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Covering the Edges of a Connected Graph by Paths

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We prove that every connected graph on n vertices can be covered by at most $n/2 + O(n^{3/4})$ paths. This implies that a weak version of a well-known conjecture of Gallai is asymptotically true. © 1996 Academic Press, Inc.

Introduction

The following classical result is due to Lovász [8].

Theorem. *Every graph G on n vertices can be covered by $\leqslant \lfloor n/2 \rfloor$ edge-disjoint paths and cycles.*

How do we get rid of the paths? We do not know. Indeed, the old conjecture of Erdős and Gallai that $O(n)$ edge-disjoint cycles and edges are sufficient to cover the edges of G is still open (see [10] for more details). On the other hand, if we do not require the cycles to be edge-disjoint then by a result of the author [9], $n-1$ cycles and edges suffice.

It is much easier to get rid of the cycles as shown by the following.

Corollary [8]. *Let G be a graph on n vertices.*

(i) *Then G can be covered by $n-1$ edge-disjoint paths.*

(ii) *If each vertex of G has odd degree then G can be covered by $n/2$ edge-disjoint paths.*

Part (i) was strengthened by Donald [2] who proved that $\lfloor (3/4)n \rfloor$ edge-disjoint paths suffice.

Our first result is an extension of (ii).

Theorem 0. *Suppose that each cycle of G contains a vertex of odd degree. Then G can be covered by $\leqslant \lfloor n/2 \rfloor$ edge-disjoint paths.*

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EXAMPLE. Let G be the graph obtained from the complete graph K_{2m+1} by the deletion of $m-1$ independent edges. Then G has exactly $(n-1)\lfloor n/2 \rfloor + 1$ edges ($n = 2m+1$) and therefore we need $\lfloor n/2 \rfloor + 1$ paths to cover G. On the other hand, the only cycle of G with all degrees even is a triangle.

This example shows that Theorem 0 is best possible.

Lovász's results were motivated by a classical conjecture of Gallai.

Conjecture. Every connected graph G on n vertices can be covered by $\lfloor (n+1)/2 \rfloor$ edge-disjoint paths.

This would, of course, be best possible. Although there are strong partial results, it appears to be very difficult to obtain a full proof of this conjecture.

Chung [1] suggested the investigation of the case when the covering paths are not necessarily edge-disjoint. Our main result is the following.

THEOREM I. *Every connected graph G on n vertices can be covered by $n/2 + O(n^{3/4})$ paths.*

In the case when the number of edges of G is small we have a stronger bound.

THEOREM II. *Every connected graph on n vertices with e edges can be covered by $n/2 + 4(e/n)$ paths.*

It is curious to note that one cannot prove an asymptotic version of Gallai's conjecture without proving the conjecture itself as shown by the following.

EXAMPLE. Suppose H is an m-vertex counterexample to Gallai's conjecture; i.e., any path partition of H contains at least $m/2 + 1$ elements. Let the n-vertex graph G consist of a vertex v and k vertex-disjoint copies of H, each of these connected to v by an edge. It is straightforward to see that we need at least $k(m/2+1) - \lfloor k/2 \rfloor$ paths to partition G. This number is at least $n/2 + n/(2m+1)$ for $k \geq 2m+3$, which cannot be bounded by $n/2 + o(n)$ for $n \to \infty$.

1. EDGE-DISJOINT COVERINGS

Suppose we are given a path and cycle partition Σ of a graph G where the number of cycles cannot be reduced. In this section we obtain some rather weak-looking results about the properties of such a partition Σ.

These results will be used in the next section to prove our theorems about path coverings.

The proof of the next lemma uses the method of Lovász [8] (for other similar proofs see [2, 7, 9]).

LEMMA 1.1. *Let Σ be a path and cycle partition of a graph G with the number of cycles minimal among path and cycle partitions having at most $|\Sigma|$ elements. Let C be a cycle of Σ and x an arbitrary vertex of C. There exist two vertices $y, z \in V(C)$ such that $(x, y), (x, z) \in E(G)$ and both y and z have even degree in G.*

Proof. Let a_1 and a_2 be the C-neighbours of x. For $i=1, 2$ we define a sequence of G-neighbours $a_{i,0}, ..., a_{i,r_i}$ of x recursively as follows. Let $a_{i,0}=a_i$. Suppose $a_{i,\mu}$ is defined. If $a_{i,\mu}$ has even degree then end the sequence. If $a_{i,\mu}$ has odd degree then it is the endvertex of a path $P_{i,\mu} \in \Sigma$. If $P_{i,\mu}$ does not pass through x then end the sequence (we will show that this is impossible). If x lies on $P_{i,\mu}$, trace the path from $a_{i,\mu}$ to x and let $a_{i,\mu+1}$ be the last vertex on the path before it reaches x.

First we prove that all the vertices $a_{i,\mu}$ are different. For suppose $a_{i,\mu}=a_{j,\nu}$ and $\mu \leqslant \nu$. If $\mu=0$ then $a_{i,\mu}=a_i$, and as $(a_i, x) \in E(C)$ it cannot be an edge of a path $P_{j,\nu-1}$; i.e., we have $\nu=0$ and $i=j$. If $\mu \geqslant 1$ then $a_{i,\mu}$ lies on $P_{i,\mu-1}$, which has an endvertex at $a_{i,\mu-1}$ and contains the edge $(x, a_{i,\mu})$. Similarly, $a_{i,\mu}=a_{j,\nu}$ lies on $P_{j,\nu-1}$, which contains $(x, a_{j,\nu})=(x, a_{i,\mu})$. Since the paths of Σ are edge-disjoint, we have $P_{i,\mu-1}=P_{j,\nu-1}$ and $a_{i,\mu-1}=a_{j,\nu-1}$. By repeating this argument we obtain $a_{i,0}=a_{j,\nu-\mu}$ and $\nu-\mu=0$. Therefore, $\mu=\nu$ and $i=j$.

This implies that the above sequences are finite.

We show next that the last vertices of the sequences, a_{1,r_1} and a_{2,r_2}, have even degrees. Otherwise, by definition there exists a path P_{i,r_i} starting at a_{i,r_i} that does not pass through x. Exchange $C, P_{i,0}, ..., P_{i,r_i-1}, P_{i,r_i}$ with the paths

$$C\backslash(a_{i,0}, x),\ P_{i,0} \cup (a_{i,0}, x)\backslash(a_{i,1}, x), ...,$$

$$P_{i,r_{i-1}} \cup (a_{i,r_{i-1}}, x)\backslash(a_{i,r_i}, x),\ P_{i,r_i} \cup (a_{i,r_i}, x).$$

This way we obtain a path and cycle partition Σ' with $|\Sigma'|=|\Sigma|$, having fewer cycles, a contradiction.

Finally we observe that the a_{i,r_i} are vertices of C. Otherwise, exchange C, $P_{i,0}, ..., P_{i,r_{i-1}}$ with the paths

$$C \cup (a_{i,r_i}, x)\backslash(a_{i,0}, x),\ P_{i,0} \cup (a_{i,0}, x)\backslash(a_{i,1}, x), ...,$$

$$P_{i,r_{i-1}} \cup (a_{i,r_{i-1}}, x)\backslash(a_{i,r_i}, x).$$

Again this would give a path and cycle partition Σ' with $|\Sigma'| = |\Sigma|$, having fewer cycles, a contradiction.

Therefore $y = a_{1, r_1}$ and $z = a_{2, r_2}$ are vertices of C with the required properties. ∎

COROLLARY 1.2. *Let Σ be a cycle and path partition as in Lemma* 1.1. *Then for each cycle $C \in \Sigma$ there is a cycle K of G such that $V(K) \subseteq V(C)$ and the vertices of K have even degree in G.*

Proof. Consider the subgraph H of G induced by vertices of C having even degree in G. By Lemma 1.1 the minimum degree of vertices in H is at least two and our statement follows. ∎

Theorem 0 is an obvious consequence of the previous corollary and Lovász' theorem.

COROLLARY 1.3. *Let G be a k-connected graph on n vertices. Then G can be covered by $\lfloor n/2 \rfloor + \lceil n/2k \rceil$ paths.*

Proof. Let I be a maximal set of independent edges in G between vertices of even degree. By Theorem 0, the edges of $G \backslash I$ can be covered by $\lfloor n/2 \rfloor$ paths. By a result of Häggkvist and Thomassen [4], every set of $k-1$ independent edges of a k-connected graph G is contained in some cycle of G. It follows that any set of k edges from I is contained in some path of G. We have $|I| \leqslant n/2$ and therefore the edges in I can be covered by at most $\lceil n/2k \rceil$ paths of G. ∎

LEMMA 1.4. *Let the graph H be an edge-disjoint but not vertex-disjoint union of two cycles C_1 and C_2. Suppose H cannot be covered by two paths. Then C_1 has an edge e with both endvertices in $V(C_2)$.*

Proof. Suppose there is no such edge e. Choose $x \in V(C_1) \cap V(C_2)$. Denote the C_1-neighbours of x by a_1 and b_1 and the C_2-neighbours by a_2 and b_2. By the indirect hypothesis, $a_1, b_1 \notin V(C_2)$ and therefore a_1 and b_1 are vertices of degree two in H.

We claim that a_2 and b_2 are both vertices of C_1. For if, say, $a_2 \notin V(C_1)$ then H is the union of the paths $C_1 \cup (x, a_2) \backslash (x, a_1)$ and $C_2 \cup (x, a_1) \backslash (x, a_2)$.

By a simple induction argument we obtain that $V(C_2) \subset V(C_1)$.

The vertices of C_2 "cut" C_1 into paths. Replacing these paths by edges we obtain a 4-regular multigraph H' with a Hamiltonian decomposition C_1', C_2. By a beautiful result of Thomason [12], H' has another decomposition. Therefore H has another decomposition C_3, C_4 into the union of edge-disjoint cycles. It follows that there is a vertex $x \in V(C_2)$ with C_1-neighbours a_1, b_1 such that, say, $(a_1, x) \in E(C_3)$ and $(b_1, x) \in E(C_4)$. As

a_1 and b_1 have degree two in H we have $a_1 \notin V(C_4)$ and $b_1 \notin V(C_3)$. Now H is the union of the paths $C_3 \cup (b_1, x) \backslash (a_1, x)$ and $C_4 \cup (a_1, x) \backslash (b_1, x)$, a contradiction. ∎

In order to improve the results of this paper it would be crucial to find a much stronger version of the above lemma. Perhaps it is possible to characterize pairs of edge-disjoint cycles C_1, C_2 that cannot be covered by two paths.

LEMMA 1.5. *Let the graph H be an edge-disjoint but not vertex-disjoint union of a cycle C and a path P. Suppose H can not be covered by two paths. Then C has an edge e with both endvertices in P.*

Proof. Let a be an endvertex of P, let x be the nearest common vertex of P and C, and let y be a C-neighbour of x. For the subpath P_0 of P connecting a and x we have $V(C) \cap V(P_0) = x$. If $y \notin V(P)$ then H is the union of the paths $C \cap P_0 \backslash (x, y)$ and $P \cup (x, y) \backslash P_0$, a contradiction. ∎

2. THE PROOFS OF THEOREMS I AND II

Lemma 1.1 suggests that if the subgraph induced by the even vertices of a graph G is "sparse" then it is easier to find a small path covering of G. To make this lemma applicable we need the following.

LEMMA 2.1. *Let G be a connected graph on n vertices and let $x \leqslant n$ be an even number. There exists a subgraph $R \subset G$ such that*

(i) *any path P of the graph $G \backslash R$ contains at most x vertices that have even degree in $G \backslash R$*

(ii) *there is at most one vertex w_0 of even degree greater than x^2 in $G \backslash R$, and*

(iii) *the edges of R can be covered by $2(n/x)$ paths of G.*

Proof. Let V denote the set of even vertices of G. We define a chain of subsets of V; $V_0, V_1, ..., V_r$, recursively as follows. Let $V_0 = V$. Suppose V_i is defined. If there is a path P_{i+1} of G containing x vertices from V_i, then by omitting these vertices from V_i we obtain V_{i+1} (here P_{i+1} is allowed to contain any number of vertices outside V_i). If there is no such path, end the series. Obviously we have $r \leqslant n/x$.

Let us consider the subgraph H formed by edges of G with at least one vertex in V_r. If P is a path in H, then at least $\lfloor |P|/2 \rfloor$ of its vertices belong to V_r, and therefore P has length at most $2x-1$. By a result of Erdős and Gallai [3], if a graph on n vertices has at least $x \cdot n$ edges then it contains a path of length at least $2x$. Therefore we have $|E(H)| < nx$. Let W denote the set of vertices from V_r that have degree at least x^2. It follows that $|W| \leqslant 2(n/x)$.

Now we are going to define R. For $t = \lfloor |W|/2 \rfloor$ take t paths $Q_1, ..., Q_t$ arbitrarily connecting distinct pairs of vertices from W. Each P_i $(i = 1, ..., r)$ contains x vertices $a_i^1, a_i^2, ..., a_i^x$ from V_{i-1}. Consider the subpaths $P_i^1, P_i^2, ..., P_i^{x/2}$ of $P - i$ connecting a_i^1 and a_i^2, a_i^3 and $a_i^4, ..., a_i^{x-1}$ and a_i^x. Define R as the mod 2 sum of the paths $Q_1, ..., Q_t$ and all the paths P_i^j.

In $G \backslash R$ only the vertices in $V_r \backslash W$ have even degree, apart from at most one vertex from W. Now (i) and (ii) are satisfied by the definitions of V_r and W. The paths $P_1, ..., P_r, Q_1, ..., Q_t$ cover the edges of R and therefore (iii) is also satisfied. ∎

Proof of Theorem I. Choose x to be the smallest even number with $x \geqslant n^{1/4}$ and consider a subgraph $R \subset G$ as in Lemma 2.1.

Consider a covering Σ of $G \backslash R$ by $\leqslant \lfloor n/2 \rfloor$ edge-disjoint paths and cycles such that the number of cycles is minimal.

We claim that, any cycle C of Σ has length at most x^3.

Let W denote the set of vertices w of C having even degree in $G \backslash R$, $w \neq w_0$ (it is of course possible that $w_0 \notin V(C)$). By 2.1(i) we have $|W| \leqslant x$ and by 2.1(ii) the vertices in W have at most $|W| \cdot x^2 \leqslant x^3$ $G \backslash R$-neighbours in C. On the other hand, by Lemma 1.1 each vertex of C has a $G \backslash R$-neighbour in W and this proves $|C| \leqslant x^3$.

If C_1 and C_2 are two vertex-disjoint cycles from Σ then, as G is connected, it is obvious that C_1 and C_2 can be covered by two paths of G. As long as possible we exchange pairs of (not necessarily vertex-disjoint) cycles from Σ with pairs of paths of G covering these cycles. Finally we end up with a path and cycle covering Σ_1 of $G \backslash R$ with $|\Sigma_1| \leqslant n/2$, such that the cycles $C_1, ..., C_t$ are pairwise intersecting cycles from Σ and the union of C_1 and C_i $(i = 2, ..., t)$ cannot be covered by two paths.

By the above claim C_1 has at most x^3 vertices.

By Lemma 1.4 each C_i has an edge e_i with both endvertices in $V(C_1)$. These edges e_i form a graph H on $\leqslant x^3$ vertices. By the corollary to Lovász's theorem, H can be covered by at most x^3 paths.

The cycles $C_1, ..., C_t$ are covered by the paths $C_i \backslash e_i$ and by the paths covering H. It follows that $G \backslash R$ can be covered by $n/2 + x^3 \leqslant n/2 + (n^{1/4} + 2)^3$ paths of G. R is covered by $2\lfloor n/x \rfloor \leqslant 2 \cdot n^{3/4}$ paths of G and Theorem I follows. ∎

Proof of Theorem II. Let G be a connected graph with e edges. Clearly G has a covering by exactly $\lceil n/2 \rceil$ disjoint paths and cycles (we can break up cycles or paths to obtain exactly $\lceil n/2 \rceil$ elements). Consider such a covering Σ with a minimal number of cycles.

Call an element of Σ "long" if it has at least $4e/n$ edges, otherwise call it "short." It is obvious that there are at most $n/4 \leqslant |\Sigma|/2$ long elements.

Just as in the previous proof we use the fact that the vertex-disjoint union of two cycles or a cycle and a path can always be covered by two paths of G.

As long as possible we exchange pairs consisting of a long cycle and a short element with pairs of paths covering their edges. This way we obtain a covering Σ_1 of G by $\lceil n/2 \rceil$ paths and cycles such that all the cycles are from Σ. Also, the remaining long cycles intersect each of the remaining short elements. (If there is a long cycle in Σ_1 then there is a short element in Σ_1, as the number of long elements is $\leqslant |\Sigma|/2$.)

We continue exchanging pairs of short cycles with pairs of paths covering their edges as long as possible. We obtain a covering Σ_2 with $\lceil n/2 \rceil$ elements.

We claim that all the cycles $C_1, ..., C_t$ of Σ_2 (which are originally from Σ) intersect a certain short element S of Σ. This is clear if there is a short cycle C_j in Σ_2 (in this case set $S = C_j$). Otherwise all the cycles of Σ_2 are long cycles from Σ_1 and these intersect all the short elements in Σ_1.

By the choice of Σ_2 and S it is also true that the union of S and C_i $(i = 1, ..., t)$, $S \neq C_i$, cannot be covered by two paths.

By Lemma's 1.4 and 1.5 each of the C_i has an edge e_i with both endvertices in $V(S)$. We have $|V(S)| \leqslant 4(e/n) + 1$, and it follows from the corollary to Lovász's theorem that the edges e_i can be covered by at most $4e/n$ paths of G. These paths and the pairs $C_i \backslash e_i$ cover the cycles C_i, and Theorem II follows. ∎

3. Infinite Graphs

Lovász [8] observed that the second part of his corollary (see the Introduction) has an extension to infinite graphs.

Theorem. *Let a locally finite graph have only vertices of odd degree. Then it can be covered by edge-disjoint finite paths such that every vertex is the endvertex of just one covering path.*

Here we observe that there is an equivalent version of the first part of the corollary that may also have an extension to infinite graphs. To prove the equivalence, we use a well-known result of Rado [11] on matroids (for definitions see [13]).

Theorem. *If M is a matroid on the set S with rank function r then a finite family of subsets $(A_i; i \in I)$ of S has a transversal which is independent in the matroid M if and only if for all $J \subset I$*

$$r\left(\bigcup_{j \in J} A_j\right) \geqslant |J|.$$

A *cycle-free transversal* of a covering $P_1, ..., P_t$ of a graph G is a set of distinct edges $e_i \in P_i$ such that the subgraph formed by the edges e_i contains no cycle.

THEOREM. *Let G be a finite graph. Then G has a covering $\mathscr{P}$ by edge-disjoint paths such that $\mathscr{P}$ has a cycle-free transversal.*

Proof. Let $\mathscr{P} = \{P_1, ..., P_t\}$ be a covering of G be edge-disjoint paths such that t is minimal. If the union of k elements of $\mathscr{P}$ is a connected subgraph H, then by the minimality of $\mathscr{P}$ we have $|V(H)| \leqslant k+1$ and therefore H contains a tree with k edges. It follows that the union of any k elements of $\mathscr{P}$ contains a forest with k edges; i.e., a k element independent set of the cycle-matroid $\mathscr{C}$ of G.

By applying Rado's theorem to $\mathscr{C}$ and the sets $E(P_i)$, we obtain our result. ▮

It would be interesting to extend the above result to infinite graphs. Note that the extension is obvious if the paths are not required to be edge-disjoint. (For let H be an infinite connected component of a graph G and T a spanning tree of H. There is a bijection $f: E(H) \to E(T)$ and for $e \in E(H)$ we can choose a path P_e containing e and $f(e)$. The edges $f(e)$ form a cycle-free transversal of the covering $\{P_e\}$.)

For similar problems concerning coverings of infinite graphs by cycles, triangles, and subdivisions of K_n see [5, 6, 9, 10].

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