132, 176\}\}.
\end{aligned}$$

Then, $\mathscr{D}$ is a perfect $(12 \times 34 + 1, 4, 1)$-DF.

Let $s = 1$, a perfect $(12s+1, 4, 1)$-DF exists from Theorem 1.1, and hence a perfect $(348s+73, 4, 1)$-DF exists from Theorem 1.3. Since $348s+73 = 421 = 12 \times 35 + 1$, then a perfect $(12 \times 35 + 1, 4, 1)$-DF exists. A perfect $(12 \times 9 + 1, 4, 1)$-DF exists from Theorem 1.1. Since $12 \times 46 + 1 = 553 = 60 \times 9 + 13$, then a perfect $(12 \times 46 + 1, 4, 1)$-DF exists from Theorem 1.1. This completes the proof. □

3. Application to properly centered permutation matrices

In [12], Ge, Ling and Miao put forward the definition of homogeneous uniform difference matrix ($HUDM(n, L)$ in short). Let $D = (d_{ij})$, $0 \le i \le k-1$, $0 \le j \le v-1$, be a $k \times v$ matrix with entries from the set $I_v = \{0, 1, \dots, v-1\}$. D is called a *uniform difference matrix*, denoted by $UDM(k, v)$, if for all $0 \le s < t \le k-1$, the sets D_{ts} of differences $\{d_{tj} - d_{sj} : 0 \le j \le v-1\}$ are all identical, and for any two distinct differences $d_1, d_2 \in D_{ts}$, $d_1 - d_2 \not\equiv 0 (\mathrm{mod}\ v)$ always holds. A $k \times v$ uniform difference matrix $D = (d_{ij})$ over $I_v = \{0, 1, \dots, v-1\}$ is said to be *homogeneous*, denoted by $HUDM(k, v)$, if the entries of each row of D comprise all the elements of $I_v = \{0, 1, \dots, v-1\}$. For our purpose, an $HUDM(k, v)$ with the property that for all $0 \le s < t \le k-1$, the sets D_{ts} of differences $\{d_{tj} - d_{sj} : 0 \le j \le v-1\}$ are all equal to $\{0, \pm 1, \pm 2, \dots, \pm \frac{v-1}{2}\}$ is called *perfect*, and denoted by $PHUDM(k, v)$.

It is not difficult to see that the transpose of the $L \times n$ array obtained from a set of n pairwise properly centered $L \times L$ permutation matrices is equivalent to an $PHUDM(n, L)$.

In the following, we will construct $PHUDM(n, L)$s from ASPs. In [3], it is shown that there exists an ASP$(v, 3)$ for each $v \in \{5, 7, 13, 15, 17, 19, 45, 121, 161, 201\}$. It is also mentioned that by using the "multiplication" of additive sequence of permutations as outlined in [15], there exists an ASP$(w, 3)$, where $w = 5^a 7^b 13^c 15^d 17^e 19^f 45^g 121^h 161^i 201^j$, $a, b, \ldots, j$ are non-negative integers, not all equal to zero.

Lemma 3.1. *If there exists an* ASP(v, k), *then there exists a* $PHUDM(k, v)$, *and hence there exist sets of k pairwise properly centered* $v \times v$ *permutation matrices.*

Proof. Suppose $X^1, X^2, \ldots, X^k$ be an ASP(v, k), where $X^i = (x_1^i, x_2^i, \ldots, x_v^i)$, $1 \le i \le v$, and $v = 2r + 1$. Let $d_{ij} = x_1^j + x_2^j +, \ldots, +x_i^j + r$, $1 \le i \le k$, $1 \le j \le v$. According to the definition of ASP, it is obvious that $D = (d_{ij})$, $1 \le i \le k$, $1 \le j \le v$ is a $k \times v$ matrix, with entries of each row comprise all the elements of $I_v = \{0, 1, \ldots, v-1\}$. Furthermore, for any $1 \le s < t \le k$, the set $D_{ts} = \{d_{tj} - d_{sj} : 1 \le j \le v\} = \{x | x \in X^{s+1} + X^{s+2}, \ldots, X^t\} = \{-r, -r+1, \ldots, -1, 0, 1, \ldots, r-1, r\}$. Since $v = 2r + 1$, then it is clear that for any two distinct differences $d_1, d_2 \in D_{ts}$, $d_1 - d_2 \not\equiv 0 (\mathrm{mod}\ v)$. So, $D = (d_{ij})$ is a $PHUDM(m, v)$. This completes the proof. $\square$

Example. A $PHUDM(3, 21)$ constructed from the ASP(21, 3) in Lemma 2.6.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 0 & 2 & 1 & 5 & 7 & 3 & 10 & 12 & 16 & 18 & 20 & 17 & 19 & 4 & 6 & 8 & 11 & 13 & 15 & 9 & 14 \\ 6 & 9 & 2 & 10 & 0 & 3 & 1 & 4 & 18 & 8 & 15 & 20 & 16 & 14 & 5 & 17 & 19 & 7 & 11 & 13 & 12 \end{pmatrix}.$$

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. From [3], there exists an ASP$(v_1, 3)$ for each $v_1 \in \{5, 7, 13, 15, 17, 19, 45, 121, 161, 201\}$. From [16], there exists an ASP$(v_2, 3)$ for each $v_2 \in \{23, 25\}$. From Lemma 2.6, there exists an ASP$(v_3, 3)$ for each $v_3 \in \{21, 27, 29\}$. So, from Theorem 2.4, there exists an ASP$(v, 3)$ for each $v = \prod_{i=6}^{14} (2i+1)^{a_{i-5}} 5^{b_1} 7^{b_2} 45^{b_3} 121^{b_4} 161^{b_5} 201^{5_6}$, where $a_1, \ldots, a_9, b_1, \ldots, b_6$ are non-negative integers, not all equal to zero. Conclusion (1) of Theorem 1.4 comes from Lemma 3.1.

It is shown in [3] that there exists an ASP(5, 4). ASP$(v_4, 4)$s for $v_4 \in \{121, 161, 201\}$ are from Theorem 1.1 and Lemma 2.5. So, there exists an ASP$(w, 4)$ for each $w = 5^{a_1} 121^{a_2} 161^{a_3} 201^{a_4}$, where $a_1, \ldots, a_4$ are non-negative integers, not all equal to zero. Conclusion (2) of Theorem 1.4 comes from Lemma 3.1. This completes the proof. $\square$

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