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Global existence for defocusing cubic NLS and Gross–Pitaevskii equations in three dimensional exterior domains

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Abstract

We prove global wellposedness in the energy space of the defocusing cubic nonlinear Schrödinger and Gross–Pitaevskii equations on the exterior of a nontrapping domain in dimension 3. The main ingredient is a Strichartz estimate obtained combining a semi-classical Strichartz estimate [R. Anton, Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains, arxiv:math.AP/0512639, Bull. Soc. Math. France, submitted for publication] with a smoothing effect on exterior domains [N. Burq, P. Gérard, N. Tzvetkov, On nonlinear Schrödinger equations in exterior domains, Ann. I.H.P. (2004) 295–318].

Résumé

On démontre l'existence et l'unicité des solutions globales dans l'espace d'énergie pour les équations de Schrödinger et de Gross–Pitaevskii cubiques à l'extérieur des obstacles non captants de dimension 3. La démonstration répose sur une inégalité de Strichartz obtenue en combinant une inégalité de Strichartz semi-classique [R. Anton, Strichartz inequalities for Lipschitz metrics on manifolds and nonlinear Schrödinger equation on domains, arxiv:math.AP/0512639, Bull. Soc. Math. France, submitted for publication] avec l'effet régularisant à l'extérieur des obstacles non captants [N. Burq, P. Gérard, N. Tzvetkov, On nonlinear Schrödinger equations in exterior domains, Ann. I.H.P. (2004) 295–318].

Keywords: Gross–Pitaevskii; Defocusing; Strichartz estimate

1. Introduction

Let $\Theta \neq \emptyset$, $\Theta \subset \mathbb{R}^3$, a nontrapping obstacle with compact boundary and let $\Omega = \complement\Theta$. In this paper we are interested in the Cauchy problem for the cubic defocusing NLS equation (here written with Dirichlet boundary conditions) on Ω:

$$\begin{cases} \mathrm{i}\partial_t u + \triangle u = |u|^2 u, & \text{on } \mathbb{R} \times \Omega, \\ u|_{t=0} = u_0, & \text{on } \Omega, \\ u|_{\mathbb{R}\times\partial\Omega} = 0. \end{cases} \tag{1}$$

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This equation appears in the nonlinear optics and more generally in propagation of nonlinear waves. For more details on nonlinear Schrödinger equations, see, for example, the books of C. Sulem, P.L. Sulem [25], T. Cazenave [12] and the references therein. This equation preserves, at least formally, the mass and the energy:

$$\|u(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)},$$
$$\mathcal{E}\big(u(t)\big) = \frac{1}{2}\int_\Omega |\nabla_x u|^2(t,x)\,\mathrm{d}x + \frac{1}{4}\int_\Omega |u|^4(t,x)\,\mathrm{d}x = \mathcal{E}(u_0).$$

There is a wide literature on the Cauchy problem in the Euclidean space. One of the main tools in addressing this problem is the Strichartz inequality, which translates the dispersive property of the linear Schrödinger flow. We refer to the work of Strichartz [27], Ginibre–Velo [15] and Keel–Tao [21].

Recently, the question of the influence of the geometry on the solution has been studied. Let us mention the work of J. Bourgain [9] on the tori $\mathbb{T}^d$ for $d = 2, 3$ and of N. Burq–P. Gérard–N. Tzvetkov [10,11] on compact manifold and exterior of nontrapping obstacles.

In recent works on superfluidity and Bose–Einstein condensates (see, for example, the book of A. Aftalion [2]) the following variant of NLS (1) is studied:

$$\begin{cases} \mathrm{i}\partial_t u + \triangle u = (|u|^2 - 1)u, & \text{on } \mathbb{R}\times\Omega, \\ u|_{t=0} = u_0, & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu}|_{\mathbb{R}\times\partial\Omega} = 0. \end{cases} \tag{2}$$

This is called the cubic Gross–Pitaevskii equation with Neumann boundary conditions. This is also a Hamiltonian equation, for the Hamiltonian:

$$\mathcal{E}_{GL}(u) = \frac{1}{2}\int_\Omega |\nabla_x u|^2(x)\,\mathrm{d}x + \frac{1}{4}\int_\Omega \big(|u|^2(x) - 1\big)^2\,\mathrm{d}x,$$

also called Ginzburg–Landau energy.

The main difference between the NLS (1) and the Gross–Pitaevskii equation (2) is in their energy space. For Gross–Pitaevskii it reads

$$E = \big\{u \in H^1_{\mathrm{loc}}(\Omega),\ \nabla u \in L^2(\Omega),\ |u|^2 - 1 \in L^2(\Omega)\big\}.$$

Namely, the initial datum in the energy space, $u_0 \in E$, is not an $L^2(\Omega)$ function. In [6,5,3,17–19,13] the question of existence of traveling waves and vortices is studied. We are interested in showing global wellposedness in the energy space. There have been previous works on the Cauchy problem for the Gross–Pitaevskii equation: P.E. Zhidkov [28,29] in Zhidkov spaces $X^1(\mathbb{R})$, F. Béthuel–J.C. Saut [6] in the space of functions $1 + H^1(\mathbb{R}^d)$, for $d = 2, 3$, P. Gérard in [16] in the energy space on the whole Euclidean space $\mathbb{R}^d$, for $d = 2, 3, 4$, C. Gallo [14] in the energy space $u_0 + H^1(\Omega)$ for exterior domains in $d = 2$.

In dimension 2 the smoothing effect [11] provides wellposedness for both NLS [11] and Gross–Pitaevskii [14], with all power nonlinearities. In dimension 3 the smoothing effect only provides wellposedness of subcubic nonlinearities [11,14]. In order to handle the cubic nonlinearity in the energy space in dimension 3, one has to gain more than $\frac{1}{2}$ of derivatives. For both (1) and (2) the method we use is based on a new Strichartz estimate obtained combining a smoothing effect in exterior domains shown by Burq–Gérard–Tzvetkov [11] with a semiclassical Strichartz estimate on small intervals of time depending on the frequencies where the flow is localized [4]. This idea appears in the work of Staffilani–Tataru [26].

Having a Strichartz inequality we obtain classically a local existence theorem for (1) by Picard iteration scheme. These also enables propagation of the regularity of the initial data. We obtain an existence time that only depends on the H^1 norm of the initial data. Local existence in the energy space $H^1_0(\Omega)$ combined with the conservation of the energy (and for defocusing nonlinearity of the $H^1_0(\Omega)$ norm) enables us to conclude that the solution to (1) is global in time.

Theorem 1.1. *For all $u_0 \in H^1_0(\Omega)$ there exists an unique solution,*

$$u \in C\big(\mathbb{R}, H^1_0(\Omega)\big) \cap L^p_{\mathrm{loc}}\big(\mathbb{R}, L^\infty(\Omega)\big),$$

(*for every* $2 < p < 3$) *of Eq.* (1). *Moreover, for every* $T > 0$ *and for every bounded subset* B *of* $H_0^1(\Omega)$, *the flow* $u_0 \mapsto u$ *is Lipschitz from* B *to* $C([-T,T], H_0^1(\Omega))$. *For* $1 < \sigma \leqslant 2$ *and* $u_0 \in H_0^1(\Omega) \cap H^\sigma(\Omega)$ *we have* $u \in C([-T,T], H_0^1(\Omega) \cap H^\sigma(\Omega))$. *The conservation of the mass and of the energy hold*: *for all* $t \in \mathbb{R}$, $\|u(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}$ *and* $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$.

For Gross–Pitaevskii equation, as $u_0 \in E$ is not an $L^2(\Omega)$ function, the Strichartz inequality does not apply directly. We adapt the arguments of [16] to the boundary case for the description of the structure of E: in dimension 3,

$$E = \{c + v,\ c \in \mathbb{C},\ |c| = 1,\ v \in \dot{H}^1(\Omega),\ |v|^2 + 2Re(c^{-1}v) \in L^2(\Omega)\}.$$

The nontrapping assumption does not have a particular influence into the analysis, the same holds for any compact obstacle with smooth boundary. The natural metric,

$$\delta_E(c + v, \tilde{c} + \tilde{v}) = |c - \tilde{c}| + \|\nabla v - \nabla \tilde{v}\|_{L^2(\Omega)} + \big\||v|^2 + 2Re(\bar{c}v) - |\tilde{v}|^2 - 2Re(\bar{\tilde{c}}\tilde{v})\big\|_{L^2(\Omega)},$$

defines a structure of complete metric space on E. We show that E is stable under the action of the linear flow: for all $u_0 \in E$, $e^{it\triangle}u_0 \in \{u_0\} + H^1(\Omega) \subset E$. For the nonlinear term in the Duhamel formula we use, thanks to the Strichartz inequality on Ω, a fixed point method in the space $C([-T,T], H^1(\Omega)) \cap S$, where $S = L_T^p(L^\infty(\Omega))$ is some Strichartz space, for the functional:

$$\Phi(w) = -i \int_0^t e^{i(t-\tau)\triangle_N} F(u_L + w)(\tau)\, d\tau.$$

We have denoted by $u_L(t) = e^{it\triangle}u_0$, by $w = u - u_L$ and by $F(u) = (|u|^2 - 1)u$. We obtain that the energy is conserved and the existence time depends on $\mathcal{E}_{GL}(u_0)$. Therefore, the global existence theorem for the Gross–Pitaevskii equation (2) follows.

Theorem 1.2. *For all* $u_0 \in E$ *there exists an unique solution,*

$$u \in C(\mathbb{R}, E) \cap L^p_{\mathrm{loc}}(\mathbb{R}, L^\infty(\Omega)),$$

(*for every* $2 < p < 3$) *of Eq.* (2). *Moreover, the following properties hold*: *for every bounded subset* B *of* E *there exists* $T > 0$ *such that for all* $u_0 \in B$ *the flow* $u_0 \mapsto u$ *is Lipschitz from* B *to* $C([-T,T], E)$; *we have* $u - u_L \in C(\mathbb{R}, H^1(\Omega))$; *if* $u_0 \in E$ *is such that* $\triangle u_0 \in L^2(\Omega)$ *and* $\frac{\partial u_0}{\partial \nu} = 0$, *then* $\triangle u \in C(\mathbb{R}, L^2(\Omega))$; *for all* $t \in \mathbb{R}$, $\mathcal{E}_{GL}(u(t)) = \mathcal{E}_{GL}(u_0)$.

Remark 1. After the completion of this work Blair–Smith–Sogge [7] announced an improved Strichartz inequality on boundary domains. They prove a Strichartz inequality with a loss of $\frac{4}{3p}$ derivatives as opposed to the Strichartz inequality [4] with a loss of $\frac{3}{2p}$ derivatives we used here. Using Blair–Smith–Sogge inequality improves our Strichartz inequality (3), obtaining a loss of $\frac{1}{3p} + \varepsilon$ derivatives.

The structure of the paper is as follows: in Section 2 we show how we obtain the Strichartz estimate (3). In Section 3 we give the proof of Theorem 1.1. In Section 4 we deal with the Gross–Pitaevskii equation (1) and we give the proof of Theorem 1.2.

2. Strichartz estimate in exterior domains

The idea is to combine Strichartz inequality on exterior domains [4] with the gain of $\frac{1}{2}$ derivative from the smoothing effect [11]. This idea appears in the work of Staffilani–Tataru [26] and has been used by many authors since (e.g. [10,24,20,8]). Let us recall the definition of an admissible pair.

Definition 1. A pair (p,q) is called admissible in dimension 3 if $p \geqslant 2$, and

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}.$$

The Strichartz inequality we obtain is the following (see also Remark 1):

Proposition 2.1. *For (p,q) an admissible couple in dimension 3 and $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that for all $u_0 \in H_D^{\frac{1}{2p}+\varepsilon}(\Omega)$,*

$$\left\| e^{it\triangle_D} u_0 \right\|_{L^p(I,L^q(\Omega))} \leqslant c_\varepsilon \|u_0\|_{H_D^{\frac{1}{2p}+\varepsilon}(\Omega)}. \tag{3}$$

For (p,q) and $(\tilde{p},\tilde{q})$ admissible couples in dimension 3 and $\varepsilon > 0$, there exists $c > 0$ such that, for all $f \in L^{p'}(I, W^{\frac{1}{2p}+\frac{1}{2\tilde{p}}+\varepsilon,q'}(\Omega))$, where p' denotes the dual exponent of p: $\frac{1}{p}+\frac{1}{p'}=1$, we have:

$$\left\| \int_0^t e^{i(t-\tau)\triangle_D} f(\tau,x)\,d\tau \right\|_{L_t^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} \leqslant c\|f\|_{L^{p'}(I,W^{\frac{1}{2p}+\frac{1}{2\tilde{p}}+\varepsilon}(\Omega))}. \tag{4}$$

A similar result holds for the linear Schrödinger flow with Neumann boundary conditions.

Rather than using the Strichartz estimate with loss of $\frac{3}{2p}+\varepsilon$ derivatives (45) of [4], we prefer to use the Strichartz estimate without loss of derivatives (Proposition 4.13 of [4]), that holds for frequency localized initial data and small intervals of time depending on the frequency.

In order to do that here, we need to recall some of the notations and results from [4]. That is done in Section 2.1. In Section 2.2 we recall the results of N. Burq, P. Gérard and N. Tzvetkov [11] concerning the smoothing effect and Strichartz estimate away from the obstacle. Section 2.3 is the core of this section. We prove a new Strichartz estimate close to the obstacle by combining semiclassical Strichartz estimate and smoothing effect. In Section 2.4 we deduce the proof of Proposition 2.1.

2.1. Preliminaries

We recall here the classical mirror reflection that allows us to pass from a manifold with boundary to a boundaryless manifold. This method consists in taking a copy of the domain and glue it to the initial one by identifying the points of the boundary. In the particular case of a half space, $\mathbb{R}^{d-1}\times\mathbb{R}_+$, the double manifold is $\mathbb{R}^d$ and we can extend the metric symmetrically with respect to the boundary. In the general case, we have to choose the coordinates carefully in order to obtain a manifold. Taking normal coordinates at the boundary is like straightening a neighborhood of the boundary into a cylinder $\partial\Omega\times[0,1)$. Gluing the two cylinders along the boundary makes a nice smooth manifold. This can be properly done using, for example, tubular neighborhoods (e.g., [22, pp. 468 and 74]). Let $M = \Omega\times\{0\}\cup_{\partial\Omega}\Omega\times\{1\}$, where we identify $(p,0)$ with $(p,1)$ for $p\in\partial\Omega$.

Lemma. *(See [22].) There is a unique C^∞ structure on M such that $\Omega\times\{k\}\hookrightarrow M$, $k\in\{0,1\}$, is C^∞ and $\tilde{\chi}: U\times\{0\}\cup_{\partial\Omega} U\times\{1\}\to\partial\Omega\times(-1,1)$ is a diffeomorphism, where U is a small neighborhood of $\partial\Omega$ for which there are deformation retractions onto $\partial\Omega$.*

On M we define the metric G induced by the new coordinates. As we have chosen coordinates in the normal direction close to the boundary, the metric is well defined over the boundary, its coefficients are Lipschitz in local coordinates and diagonal by blocs (no interaction between the normal and the tangent components). Moreover,

$$G\big(r(y)\big) = G(y),$$

where $r: M\to M$, $r(x,0)=(x,1)$, $r^2 = Id$ is the reflection with respect to the boundary $\partial\Omega$.

For the *Dirichlet* problem we introduce the space H^1_{AS} of functions of $H^1(M)$ which are anti-symmetric with respect to the boundary. Let

$$H^1_{AS} = \big\{v: M\to\mathbb{C},\ v\in H^1(M),\ v(y) = -v\big(r(y)\big)\big\}.$$

Note that for $v\in H^1_{AS}$ the restriction $v|_{\Omega\times 0}$ is in $H^1_0(\Omega)$ and every function from H^1_{AS} is obtained from a function of $H^1_0(\Omega)$. We shall prove the stability of H^1_{AS} under the action of $e^{it\triangle_G}$.

By complex interpolation define H^s_{AS} for $s \in [0,1]$ and deduce its stability under the action of $e^{it\triangle_G}$. Moreover, the restriction to Ω of functions in H^s_{AS} belongs to $H^s_D(\Omega)$ and vice versa. This allows us to deduce the Strichartz inequality for $e^{it\triangle_D}$ on Ω from the Strichartz inequality for $e^{it\triangle_G}$ on M.

Similarly, we can define for the *Neumann* problem the space H^1_S of symmetric functions with respect to the boundary. This space is also stable under the action of $e^{it\triangle_G}$,

$$H^1_S = \{v : M \to \mathbb{C},\ v \in H^1(M),\ v(y) = v(r(y))\}.$$

Let us prove the stability of H^1_{AS} under the action of $e^{it\triangle_G}$. Let $v_0 \in H^1_{AS}$ and $v(t,y) = e^{it\triangle_G} v_0$. Then v satisfies to $i\partial_t v(t,y) + \triangle_{G(y)} v(t,y) = 0$, $v(0) = v_0$. Let $\tilde{v}(t,y) = v(t, r(y))$. We shall look for the equation verified by $\tilde{v}$. First note that $\tilde{v}(0) = -v_0$ and $\partial_t \tilde{v}(t,y) = \partial_t v(t,y)$. As G is diagonal by blocks, having no interactions between the normal and tangent components, so is G^{-1}. Thus in $\triangle_{G(y)}$ there is no crossed term. Consequently $\triangle_{G(r(y))}\tilde{v}(t,y) = \triangle_{G(y)} v(t,y)$. We see thus that $\tilde{v}$ satisfies to the linear Schrödinger equation with initial data $-v_0(y)$. But $-v(t,y)$ satisfies the same equations. By uniqueness we conclude that

$$v(t, r(y)) = -v(t,y).$$

Moreover, if $v_0(t,y) = u_0(t,y)$ for all $y \in \Omega$, then $v(t,y) = u(t,y)$ for all t and for all $y \in \Omega$, where $u(t) = e^{it\triangle_D} u_0$.

We prepare the frequency decomposition. We begin with a partition of unity on M. Since M is flat outside a compact set, let $(U_j, \kappa_j)_{j\in J}$ be a covering of the area of M where $G \neq \mathbb{I}\mathrm{d}$. This area is compact, so we can choose J of finite cardinal. We have $M = \bigcup_{j\in J} U_j \cup U_{1,\infty} \cup U_{2,\infty}$, where $U_{1,\infty}$ and $U_{2,\infty}$ are two disjoint neighborhood of infinity, diffeomorphic to $\mathbb{R}^d \setminus \bar{B}$. Let $(\chi_j)_{j\in J}, \chi_{1,\infty}, \chi_{2,\infty} : M \to [0,1]$ be a partition of unity subordinated to the previous covering. For all $j \in J$ let $\tilde{\chi}_j : M \to [0,1]$ be a C^∞ function such that $\tilde{\chi}_j = 1$ on the support of χ_j and the support of $\tilde{\chi}_j$ is contained in U_j. Similarly we define $\tilde{\chi}_{1,\infty}, \tilde{\chi}_{2,\infty} : M \to [0,1]$. Let $\varphi_0 \in C^\infty(\mathbb{R}^d)$ be supported in a ball centered at origin and $\varphi \in C^\infty(\mathbb{R}^d)$ be supported in an annulus such that for all $\lambda \in \mathbb{R}^d$,

$$\varphi_0(\lambda) + \sum_{k\in\mathbb{N}} \varphi(2^{-k}\lambda) = 1. \tag{5}$$

On $\mathcal{S}(\mathbb{R}^d)$, the Schwartz space, we define the Fourier multiplier $\varphi(hD)$ by $(\widehat{\varphi(hD)q})(\xi) = \varphi(h\xi)\hat{q}(\xi)$, for $q \in \mathcal{S}(\mathbb{R}^d)$, $h \in (0,1)$ and $\xi \in \mathbb{R}^d$. In order to define a spectral truncation for $f \in \mathcal{S}(M)$ (M flat outside a compact set), we use the partition of unity to restrict f to a coordinate neighborhood. Then we use the pullback to read it in local coordinates in $\mathbb{R}^d$. Here we apply $\varphi(hD)$. We cut the result with a cut-off function slightly larger than the pullback of the initial one such that we are still in the coordinate neighborhood. We go back on the manifold and sum using the partition of unity over all such neighborhoods. Thus, we define a family of spectral truncations on M: for $f \in C^\infty(M)$ and $h \in (0,1)$, let

$$J_h f = \sum_{j\in J} (\kappa_j)^* \left(\tilde{\chi}_j \varphi(hD) \left(\kappa_j^{-1}\right)^* (\chi_j f)\right) + F_{1,h,\infty} f + F_{2,h,\infty} f, \tag{6}$$

and

$$J_0 f = \sum_{j\in J} (\kappa_j)^* \left(\tilde{\chi}_j \varphi_0(D) \left(\kappa_j^{-1}\right)^* (\chi_j f)\right) + F_{1,0,\infty} f + F_{2,0,\infty} f, \tag{7}$$

where * denotes the usual pullback operation and $F_{l,h,\infty} f = \tilde{\chi}_{l,\infty} \varphi(hD) \chi_{l,\infty} f(x)$, for $l \in \{1,2\}$ correspond to spectral truncations on the flat regions of M (neighborhoods of infinity). For more on pseudo-differential operators see for example [1]. The following identity holds:

$$J_0 f(x) + \sum_{k=0}^{\infty} J_{2^{-k}} f(x) = f(x). \tag{8}$$

This will be useful for decomposing f in spectrally localized functions $J_h f$.

In order to construct a parametrix we need more regularity on the coefficients of the metric than the Lipschitz regularity. Therefore we define a regularized metric G_h as follows: let ψ be a $C_0^\infty(\mathbb{R}^d)$ radially symmetric function with $\psi \equiv 1$ near 0. Let

$$G_h = \sum_{j\in J} (\kappa_j)^* \left(\tilde{\chi}_j \psi\left(h^{\frac{1}{2}} D\right) \left(\kappa_j^{-1}\right)^* (\chi_j G)\right). \tag{9}$$

The transformation of G into G_h does not spoil the symmetry. Note also that G_h converges uniformly in x to G, and thus, for h sufficiently small, G_h is positive definite. Therefore, G_h is still a metric. We present some properties of metrics G and G_h.

Lemma. *The metric $G : M \to M_d(\mathbb{R})$ is symmetric, positive definite and its coefficients are Lipschitz: there exist $c, C, c_1 > 0$ such that for all $j \in J \cup \{(1,\infty),(2,\infty)\}$, for all $x \in U_j$,*

$$c\,\mathbb{I}\mathrm{d} \leqslant G(x) \leqslant C\,\mathbb{I}\mathrm{d}, \quad |\partial G| \leqslant c_1,$$

where we have denoted by ∂G the derivatives of the coefficients of the metric in the sense of distributions on U_j. The coefficients of the regularized metric G_h are C^∞ functions that verify the followings: there exists $c, C > 0$ and $c_\gamma > 0$ such that for all $j \in J \cup \{(1,\infty),(2,\infty)\}$, for all $x \in U_j$ and every $\gamma \in \mathbb{N}^d$:

$$c\,\mathbb{I}\mathrm{d} \leqslant G_h(x) \leqslant C\,\mathbb{I}\mathrm{d}, \quad \left|\partial^\gamma G_h(x)\right| \leqslant c_\gamma h^{-\alpha \max(|\gamma|-1,0)}.$$

Notice that the constants can be chosen independent of the neighborhood of coordinates as we have a finite system of coordinate neighborhoods and outside a compact set, the metric is the Euclidean metric.

We present next a collection of estimates on J_h. There exist constants $c > 0$ such that, for all $h \in (0,1)$:

- $\|J_h\|_{L^p \to L^p} \leqslant c_p$, for all $1 \leqslant p \leqslant \infty$.
- $\|[J_h, \triangle_{G_h}]\|_{L^2 \to L^2} \leqslant \frac{c}{h}$ and $\|[J_h, \triangle_{G_h}]\|_{H^1 \to L^2} \leqslant c$.

As G is only Lipschitz, the similar statement for $[F_h, \triangle_G]$ only holds for the $H^1 \to L^2$ norm: $\|[F_h, \triangle_G]\|_{H^1 \to L^2} \leqslant c$.

- $\|J_h(\triangle_{G_h} - \triangle_G)\|_{H^1 \to L^2} \leqslant c h^{-\frac{1}{2}}$.

We define also a spectral cut-off slightly larger than J_h. Let $\tilde{\varphi}$ be a C^∞ function supported in an annulus such that $\tilde{\varphi} = 1$ on a neighborhood of the support of φ. We define $\tilde{J}_h$ just like J_h, replacing φ par $\tilde{\varphi}$ in (6):

$$\tilde{J}_h f = \sum_{j \in J} (\kappa_j)^* \big(\tilde{\chi}_j \tilde{\varphi}(hD) \big(\kappa_j^{-1}\big)^* (\chi_j f)\big) + \tilde{F}_{1,h,\infty} f + \tilde{F}_{2,h,\infty} f. \tag{10}$$

Then the action of $\tilde{J}_h$ on J_h and $[J_h, \triangle_{G_h}]$ is close to identity in $L^p \to L^p$ norm, $p \geqslant 2$, and $L^2 \to L^2$ norm, respectively.

- $\|\tilde{J}_h J_h - J_h\|_{L^p \to L^p} \leqslant c_N h^N$.
- $\|[J_h, \triangle_{G_h}] - [J_h, \triangle_{G_h}]\tilde{J}_h\|_{L^2 \to L^2} \leqslant c_N h^N$, for all $N \in \mathbb{N}$.

Let us recall also the Strichartz estimate we use from [4].

Lemma. *(See 4.13 of [4].) For all couples (p,q) admissible in dimension 3 and I_h an interval of time such that $|I_h| = ch^{\frac{3}{2}}$, we have:*

$$\big\|J_h^* e^{it\triangle_{G_h}} u_0\big\|_{L^p(I_h, L^q(M))} \leqslant c\|u_0\|_{L^2}. \tag{11}$$

We prefer to go back to the estimate on $e^{it\triangle_{G_h}}$ since the form of the Strichartz estimate for $e^{it\triangle_G}$ is more difficult to handle ((45) of [4]):

$$\big\|J_h^* e^{it\triangle_G} u_0\big\|_{L^p(I_h, L^q(M))} \leqslant ch\|u_0\|_{H^1}.$$

This is due to the fact that $\triangle_G$ and $\triangle_{G_h}$ are not both self-adjoint in the same space, because of the volume density $\frac{1}{\sqrt{\det G(x)}}$.

2.2. Smoothing effect and Strichartz estimate away from the obstacle

In this section we recall two results of N. Burq, P. Gérard and N. Tzvetkov [11] on the smoothing effect for the Schrödinger flow on exterior domains and the Strichartz estimate away from the obstacle. The smoothing effect was obtained via resolvent bounds. For Strichartz estimate they used a strategy inspired by G. Staffilani and D. Tataru's paper [26] on C^2 short range perturbation of the free Laplacian on $\mathbb{R}^d$. Thus, they proved that away from the obstacle the linear Schrödinger flow satisfies the usual Strichartz estimates. We present an equivalent statement on the double manifold.

Proposition. *(See 2.7 of [11].) Assume that $\Theta \neq \emptyset$. Then for every $T > 0$, for every $\chi \in C_0^\infty(\mathbb{R}^d)$, $d \geqslant 2$,*

$$\left\| \chi e^{it\triangle_D} u_0 \right\|_{L_T^2 H_D^{s+\frac{1}{2}}(\Omega)} \leqslant c \|u_0\|_{L_T^2 H_D^s(\Omega)}, \quad \textit{for } s \in [0,1].$$

Proposition. *(See 2.10 of [11].) For every $T > 0$, for every $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi = 1$ close to Θ, there exists $C > 0$ such that*

$$\left\| (1-\chi) u \right\|_{L_T^p W^{s,q}(\Omega)} \leqslant C \|u_0\|_{H_D^s}(\Omega), \tag{12}$$

where $s \in [0,1]$, $u(t) = e^{it\triangle_D} u_0$ and (p,q) any Strichartz admissible pair.

The proof relies on the use of the smoothing effect and the fact that $(1-\chi) e^{it\triangle_D} u_0$ can be seen as a solution to some nonlinear Schrödinger equation on $\mathbb{R}^d$.

Although the properties are written for the Dirichlet Laplacian, Remark 1.2 of [11] ensures that the results hold for the Neumann conditions as well. From the way we constructed the double manifold and flows, we deduce that those results extend easily on the double manifold.

Proposition 2.2. *Assume that $\Theta \neq \emptyset$. Then for every $T > 0$, for every $\chi \in C_0^\infty(M)$,*

$$\left\| \chi e^{it\triangle_G} u_0 \right\|_{L_T^2 H^{s+\frac{1}{2}}(M)} \leqslant c \|u_0\|_{L_T^2 H^s(M)}, \quad \textit{for } s \in [0,1].$$

Proposition 2.3. *For every $T > 0$, for every $\chi \in C_0^\infty(M)$, $\chi = 1$ close to $\mathcal{D}\Theta$, where $\mathcal{D}\Theta$ represents the double of Θ, there exists $C > 0$ such that*

$$\left\| (1-\chi) e^{it\triangle_G} u_0 \right\|_{L_T^p W^{s,q}(M)} \leqslant C \|u_0\|_{H^s(M)}, \tag{13}$$

where $s \in [0,1]$ and (p,q) any Strichartz admissible pair.

2.3. Strichartz estimate near the obstacle

We want to combine Strichartz estimate on domains [4] with smoothing effect [11]. This idea goes back to the work of Staffilani–Tataru [26]. For this we use the Strichartz estimate of the frequency localized linear flow, without loss of derivatives, which holds on a small interval of time (see estimate (11) from Section 2.1).

Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$, $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp} \varphi \subset [-\frac{1}{2}, \frac{1}{2}]$ and there exists $J \subset \mathbb{R}$ a discrete set such that $\sum_{t_0 \in J} \varphi^2(t - t_0) = 1$ for all $t \in \mathbb{R}$. Let $\delta > 0$ be a small number. If we consider $\bar{J} = [-\delta, 1+\delta] \cap ch^{\frac{3}{2}} J$ then, for $t \in [-\frac{\delta}{2}, 1 + \frac{\delta}{2}]$,

$$\sum_{t_0 \in \bar{J}} \varphi^2\left(\frac{t - t_0}{ch^{\frac{3}{2}}}\right) = 1. \tag{14}$$

Let us denote by $I_h(t_0) = [t_0 - \frac{ch^{3/2}}{4}, t_0 + \frac{ch^{3/2}}{4}]$, $I'_h(t_0) = [t_0 - \frac{ch^{3/2}}{2}, t_0 + \frac{ch^{3/2}}{2}]$, $u_L(t) = e^{it\triangle_G} u_0$ and by:

$$v(t) = \varphi\left(\frac{t - t_0}{ch^{\frac{3}{2}}}\right) J_h^* \chi e^{it\triangle_G} u_0. \tag{15}$$

Notice that $v(t) = J_h^* \chi e^{it\triangle_G} u_0$ for $t \in I_h(t_0)$ and $\operatorname{supp}_t v \subset I'_h$. We write the end-point Strichartz estimate for v on I'_h, that is for the couple $(p, q) = (2, 6)$.

Lemma 2.4. *For $t_0 \in \mathbb{R}$, $\tilde{\chi} \in C_0^\infty(M)$ such that $\tilde{\chi}\chi = \chi$ and $\tilde{J}_h$ a spectral cut-off slightly larger than J_h, for definition see* (10)*, we have*:

$$\begin{aligned}
&\left\| \varphi\left(\frac{t - t_0}{ch^{\frac{3}{2}}}\right) J_h^* \chi e^{it\triangle_G} u_0 \right\|_{L^2(I'_h, L^6(M))} \\
&\quad \leqslant ch^{-\frac{3}{4}} \left\| J_h^* \chi u_L \right\|_{L^2(I'_h, L^2(M))} + h^{\frac{3}{4}} \left\| \tilde{J}_h^* \tilde{\chi} u_L \right\|_{L^2(I'_h, H^1(M))} + ch^{\frac{1}{4}} \|\tilde{\chi} u_L\|_{L^2(I'_h, H^1(M))}.
\end{aligned} \tag{16}$$

Proof. For simplicity, let us suppose that $I'_h(t_0) = [0, T]$, where $T = ch^{3/2}$. Then $v(t)$ verifies, for $t \in I'_h(t_0)$, the equation,

$$\begin{cases} i\partial_t v + \triangle_{G_h} v = f_1 + f_2 + f_3, \\ v|_{t=0} = 0, \end{cases}$$

where $f_1 = \frac{i}{ch^{3/2}} \varphi'(\frac{t-t_0}{ch^{3/2}}) J_h^* \chi u_L$, $f_2 = \varphi(\frac{t-t_0}{ch^{3/2}})[\triangle_{G_h}, J_h^* \chi] u_L$ and $f_3 = \varphi(\frac{t-t_0}{ch^{3/2}}) J_h^* \chi (\triangle_G - \triangle_{G_h}) u_L$. By the Duhamel formula and using that $\tilde{J}_h^* J_h^* = J_h^* + c_N h^N$ in $L^p \to L^p$ norm, for $p \geqslant 2$, we have:

$$v(t) = v_1(t) + v_2(t) + v_3(t) + c_N h^N,$$

where we define,

$$\begin{aligned}
v_1(t) &= \frac{1}{ch^{\frac{3}{2}}} \int_0^t \tilde{J}_h^* e^{i(t-\tau)\triangle_{G_h}} \varphi'\left(\frac{\tau - t_0}{ch^{\frac{3}{2}}}\right) J_h^* \chi u_L(\tau)\, d\tau, \\
v_2(t) &= -i \int_0^t \tilde{J}_h^* e^{i(t-\tau)\triangle_{G_h}} \varphi\left(\frac{\tau - t_0}{ch^{\frac{3}{2}}}\right) \left[\triangle_{G_h}, J_h^* \chi\right] u_L(\tau)\, d\tau, \\
v_3(t) &= -i \int_0^t \tilde{J}_h^* e^{i(t-\tau)\triangle_{G_h}} \varphi\left(\frac{\tau - t_0}{ch^{\frac{3}{2}}}\right) J_h^* \chi (\triangle_G - \triangle_{G_h}) u_L(\tau)\, d\tau.
\end{aligned}$$

By Minkowski inequality and estimate (11), we have:

$$\begin{aligned}
\|v_1\|_{L_t^2(I'_h, L^6(M))} &\leqslant ch^{-\frac{3}{2}} \int_0^T \left| \varphi'\left(\frac{\tau - t_0}{ch^{\frac{3}{2}}}\right) \right| \left\| \mathbb{1}_{\tau < t} \tilde{J}_h^* e^{i(t-\tau)\triangle_{G_h}} J_h^* \chi u_L(\tau) \right\|_{L_t^2(L_x^6(M))} d\tau \\
&\leqslant ch^{-\frac{3}{2}} \int_0^T \left| \varphi'\left(\frac{\tau - t_0}{ch^{\frac{3}{2}}}\right) \right| \left\| J_h^* \chi u_L(\tau) \right\|_{L_x^2(M)} d\tau.
\end{aligned}$$

Using Cauchy–Schwarz inequality and $\|\varphi'(\frac{\cdot}{ch^{3/2}})\|_{L^2} = ch^{3/4}$, we obtain:

$$\|v_1\|_{L_t^2(I'_h, L^6(M))} \leqslant ch^{-\frac{3}{4}} \left\| J_h^* \chi u_L \right\|_{L^2(I'_h \times M)}. \tag{17}$$

Similarly, we have $\|v_2\|_{L_t^2(I'_h, L^6(M))} \leqslant ch^{3/4} \|[\triangle_{G_h}, J_h^* \chi] u_L\|_{L^2(I'_h \times M)}$. Using that $\|[\triangle_{G_h}, J_h^* \chi]\|_{H^1 \to L^2} \leqslant c$ and $\|[\triangle_{G_h}, J_h^* \chi]\|_{H^1 \to L^2} \sim \|[\triangle_{G_h}, J_h^* \chi] \tilde{J}_h^* \tilde{\chi}\|_{H^1 \to L^2}$ modulo ch^N, we obtain:

$$\|v_2\|_{L_t^2(I'_h, L^6(M))} \leqslant ch^{\frac{3}{4}} \left\| \tilde{J}_h^* \tilde{\chi} u_L \right\|_{L^2(I'_h, H^1(M))}. \tag{18}$$

We estimate the third term v_3 in $L^2(I'_h, L^6(M))$ norm in a similar manner. We get: $\|v_3\|_{L_t^2(I'_h, L^6(M))} \leqslant ch^{\frac{3}{4}} \|J_h^* \chi (\triangle_G - \triangle_{G_h}) u_L\|_{L^2(I'_h \times M)}$. Using the estimate $\|J_h^* \chi (\triangle_G - \triangle_{G_h}) f\|_{L^2(M)} \leqslant ch^{-\frac{1}{2}} \|\tilde{\chi} f\|_{H^1(M)}$, we obtain:

$$\|v_3\|_{L_t^2(I'_h, L^6(M))} \leqslant ch^{\frac{1}{4}} \|\tilde{\chi} u_L\|_{L^2(I'_h, H^1(M))}. \tag{19}$$

Recalling that $v(t) = v_1(t) + v_2(t) + v_3(t) + c_n h^N$, the result follows from the triangle inequality and the sum of (17), (18) and (19). □

We proceed to the summation over the intervals of time in order to obtain a Strichartz inequality (for the frequency localized flow) on a fixed interval of time. Let us denote by $I = [0, 1]$ and by $I_\delta = I + [-\delta, \delta]$, where δ is chosen like in (14).

Lemma 2.5. *Under the same notations as in Lemma* 2.4*, we have*:

$$\begin{aligned} &\left\| J_h^* \chi u_L \right\|_{L^2(I,L^6(M))} \\ &\quad \leqslant ch^{-\frac{3}{4}} \left\| J_h^* \chi u_L \right\|_{L^2(I_\delta,L^2(M))} + h^{\frac{3}{4}} \left\| \tilde{J}_h^* \tilde{\chi} u_L \right\|_{L^2(I_\delta,H^1(M))} + ch^{\frac{1}{4}} \|\tilde{\chi} u_L\|_{L^2(I_\delta,H^1(M))}. \end{aligned} \tag{20}$$

Proof. We sum the square of (16) over $t_0 \in \bar{J}$, where $\bar{J}$ was defined for the identity (14). From (14) and the definition of φ we deduce that the reunion of intervals $I'_h(t_0)$, for $t_0 \in \bar{J}$, recovers I_δ at most twice. Thus, $\sum_{t_0 \in \bar{J}} \|f\|^2_{L^2(I'_h)} \leqslant 2\|f\|^2_{L^2([-\delta,1+\delta])}$. The result follows by merely observing that $\|J_h^* \chi u_L\|_{L^2(I,L^6(M))} \leqslant \|J_h^* \chi u_L\|_{L^2(I_\delta,L^6(M))}$. □

From (20) we get the Strichartz inequality near the obstacle by means of summation.

Proposition 2.6. *For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for (p, q) admissible in dimension* 3*,*

$$\left\| \chi e^{it\triangle_G} u_0 \right\|_{L^p([0,1],W^{\frac{1}{2p}-\varepsilon,q}(M))} \leqslant c_\varepsilon \|u_0\|_{H^{\frac{1}{p}}(M)}. \tag{21}$$

Proof. Notice that the last term in (20) is not localized in frequency. Therefore, when we sum the frequencies we necessarily lose ε derivatives. Thus, a triangle inequality suffices to sum the first terms of (20) as well. We give the details of this argument. The inequality (20) also reads:

$$\left\| J_h^* \chi u_L \right\|_{L^2(I,W^{\frac{1}{4}-\varepsilon,6})} \leqslant ch^\varepsilon \left(\left\| J_h^* \chi u_L \right\|_{L^2(I_\delta,H^1)} + \left\| \tilde{J}_h^* \tilde{\chi} u_L \right\|_{L^2(I_\delta,H^{\frac{1}{2}})} + \|\tilde{\chi} u_L\|_{L^2(I_\delta,H^1)} \right).$$

For $j \in \mathbb{N}$, let $h = 2^{-j}$. We want to sum for $j \geqslant 0$ the previous inequality. Using the frequency decomposition (8) and triangle inequality, we have:

$$\|\chi u_L\|_{L^2(I,W^{\frac{1}{4}-\varepsilon,6})} \leqslant \sum_{j=0}^{\infty} \left\| J_{2^{-j}}^* \chi u_L \right\|_{L^2(I,W^{\frac{1}{4}-\varepsilon,6})}.$$

From the previous inequality, we deduce that this is bounded by a geometric series with base $2^{-\varepsilon}$. Therefore, we obtain,

$$\|\chi u_L\|_{L^2(I,W^{\frac{1}{4}-\varepsilon,6}(M))} \leqslant c\|\chi u_L\|_{L^2(I_\delta,H^{\frac{1}{2}}(M))} + c\|\tilde{\chi} u_L\|_{L^2(I_\delta,H^1(M))}.$$

We apply the smoothing effect (see Proposition 2.7 of [11] and the translation onto the double). Thus,

$$\|\chi u_L\|_{L^2(I,W^{\frac{1}{4}-\varepsilon,6}(M))} \leqslant c\|u_0\|_{H^{\frac{1}{2}}(M)}.$$

We want to perform a complex interpolation between this estimate and the conservation of the L^2 norm (we used also $0 \leqslant \chi \leqslant 1$):

$$\|\chi u_L\|_{L^\infty(I,L^2(M))} \leqslant c\|u_0\|_{L^2(M)}.$$

Using a weight of $\frac{2}{p}$, respectively, $1 - \frac{2}{p}$, we get an estimate of Strichartz type with loss of derivatives:

$$\|\chi u_L\|_{L^p(I,W^{\frac{1}{2p}-2\varepsilon,q}(M))} \leqslant c\|u_0\|_{H^{\frac{1}{p}}(M)},$$

where (p, q) satisfy $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$, i.e. they form an admissible couple in dimension 3. □

2.4. Proof of Proposition 2.1

Combining estimates (21) (Strichartz estimate near the boundary of Ω) with (13) (Strichartz estimate away from the boundary) for $s = \frac{1}{2p} - \varepsilon$, we obtain, using that $\|v_0\|_{H^{\frac{1}{2p}-\varepsilon}(M)} \leqslant \|v_0\|_{H^{\frac{1}{p}}(M)}$,

$$\left\|e^{it\triangle_G} v_0\right\|_{L^p([0,1],W^{\frac{1}{2p}-\varepsilon,q}(M))} \leqslant c_\varepsilon \|v_0\|_{H^{\frac{1}{p}}(M)}.$$

Let $u_0 \in H_D^{\frac{1}{2p}+\varepsilon}(\Omega)$ and let $v_0 \in H_{AS}^{\frac{1}{2p}+\varepsilon}(\Omega)$ be such that $v_0|_\Omega = u_0$. By uniqueness and stability at reflexion over the boundary of Ω of the linear flow (see Section 2.1), we have $e^{it\triangle_G} v_0|_\Omega = e^{it\triangle_D} u_0$. Thus,

$$\left\|e^{it\triangle_G} v_0\right\|_{L^p([0,1],W^{s,q}(M))} \approx \left\|e^{it\triangle_D} u_0\right\|_{L^p([0,1],W^{s,q}(\Omega))},$$

and $\|v_0\|_{H^s(M)} \approx \|u_0\|_{H^s(\Omega)}$. We obtain,

$$\left\|e^{it\triangle_D} u_0\right\|_{L^p([0,1],W^{\frac{1}{2p}-\varepsilon,q}(\Omega))} \leqslant c_\varepsilon \|u_0\|_{H^{\frac{1}{p}}(\Omega)}.$$

We apply the ellipticity of the Laplacian $\triangle_D$ to deduce a whole range of Strichartz inequalities: let $\tilde{u}_0 = (1-\triangle_D)^{-\sigma/2} u_0$, where $\sigma = \frac{1}{2p} - \varepsilon - s$, $s \in [0,1]$. If $u_0 \in H^{s_0}(\Omega)$ then $\tilde{u}_0 \in H^{s_0-\sigma}(\Omega)$. We obtain the following inequality for $e^{it\triangle_D} u_0$:

$$\left\|e^{it\triangle_D} u_0\right\|_{L^p([0,1],W^{s,q}(\Omega))} \leqslant c_\varepsilon \|u_0\|_{H^{s+\frac{1}{2p}+\varepsilon}(\Omega)}. \tag{22}$$

In the case of Neumann boundary conditions, for $u_0 \in H_N^{\frac{1}{2p}+\varepsilon}(\Omega)$, we consider $v_0 \in H_S^{\frac{1}{2p}+\varepsilon}(M)$ be such that $v_0|_\Omega = u_0$. We deduce as above the Strichartz inequality for the linear Schrödinger flow with Neumann Laplacian.

3. Global existence for NLS

Having a Strichartz inequality we obtain classically a local existence theorem by Picard iteration scheme. These also enables propagation of the regularity of the initial data. Local existence in the energy space $H_0^1(\Omega)$ combined with the conservation of the energy (and for defocusing nonlinearity of the $H_0^1(\Omega)$ norm) enables us to conclude that the solution to (1) is global in time.

Proof of Theorem 1.1. Let us denote by $X_T = C([-T,T], H_0^1(\Omega)) \cap L^p([-T,T], L^\infty(\Omega))$ and, for a fix $u_0 \in B \subset H_0^1(\Omega)$, by $\Phi : X_T \to X_T$ the functional,

$$\Phi(u)(t) = e^{it\triangle} u_0 - i \int_0^t e^{i(t-\tau)\triangle} \left|u(\tau)\right|^2 u(\tau)\, d\tau.$$

The space X_T is a complete Banach space for the following norm:

$$\|u\|_{X_T} = \max_{|t|\leqslant T} \left\|u(t)\right\|_{H^1(\Omega)} + \|u\|_{L^p([-T,T],L^\infty(\Omega))}.$$

We prove that for a $T > 0$ and $R > 0$ small enough, Φ is a contraction from $B(0,R) \subset X_T$ into itself. We begin by estimating the H^1 norm of $\Phi(u)$:

$$\left\|\Phi(u)(t)\right\|_{H^1} \leqslant \|u_0\|_{H^1} + cT^{1-\frac{2}{p}} \|u\|^2_{L^p(L^\infty)} \|u\|_{L^\infty_T(H^1)} \leqslant \|u_0\|_{H^1} + cT^{1-\frac{2}{p}} \|u\|^3_{X_T}.$$

We have considered $2 < p < 3$. Thus, there exists $\varepsilon > 0$ such that $\varepsilon < \frac{3}{2p} - \frac{1}{2}$. Therefore, by Sobolev imbedding theorem we have, for (p,q) admissible in dimension 3, that $W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega) \subset L^\infty(\Omega)$:

$$\left\|\Phi(u)\right\|_{L^p_T L^\infty(\Omega)} \leqslant c \left\|\Phi(u)\right\|_{L^p_T W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega)}.$$

Using the Strichartz estimate (3) and Minkowski inequality (like in the proof of (16)), we have:

$$\begin{aligned}\left\|\Phi(u)\right\|_{L^p(L^\infty)} &\leqslant \left\|e^{it\triangle}u_0\right\|_{L^p_T W^{1-\frac{1}{2p}-\varepsilon,q}} + \left\|\int_0^t e^{i(t-\tau)\triangle}|u|^2(\tau)u(\tau)\,d\tau\right\|_{L^p_T W^{1-\frac{1}{2p}-\varepsilon,q}} \\ &\leqslant c\|u_0\|_{H^1} + c\int_0^T \left\||u|^2u(\tau)\right\|_{H^1(\Omega)}\,d\tau.\end{aligned}$$

Using that $\||u|^2(\tau)u(\tau)\|_{H^1(\Omega)} \leqslant c\|u(\tau)\|_{H^1}\|u(\tau)\|^2_{L^\infty}$, we obtain:

$$\left\|\Phi(u)\right\|_{L^p(L^\infty)} \leqslant c\|u_0\|_{H^1} + cT^{1-\frac{2}{p}}\|u\|_{L^\infty(H^1)}\|u\|^2_{L^p(L^\infty)} \leqslant c\|u_0\|_{H^1} + cT^{1-\frac{2}{p}}\|u\|^3_{X_T}.$$

Thus, $\|\Phi(u)\|_{X_T} \leqslant c\|u_0\|_{H^1} + cT^{1-\frac{2}{p}}\|u\|^3_{X_T}$.

Consequently, as $p > 2$, there exist $T, R > 0$, depending only on $B \subset H^1_0(\Omega)$ $(u_0 \in B)$, such that, for $u \in X_T$ with $\|u\|_{X_T} \leqslant R$, we have $\|\Phi(u)\|_{X_T} < R$.

As above, we prove that, for $u, v \in X_T$ such that $u(0) = u_0 = v(0)$,

$$\left\|\Phi(u) - \Phi(v)\right\|_{X_T} \leqslant cT^{1-\frac{2}{p}}\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\|u - v\|_{X_T}.$$

Choosing T eventually smaller, we ensure that Φ is a contraction on the ball $B(0,R) \subset X_T$, $B(0,R) = \{u \in X_T,\ \|u\|_{X_T} < R\}$. Consequently, there exists a fix point of Φ, which is therefore solution to (1).

For the Lipschitz property of the flow let us consider $u, v \in B(0,R) \subset X_T$ two solutions of (1) with initial data, respectively, $u_0, v_0 \in B$. As above, we have:

$$\|u - v\|_{X_T} \leqslant c\|u_0 - v_0\|_{H^1} + cT^{1-\frac{2}{p}}\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\|u - v\|_{X_T}.$$

For $T, R > 0$ chosen before we have $cT^{1-\frac{2}{p}}(\|u\|^2_{X_T} + \|v\|^2_{X_T}) < 1$ and therefore, $\exists \tilde{c} > 0$ such that $\|u - v\|_{X_T} \leqslant \tilde{c}\|u_0 - v_0\|_{H^1}$. We conclude that the flow $u_0 \mapsto u$ is Lipschitz on $B \subset H^1_0$.

Let $\sigma \geqslant 1$ and suppose $u_0 \in H^\sigma(\Omega) \cap H^1_0(\Omega)$. Let us estimate $\Phi(u)$ in $Y_T = C([-T,T], H^\sigma(\Omega)) \cap L^p([-T,T], L^\infty(\Omega))$ norm:

$$\|u\|_{Y_T} = \max_{|t|\leqslant T}\left\|u(t)\right\|_{H^\sigma(\Omega)} + \|u\|_{L^p([-T,T],L^\infty(\Omega))}.$$

As above, we obtain:

$$\left\|\Phi(u)\right\|_{L^\infty_T H^\sigma} \leqslant c\|u_0\|_{H^\sigma} + cT^{1-\frac{2}{p}}\|u\|^2_{X_T}\|u\|_{L^\infty_T H^\sigma}.$$

We have chosen $T > 0$ such that $cT^{1-\frac{2}{p}}(\|u\|^2_{X_T} + \|v\|^2_{X_T}) < 1$. Consequently, the H^σ norm does not blow up for $|t| \leqslant T$:

$$\|u\|_{L^\infty_T H^\sigma} \leqslant \tilde{c}\|u_0\|_{H^\sigma}.$$

Therefore we can conclude that regularity propagates up to time T.

The semi-linear Schrödinger equation (1) has a Hamiltonian structure with gauge invariance and thus conservation laws hold for H^2 initial data. For $u_0 \in H^1$ we deduce them by density: the solution of (1) constructed above satisfies, for $|t| \leqslant T$, to

$$\begin{cases}\int |u(t)|^2\,dx = \int |u_0|^2\,dx,\\ \int |\nabla u(t)|^2 + \frac{1}{2}|u(t)|^4\,dx = \int |\nabla u_0|^2 + \frac{1}{2}|u_0|^4\,dx.\end{cases}$$

Moreover, note that $T > 0$ depends only on $\|u_0\|_{H^1}$. Therefore, conservation of H^1 norm enables us to obtain, via a bootstrap argument, the global existence. □

4. Global existence for Gross–Pitaevskii

The Gross–Pitaevskii equation (2) is associated to the energy:

$$\mathcal{E}(u)=\int_{\Omega}\frac{1}{2}|\nabla u|^2(x)+\frac{1}{4}\big(|u|^2(x)-1\big)^2\,\mathrm{d}x. \tag{23}$$

The main difference between the NLS (1) and the Gross–Pitaevskii equation (2) is their energy space. For Gross–Pitaevskii it reads,

$$E=\big\{u\in H^1_{\mathrm{loc}}(\Omega),\ \nabla u\in L^2(\Omega),\ |u|^2-1\in L^2(\Omega)\big\}.$$

Namely, the initial data in the energy space, $u_0\in E$, is not an $L^2(\Omega)$ function. Therefore we begin this section by describing the structure of E and of the action of the linear Schrödinger group on E by adapting the arguments of [16] to the boundary case. Then, we give the proof of the global existence theorem for the Gross–Pitaevskii equation (2) by combining the latter structure with dispersive estimates derived in Sections 2.2 and 2.3.

4.1. The energy space

This section is inspired from the work of P. Gérard [16]. In that paper, the Cauchy problem for Gross–Pitaevskii equation is studied in the whole Euclidean space $\mathbb{R}^d$, for $d=2,3,4$. In the special case of $d=3$, $u_0\in E$ can be expressed in an explicit form as $u_0=c+v_0$, where $c\in\mathbb{C}$ and $v_0\in\dot H^1$. We show here that the same holds on Ω and give the outline of the proof. For more details we refer to [16]. Notice that the nontrapping assumption does not influence the analysis, the same holds for the exterior of a compact obstacle with smooth boundary.

We denote by $C_0^\infty(\bar\Omega)$ the restriction to $\bar\Omega$ of $C_0^\infty(\mathbb{R}^3)$ and by $\dot H^1(\Omega)$ the completion of $C_0^\infty(\bar\Omega)$ in the norm $\|\nabla\cdot\|_{L^2(\Omega)}$. We recall that

$$\dot H^1(\Omega)=\big\{u\in L^6(\Omega),\ \nabla u\in L^2(\Omega)\big\}.$$

Moreover, we have the following approximation property.

Let $\chi\in C_0^\infty(\mathbb{R}^3)$, $\chi=1$ on the ball of radius 1 $B(0,1)$ and $\chi=0$ outside $B(0,2)$. We define $\chi_R(x)=\chi(\frac{x}{R})$. For $v\in\dot H^1(\Omega)$ we have $\chi_R v\in H^1(\Omega)$, and

$$\chi_R v\xrightarrow{R\to\infty} v \text{ in the } \|\nabla\cdot\|_{L^2(\Omega)}\text{norm}. \tag{24}$$

We prove the main result of this section.

Proposition 4.1. *The energy space E has the following structure*:

$$E=\big\{c+v,\ c\in\mathbb{C},\ |c|=1,\ v\in\dot H^1(\Omega),\ |v|^2+2Re\big(c^{-1}v\big)\in L^2(\Omega)\big\}.$$

The space E is a complete metric space with the distance function:

$$\delta_E(c+v,\tilde c+\tilde v)=|c-\tilde c|+\|\nabla v-\nabla\tilde v\|_{L^2(\Omega)}+\big\||v|^2+2Re(\bar c v)-|\tilde v|^2-2Re(\bar{\tilde c}\tilde v)\big\|_{L^2(\Omega)}.$$

Proof. The embedding "$\supset$" is obvious. For the converse we consider $R_0>0$ such that $\complement\Omega\subset B(R_0)$. For $u\in E$ we define, for every $\omega\in\mathbb{S}^2$ and $R>R_0$,

$$U_R(\omega)=u(R\omega).$$

Just as in the proof of Lemma 7 of [16], we show that U_R converges to U in $L^2(\mathbb{S}^2)$ norm and moreover $\nabla_\omega U=0$. This enables us to conclude that U is a constant $c(u)$. Since $|u|^2-1\in L^2(\Omega)$, we conclude that $c(u)=1$. Let us proceed to the proof by noticing that

$$\int_{R_0}^{\infty}R^2\|\partial_R U_R\|^2_{L^2(\mathbb{S}^2)}+\|\partial_\omega U_R\|^2_{L^2(\mathbb{S}^2)}\,\mathrm{d}R\leqslant\|\nabla u\|^2_{L^2(\Omega)}<\infty. \tag{25}$$

By Cauchy–Schwarz, $\int_{R_0}^{\infty} \|\partial_R U_R\|_{L^2(\mathbb{S}^2)} \, dR \leqslant c(\int_{R_0}^{\infty} R^2 \|\partial_R U_R\|^2_{L^2(\mathbb{S}^2)} \, dR)^{\frac{1}{2}}$ and thus $\int_R^{\infty} \partial_\rho U_\rho \, d\rho$ satisfies the Cauchy criterion for convergence in $L^2(\mathbb{S}^2)$. We conclude the existence of a limit U of U_R in $L^2(\mathbb{S}^2)$. From (25) we deduce also that $\int_R^{R+1} \|\nabla_\omega U_\rho\|_{L^2(\mathbb{S}^2)} \, d\rho$ goes to 0 as $R \to \infty$. Since $\nabla_\omega U = \lim_{R\to\infty} \int_R^{R+1} \nabla_\omega U_\rho \, d\rho$ we conclude that $\|\nabla_\omega U\|_{L^2(\mathbb{S}^2)} = 0$. Thus, $U = c$, a constant of absolute value 1.

Let us show that, if we denote by $v = u - c$, then $v \in \dot{H}^1(\Omega)$. Notice that $\nabla v = \nabla u \in L^2(\Omega)$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$, $\chi = 1$ on the ball of radius 1 $B(0,1)$ and $\chi = 0$ outside $B(0,2)$. We define $\chi_R(x) = \chi(\frac{x}{R})$. We show that v is the limit of $\chi_R v$ in the norm $\|\nabla \cdot\|_{L^2(\Omega)}$. As $\chi_R v \in H^1(\Omega)$, we obtain $v \in \dot{H}^1(\Omega)$.

Notice that we have $v(R\omega) = -\int_R^\infty \partial_\rho U_\rho \, d\rho = -\int_R^\infty \omega \cdot (\nabla u)(\rho\omega) \, d\rho$. By Cauchy–Schwarz we obtain $|v(R\omega)| \leqslant \frac{1}{\sqrt{R}} (\int_R^\infty \rho^2 |\nabla u|^2(\rho\omega) \, d\rho)^{\frac{1}{2}}$. Consequently,

$$\int_{R'}^{2R'} \int_{\mathbb{S}^2} \left|v(R\omega)\right|^2 d\omega \, dR \leqslant \int_{R'}^{2R'} \frac{1}{R} \int_R^\infty \int_{\mathbb{S}^2} \rho^2 |\nabla u|^2(\rho\omega) \, d\omega \, d\rho \, dR.$$

Let us denote by $g(R) = \int_R^\infty \int_{\mathbb{S}^2} \rho^2 |\nabla u|^2(\rho\omega) \, d\omega \, d\rho$. The function g is a decreasing function whose limit is 0 at ∞. Then $\int_{R'}^{2R'} \frac{1}{R} g(R) \, dR < g(R') \ln 2$, which goes to 0 as R' goes to ∞. Consequently,

$$\lim_{R'\to\infty} \int_{R'}^{2R'} \int_{\mathbb{S}^2} \left|v(R\omega)\right|^2 d\omega \, dR = 0.$$

This enables us to show that $\|\nabla(v - \chi_R v)\|_{L^2(\mathbb{S}^2)} \to 0$ as $R \to \infty$. Indeed, we have that

$$\nabla(v - \chi_R v) = \frac{1}{R} (\nabla\chi)_R v + (1 - \chi_R) v.$$

By writing v in polar coordinates we obtain, for $R > R_0$,

$$\int_\Omega \frac{1}{R^2} \left| (\nabla\chi)\left(\frac{x}{R}\right) v(x) \right|^2 dx \leqslant c \int_R^{2R} \int_{\mathbb{S}^2} \left|v(\rho\omega)\right|^2 d\omega \, d\rho \to 0,$$

as $R \to \infty$. The other term also goes to 0 in $L^2(\Omega)$ norm as $R \to \infty$:

$$\left\|(1 - \chi_R)\nabla v\right\|_{L^2(\Omega)} \leqslant c\|\nabla v\|_{L^2(|x|>R)} \to 0.$$

This concludes the proof of $v = u - c \in \dot{H}^1(\Omega)$ and thus of the embedding "$\subset$". The completeness of the metric space E is an easy consequence of its structure. □

We end this section by showing that $E + H^1(\Omega) \subset E$ (see also Lemma 2 of [16]).

Lemma 4.2. *Let $u \in E$ and $w \in H^1(\Omega)$. Then $u + w \in E$, and*

$$\left\| |u + w|^2 - 1 \right\|_{L^2(\Omega)} \leqslant \left(\sqrt{\mathcal{E}(u)} + \|w\|_{H^1(\Omega)}\right)\left(1 + \|w\|_{H^1(\Omega)}\right). \tag{26}$$

Moreover, for $\tilde{u} \in E$ and $\tilde{w} \in H^1(\Omega)$, we have:

$$\begin{aligned} &\delta_E(u + w, \tilde{u} + \tilde{w}) \\ &\quad \leqslant \left(1 + \|w\|_{H^1} + \|\tilde{w}\|_{H^1}\right)\delta_E(u, \tilde{u}) + \left(1 + \sqrt{\mathcal{E}(u)} + \sqrt{\mathcal{E}(\tilde{u})} + \|w\|_{H^1} + \|w\|_{H^1}\right)\|w - \tilde{w}\|_{H^1}. \end{aligned} \tag{27}$$

Proof. From Proposition 4.1 we know that $u = c + v$, $c \in \mathbb{C}$, $|c| = 1$ and $v \in \dot{H}^1(\Omega)$. Then $u + w = c + (v + w)$ and $v + w \in \dot{H}^1(\Omega) + H^1(\Omega) \subset \dot{H}^1(\Omega)$. We have to show that $|u + w|^2 - 1 \in L^2(\Omega)$. We have:

$$|u + w|^2 - 1 = |v|^2 + 2Re\left(c^{-1}v\right) + |w|^2 + 2Re\left(c^{-1}w\right) + 2Re(\bar{v}w).$$

From Proposition 4.1 we have $|v|^2 + 2Re(c^{-1}v) \in L^2(\Omega)$ and from (23) $\||v|^2 + 2Re(c^{-1}v)\|_{L^2(\Omega)} \leqslant \sqrt{\mathcal{E}(u)}$. From $w \in H^1(\Omega) \subset L^2(\Omega) \cap L^6(\Omega)$ we deduce $\||w|^2\|_{L^2(\Omega)} \leqslant c\|w\|^2_{H^1(\Omega)}$, $\|2Re(\bar{v}w)\|_{L^2(\Omega)} \leqslant c\|v\|_{L^6(\Omega)}\|w\|_{H^1(\Omega)}$ and $\|2Re(\bar{c}w)\|_{L^2(\Omega)} \leqslant c\|w\|_{H^1(\Omega)}$. Estimate (26) follows. For (27) we proceed similarly. □

4.2. *The action of $S(t) = \mathrm{e}^{it\triangle_N}$ on E*

This section is devoting to defining the action of the group $S(t) = \mathrm{e}^{it\triangle_N}$ on the energy space E. In view of the Neumann condition, $S(t)$ leaves constants invariant. We have to justify that $S(t)$ acts on $\dot{H}^1(\Omega)$. We begin by recalling some functional calculus facts (e.g., [23]).

The domain of $-\triangle_N$ in $L^2(\Omega)$ is $H^2_N(\Omega) = H^2(\Omega) \cap \{\frac{\partial v}{\partial \nu} = 0\}$. For $v \in H^2_N(\Omega)$ we have $\|\sqrt{-\triangle_N} v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$. Indeed,

$$\|\sqrt{-\triangle_N} v\|^2_{L^2(\Omega)} = (\sqrt{-\triangle_N} v, \sqrt{-\triangle_N} v)_{L^2} = (v, -\triangle_N v)_{L^2} = \|\nabla v\|^2_{L^2(\Omega)}.$$

The domain of $\sqrt{-\triangle_N}$ in $L^2(\Omega)$ is $H^1(\Omega)$. For $u \in H^1(\Omega)$ we also have the identity $\|\sqrt{-\triangle_N} u\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. Indeed, let $v \in H^2_N(\Omega)$. Then

$$(\sqrt{-\triangle_N} u, \sqrt{-\triangle_N} v)_{L^2} = (u, -\triangle_N v)_{L^2} = (\nabla u, \nabla v)_{L^2}.$$

From $\|\sqrt{-\triangle_N} v\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$ for $v \in H^2_N(\Omega)$ we deduce the same identity for $u \in H^1(\Omega)$.

Lemma 4.3. *Using the notations of* (24), *for $v \in \dot{H}^1(\Omega)$ the limit,*

$$\lim_{R\to\infty} \sqrt{-\triangle_N}(\chi_R v),$$

exists in the $L^2(\Omega)$ norm and we denote it by $\sqrt{-\triangle_N} v$. Moreover,

$$\big\|\sqrt{-\triangle_N} v\big\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}.$$

Proof. From (24) we have that $(\nabla(\chi_R v))_R$ is a Cauchy sequence in the $L^2(\Omega)$ norm. As $\chi_R v \in H^1(\Omega)$, the identity $\|\sqrt{-\triangle_N}(\chi_R v)\|_{L^2(\Omega)} = \|\nabla(\chi_R v)\|_{L^2(\Omega)}$ holds. Therefore, $(\sqrt{-\triangle_N}(\chi_R v))_R$ is also a Cauchy sequence in the $L^2(\Omega)$ norm. Denoting by $\sqrt{-\triangle_N} v$ its limit, we obtain:

$$\big\|\sqrt{-\triangle_N} v\big\|_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}. \qquad \square$$

Remark 2. Using the previous lemmas we can define a functional calculus $\varphi(\sqrt{-\triangle_N})$ on $\dot{H}^1(\Omega)$ for functions $\varphi : [0,\infty) \to \mathbb{C}$ such that $\lambda \mapsto \frac{\varphi(\lambda)}{\lambda}$ is continuous and bounded for $\lambda \in [0,\infty)$. We denote by,

$$\varphi\big(\sqrt{-\triangle_N}\big)v = \frac{\varphi(\sqrt{-\triangle_N})}{\sqrt{-\triangle_N}} \sqrt{-\triangle_N} v,$$

and this is well defined for $v \in \dot{H}^1(\Omega)$ as $\sqrt{-\triangle_N} v \in L^2(\Omega)$. An equivalent definition is: $\varphi(\sqrt{-\triangle_N})v$ is the limit, in $L^2(\Omega)$ norm, of $\varphi(\sqrt{-\triangle_N})(\chi_R v)$.

An important consequence of the previous remark is the definition of $S(t) = \mathrm{e}^{it\triangle_N}$ on $\dot{H}^1(\Omega)$. Let $v \in \dot{H}^1(\Omega)$. We have $S(t)v = v + (\mathrm{e}^{it\triangle_N} - 1)v$ and each term of the sum is well defined.

Lemma 4.4. *For all $t \in \mathbb{R}$ we have $S(t) : \dot{H}^1(\Omega) \to \dot{H}^1(\Omega)$ and moreover, for $v \in \dot{H}^1(\Omega)$, we have*:

$$\big\|S(t)v - v\big\|_{H^1(\Omega)} \leqslant c\big(1 + |t|^{\frac{1}{2}}\big)\|\nabla v\|_{L^2(\Omega)}. \tag{28}$$

Proof. By functional calculus we have that $\frac{e^{it\triangle_N}-1}{\sqrt{-\triangle_N}} = \varphi(-\triangle_N)$ acts on $L^2(\Omega)$ with a norm $\|\varphi(-\triangle_N)\|_{L^2\to L^2} \leqslant \sup_{\lambda\in\sigma(-\triangle_N)} |\varphi(\lambda)|$. Here $\varphi(\lambda) = \frac{e^{it\lambda}-1}{\sqrt{\lambda}}$, for $\lambda > 0$. We have $\|\varphi\|_{L^\infty} \leqslant c\min(|t|\sqrt{\lambda}, \sqrt{\lambda^{-1}})$. Optimizing on λ we obtain $\|\varphi\|_{L^\infty} \leqslant c|t|^{\frac{1}{2}}$ and, thus

$$\left\| \frac{e^{it\triangle_N}-1}{\sqrt{-\triangle_N}} \sqrt{-\triangle_N} v \right\|_{L^2(\Omega)} \leqslant c|t|^{\frac{1}{2}} \|\nabla v\|_{L^2(\Omega)}.$$

We have also $\|\sqrt{-\triangle_N}(e^{it\triangle_N}-1)v\|_{L^2(\Omega)} \leqslant c\|\sqrt{-\triangle_N}v\|_{L^2(\Omega)} \leqslant c\|\nabla v\|_{L^2(\Omega)}$. Thus, $S(t)v = v + (S(t)-1)v \in \dot{H}^1(\Omega) + H^1(\Omega) \subset \dot{H}^1(\Omega)$. □

From the previous lemmas we shall deduce that E is stable under the action of $S(t)$, for all $t \in \mathbb{R}$.

Proposition 4.5. *For every $t \in \mathbb{R}$ we have $S(t)E \subset E$. Moreover, for every $R > 0$, for every $T > 0$, there exists $C > 0$ such that, for $u_0, \tilde{u}_0 \in E$ with $\mathcal{E}(u_0), \mathcal{E}(\tilde{u}_0) \leqslant R$, the following holds*:

$$\sup_{|t|\leqslant T} \delta_E\big(S(t)u_0, S(t)\tilde{u}_0\big) \leqslant C\delta_E(u_0, \tilde{u}_0). \tag{29}$$

Proof. We write $S(t)u_0 = u_0 + (S(t)-1)u_0$. Writing $u_0 = c_0 + v_0$, with $v_0 \in \dot{H}^1(\Omega)$, we have that $S(t)u_0 - u_0 = S(t)v_0 - v_0$. From (28) we deduce $(S(t)-1)u_0 \in H^1(\Omega)$. From Lemma 4.2 we have $S(t)u_0 = u_0 + (S(t)-1)u_0 \in E$. Estimate (29) follows from (27), which reads in this setting:

$$\delta_E\big(S(t)u_0, S(t)\tilde{u}_0\big) \leqslant c\big(1+|t|^{\frac{1}{2}}\big)\big(1+\sqrt{\mathcal{E}(u_0)}+\sqrt{\mathcal{E}(\tilde{u}_0)}\big)\delta_E(u_0, \tilde{u}_0). \qquad \square$$

4.3. Strichartz inequality and energy space

As we mentioned in the beginning of Section 4, one of the main differences between NLS and Gross–Pitaevskii is that the initial data is not in $L^2(\Omega)$ for Gross–Pitaevskii. Therefore, it is not obvious to guess what the Strichartz inequality gives for $S(t)u_0$, when $u_0 \in E$. This is the purpose of this section. We denote by $u_L(t) = S(t)u_0$, for all $t \in \mathbb{R}$. We show in this section that for $u_0 \in E$ and $2 < p < 3$ we have $u_L \in L^p([-T,T], L^\infty(\Omega))$, for some $T > 0$. We decompose u_L in its high and low frequency parts and we treat them separately.

Let $\varphi_1 \in C_0^\infty(\mathbb{R})$ such that $\varphi_1(s) = 1$ pour $|s| \leqslant 1$ and $\varphi_1(s) = 0$ for $|s| \geqslant 2$. Let $\varphi_2 \in C^\infty(\mathbb{R})$ such that $\varphi_1 + \varphi_2 = 1$. Let $u_0 \in E$, $u_0 = c_0 + v_0$, with $c_0 \in \mathbb{C}$, $|c_0| = 1$ and $v_0 \in \dot{H}^1(\Omega)$.

We denote by $v_{20} = \varphi_2(\sqrt{-\triangle_N})v_0$. From Remark 2 and Lemma 4.3 we deduce the following properties of v_{20}.

Lemma 4.6. *Under the previous notations, we have $v_{20} \in H^1(\Omega)$, and*

$$\|v_{20}\|_{H^1(\Omega)} \leqslant c\|\nabla v_0\|_{L^2(\Omega)}.$$

In view of Lemma 4.6 we can apply the Strichartz inequality (3) (in Neumann setting) to $S(t)v_{20}$.

Lemma 4.7. *Let $v_2(t) = S(t)v_{20}$. For $T > 0$ and $2 < p < 3$, the following holds*: $v_2 \in L^p([-T,T], L^\infty(\Omega)) \cap L^\infty([-T,T], H^1(\Omega))$, *and*

$$\|v_2\|_{L_T^p(L^\infty)} + \|v_2\|_{L_T^\infty(H^1)} \leqslant C\|\nabla v_0\|_{L^2(\Omega)}.$$

Proof. From Lemma 4.6 we have $v_{20} \in H^1(\Omega)$. Let (p,q) be an admissible couple in dimension 3 and $\varepsilon > 0$. From the Strichartz inequality (3) we deduce:

$$\|v_2\|_{L_T^p W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega)} \leqslant c\|v_{20}\|_{H^1(\Omega)}.$$

For $2 < p < 3$ there exists $\varepsilon > 0$ such that $W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega) \subset L^\infty(\Omega)$ (see the proof of 1.1). Thus, $\|v_2\|_{L_T^p L^\infty(\Omega)} \leqslant c\|\nabla v_0\|_{L^2(\Omega)}$. The estimate on $\|v_2\|_{L_T^\infty(H^1)}$ follows from the conservation of the H^1 norm by the linear Schrödinger flow $e^{it\triangle_N}$. □

We denote by $v_{10} = v_0 - v_{20} = \varphi_1(\sqrt{-\triangle_N})v_0$ and by $v_1(t) = S(t)v_{10}$.

Lemma 4.8. *For $T > 0$, there exists $C > 0$ such that we have $v_1 \in L^\infty([-T,T] \times \Omega)$ satisfying,*

$$\|v_1\|_{L^\infty_{t,x}} \leqslant C\|\nabla v_0\|_{L^2}.$$

Proof. In this proof we look at v_1 separately near the obstacle and away from the obstacle. The reason is that v_1 is only an $\dot{H}^1(\Omega)$ function. Indeed, $\varphi_1(\sqrt{-\triangle_N}) : L^6(\Omega) \to L^6(\Omega)$ and $\varphi_1(\sqrt{-\triangle_N}) : L^2(\Omega) \to L^2(\Omega)$. As $S(t) : \dot{H}^1(\Omega) \to \dot{H}^1(\Omega)$ by Lemma 4.4, we obtain $v_1 \in \dot{H}^1(\Omega)$.

We consider $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi = 1$ near $\Theta = \complement\Omega$. Then $\chi v_1 \in L^\infty([-T,T], L^2(\Omega))$:

$$\big\|\chi v_1(t)\big\|_{L^\infty_T(L^2(\Omega))} \leqslant \|\chi\|_{L^3(\Omega)}\|v_1\|_{L^\infty_T(L^6(\Omega))} \leqslant C\|\nabla v_0\|_{L^2(\Omega)}.$$

Similarly, we obtain $\triangle(\chi v_1) = (\triangle\chi)v_1 + 2\nabla\chi\cdot\nabla v_1 + \chi(\triangle v_1) \in L^\infty_T(L^2(\Omega))$. Moreover, $\frac{\partial}{\partial\nu}(\chi v_1)|_{\partial\Omega} = \frac{\partial v_1}{\partial\nu}|_{\partial\Omega} = 0$ as $\chi = 1$ in the neighborhood of $\partial\Omega$. Thus, $\chi v_1 \in L^\infty_T H^2_N(\Omega)$, where $H^2_N(\Omega)$ is the domain of $-\triangle_N$ in $L^2(\Omega)$. As $H^2_N(\Omega) \subset L^\infty(\Omega)$, we obtain $\chi v_1 \in L^\infty([-T,T] \times \Omega)$.

We pass to the term $(1-\chi)v_1$. It can be seen as a function on $\mathbb{R}^3$ in the x variable extending it by 0. Since $v_1 \in L^\infty_T L^6(\Omega)$, we have $(1-\chi)v_1 \in L^\infty_T L^6(\mathbb{R}^3)$. We show that $(1-\chi)v_1 \in L^\infty_T W^{2,6}(\mathbb{R}^3)$. For that purpose, it suffices to show that $\triangle((1-\chi)v_1) \in L^\infty_T(L^6(\mathbb{R}^3))$. We have:

$$\triangle\big((1-\chi)v_1\big) = -(\triangle\chi)v_1 - 2\nabla\chi\cdot\nabla v_1 + (1-\chi)(\triangle v_1). \tag{30}$$

Clearly, the first and the last term of the right-hand side expression are in $L^\infty_T(L^6(\mathbb{R}^3))$. For $\nabla\chi\cdot\nabla v_1$ we need to do finer analysis. As $\nabla v_1 \in L^\infty_T L^2(\Omega)$ we deduce $\nabla\chi\cdot\nabla v_1 \in L^\infty_T L^2(\mathbb{R}^3)$. We show that $\nabla\chi\cdot\nabla v_1 \in L^\infty_T W^{2,2}(\mathbb{R}^3)$. We compute:

$$\triangle(\nabla\chi\cdot\nabla v_1) = (\triangle\nabla\chi)\cdot\nabla v_1 + 2\big(\nabla^2\chi\big)\cdot\big(\nabla^2 v_1\big) + \nabla\chi\cdot(\triangle\nabla v_1).$$

We have $(\triangle\nabla\chi)\cdot\nabla v_1 \in L^\infty_T L^2(\mathbb{R}^3)$ and $\nabla\chi\cdot(\triangle\nabla v_1) \in L^\infty_T L^2(\mathbb{R}^3)$. The middle term, $2(\nabla^2\chi)\cdot(\nabla^2 v_1)$ can be written as $P(x,D)(1-\triangle)v$, with $P(x,D) = 2(\nabla^2\chi)\cdot(\nabla^2(1-\triangle)^{-1})$ an pseudo-differential operator of order 0 with compact support. Its coefficients are independent of t. Consequently, $2(\nabla^2\chi)\cdot(\nabla^2 v_1) \in L^\infty_T L^6(\mathbb{R}^3)$ and since this function is compactly supported in x, it belongs also to $L^\infty_T L^2(\mathbb{R}^3)$.

We obtain $\nabla\chi\cdot\nabla v_1 \in L^\infty_T W^{2,2}(\mathbb{R}^3) \subset L^\infty_T L^6(\mathbb{R}^3)$. Going back to (30) we deduce $(1-\chi)v_1 \in L^\infty_T W^{2,6}(\mathbb{R}^3) \subset L^\infty([-T,T] \times \mathbb{R}^3)$. Taking the restriction to Ω concludes the proof. □

From the previous lemmas, we deduce easily the following:

Proposition 4.9. *For $T > 0$ and $2 < p < 3$, there exists $C > 0$ such that, for $u_0 \in E$ and $u_L(t) = \mathrm{e}^{\mathrm{i}t\triangle_N}u_0$, we have: $u_L \in L^p([-T,T], L^\infty(\Omega))$, and*

$$\|u_L\|_{L^p_T(L^\infty)} \leqslant 1 + C\|\nabla v_0\|_{L^2(\Omega)}. \tag{31}$$

Moreover, for $\tilde{u}_0 \in E$ and $\tilde{u}_L(t) = \mathrm{e}^{\mathrm{i}t\triangle_N}\tilde{u}_0$,

$$\|u_L - \tilde{u}_L\|_{L^p_T(L^\infty)} \leqslant C\delta_E(u_0,\tilde{u}_0). \tag{32}$$

Proof. We write $u_L(t) = c_0 + \mathrm{e}^{\mathrm{i}t\triangle_N}v_0 = c_0 + v_1(t) + v_2(t)$. The conclusion follows from $c_0 \in \mathbb{C}$, $v_1 \in L^p_T(L^\infty)$, $v_2 \in L^\infty_T(L^\infty)$ and their respective estimates. □

We close this section by collecting estimates which will be useful in the sequel. We consider $u_0, \tilde{u}_0 \in E$, $u_L(t) = S(t)u_0$ and $\tilde{u}_L(t) = S(t)\tilde{u}_0$, $w, \tilde{w} \in X_T = C([-T,T], H^1_0(\Omega)) \cap L^p([-T,T], L^\infty(\Omega))$ with the associated norm $\|w\|_{X_T} = \max_{|t|\leqslant T}\|w(t)\|_{H^1(\Omega)} + \|w\|_{L^p([-T,T],L^\infty(\Omega))}$. Let $u = u_L + w$ and $\tilde{u} = \tilde{u}_L + \tilde{w}$. We denote by:

$$\gamma = \delta_E(u_0,\tilde{u}_0) + \|w - \tilde{w}\|_{X_T}.$$

As a corollary of Lemmas 4.4 and 4.2 we have:

$$\big\||u_L|^2-1\big\|_{L^\infty_T L^2(\Omega)} \leqslant c\big(\sqrt{\mathcal{E}(u_0)}+\mathcal{E}(u_0)\big), \tag{33}$$

$$\big\||u|^2-1\big\|_{L^\infty_T L^2(\Omega)} \leqslant \big(1+\mathcal{E}(u_0)\big)\|w\|_{X_T}+\|w\|^2_{X_T}. \tag{34}$$

As a corollary of Proposition 4.9 we have:

$$\|u\|_{L^p_T L^\infty(\Omega)} \leqslant c\big(1+\sqrt{\mathcal{E}(u_0)}+\|w\|_{X_T}\big), \tag{35}$$

$$\|u-\tilde{u}\|_{L^p_T L^\infty(\Omega)} \leqslant \gamma. \tag{36}$$

From (35), (36) and (29) we deduce:

$$\big\||u|^2-|\tilde{u}|^2\big\|_{L^\infty_T L^2} \leqslant \gamma\big(1+\sqrt{\mathcal{E}(u_0)}+\sqrt{\mathcal{E}(\tilde{u}_0)}+\|w\|_{X_T}+\|\tilde{w}\|_{X_T}\big). \tag{37}$$

Moreover,

$$\big\||u|^2-1\big\|_{L^{\frac{p}{2}}_T L^\infty} \leqslant 1+\mathcal{E}(u_0)+\|w\|^2_{X_T}, \tag{38}$$

$$\big\||u|^2-|\tilde{u}|^2\big\|_{L^{\frac{p}{2}}_T L^\infty} \leqslant \gamma\big(1+\sqrt{\mathcal{E}(u_0)}+\sqrt{\mathcal{E}(\tilde{u}_0)}+\|w\|_{X_T}+\|\tilde{w}\|_{X_T}\big). \tag{39}$$

By simple computations we obtain:

$$\|\nabla u\|_{L^\infty_T L^2(\Omega)} \leqslant \sqrt{\mathcal{E}(u_0)}+\|w\|_{X_T}, \tag{40}$$

$$\|\nabla u-\nabla\tilde{u}\|_{L^\infty_T L^2(\Omega)} \leqslant \gamma. \tag{41}$$

The estimates (33) to (41) follow from simple computations, decomposing $u=u_L+w$ and applying Hölder and Sobolev inequalities combined with the estimates cited.

4.4. Proof of Theorem 1.2

Let $u_0\in E$. In Section 4.2 we presented the action of $S(t)=\mathrm{e}^{\mathrm{i}t\triangle_N}$ on E. We recall the notation $u_L(t)=S(t)u_0$. We call the solution of (2) the solution to the Duhamel associated formula:

$$u(t)=u_L(t)-\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\triangle_N}F(u)(\tau)\,\mathrm{d}\tau, \tag{42}$$

where $F(u)=(|u|^2-1)u$. We denote by $w=u-u_L$ and by Φ the functional:

$$\Phi(w)=-\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\triangle_N}F(u_L+w)(\tau)\,\mathrm{d}\tau. \tag{43}$$

We show the local existence of u that satisfies (42) by showing that Φ has a fixed point $\Phi(w)=w$. For that purpose we define, for $T>0$ and $2<p<3$, $X_T=C([-T,T],H^1_0(\Omega))\cap L^p([-T,T],L^\infty(\Omega))$. The space X_T is a complete Banach space for the following norm:

$$\|w\|_{X_T}=\max_{|t|\leqslant T}\big\|w(t)\big\|_{H^1(\Omega)}+\|w\|_{L^p([-T,T],L^\infty(\Omega))}.$$

We prove that, for a $T>0$ and $R>0$ small enough, Φ is a contraction from $B(0,R)\subset X_T$ into itself.

Lemma 4.10. *Using the previous notations we have, for $w\in X_T$, that*

$$\big\|\Phi(w)\big\|_{X_T} \leqslant c\big\|F(u_L+w)\big\|_{L^1_T H^1(\Omega)}.$$

Proof. From (42) we deduce, by Minkowski inequality, that

$$\big\|\Phi(w)\big\|_{L^\infty_T L^2(\Omega)} \leqslant c\big\|F(u_L+w)\big\|_{L^1_T L^2(\Omega)}.$$

As $\nabla(\Phi(w)) = -\mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\triangle_N}\nabla(F(u_L+w))(\tau)\,\mathrm{d}\tau$ we have also,

$$\big\|\nabla\big(\Phi(w)\big)\big\|_{L^\infty_T L^2(\Omega)} \leqslant c\big\|\nabla\big(F(u_L+w)\big)\big\|_{L^1_T L^2(\Omega)}.$$

We have considered $2 < p < 3$. Thus, there exists $\varepsilon > 0$ such that, for (p,q) an admissible couple in dimension 3, $W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega) \subset L^\infty(\Omega)$. From the Strichartz inequality (3) we obtain:

$$\big\|\Phi(w)\big\|_{L^p_T L^\infty(\Omega)} \leqslant \big\|\Phi(w)\big\|_{L^p_T W^{1-\frac{1}{2p}-\varepsilon,q}(\Omega)} \leqslant c\big\|F(u_L+w)\big\|_{L^1_T H^1(\Omega)}. \qquad \square$$

We have to estimate $F(u)$ in $L^1_T H^1(\Omega)$ for $u = u_L + w$, $w \in X_T$. For the fixed point method we also need to estimate $\|F(u_L+w) - F(\tilde{u}_L+\tilde{w})\|_{L^1_T H^1(\Omega)}$.

Proposition 4.11. *Under the conditions of Section* 4.4 *we have*:

$$\big\|F(u)\big\|_{L^1_T L^2} \leqslant cT^{1-\frac{1}{p}}\big(1+\mathcal{E}(u_0)+\|w\|_{X_T}\big)^2\|w\|_{X_T},$$
$$\big\|\nabla\big(F(u)\big)\big\|_{L^1_T L^2} \leqslant cT^{1-\frac{2}{p}}\big(1+\sqrt{\mathcal{E}(u_0)}+\|w\|_{X_T}\big)^3,$$

and

$$\big\|F(u)-F(\tilde{u})\big\|_{L^1_T L^2} \leqslant cT^{1-\frac{1}{p}}\gamma\big(1+\mathcal{E}(u_0)+\mathcal{E}(\tilde{u}_0)+\|w\|_{X_T}+\|\tilde{w}\|_{X_T}\big)^2,$$
$$\big\|\nabla\big(F(u)-F(\tilde{u})\big)\big\|_{L^1_T L^2} \leqslant cT^{1-\frac{2}{p}}\gamma\big(1+\sqrt{\mathcal{E}(u_0)}+\sqrt{\mathcal{E}(\tilde{u}_0)}+\|w\|_{X_T}+\|\tilde{w}\|_{X_T}\big)^2,$$

where we have denoted by $\gamma = \delta_E(u_0,\tilde{u}_0) + \|w-\tilde{w}\|_{X_T}$.

Notice that, if $u_0 = \tilde{u}_0$, then we have $\gamma = \|w-\tilde{w}\|_{X_T}$.

Proof. The conclusions follow from estimates (33) to (41). Let us explain one of the conclusions, for example the estimate on $F(u_L+w) - F(\tilde{u}_L+\tilde{w})$. We have:

$$F(u)-F(\tilde{u}) = \big(|u|^2-|\tilde{u}|^2\big)u + (u-\tilde{u})\big(|u|^2-1\big).$$

We apply the Hölder inequality combined with (37) and (35) for the first term and (36) and (34) for the second one. We bound thus $\|F(u_L+w)-F(\tilde{u}_L+\tilde{w})\|_{L^p_T L^2}$. By Hölder inequality we obtain the positive power of T:

$$\big\|F(u)-F(\tilde{u})\big\|_{L^1_T L^2} \leqslant cT^{1-\frac{1}{p}}\big\|F(u)-F(\tilde{u})\big\|_{L^p_T L^2}.$$

The other estimates follow similarly. $\square$

Combining the estimates on the nonlinear term from Proposition 4.11 with Lemma 4.10 we obtain the following:

Corollary 4.12. *Under the conditions on Lemma* 4.10 *we have*:

$$\big\|\Phi(w)\big\|_{X_T} \leqslant cT^{1-\frac{2}{p}}\big(1+\mathcal{E}(u_0)+\|w\|_{X_T}\big)^3, \tag{44}$$
$$\big\|\Phi(w)-\Phi(\tilde{w})\big\|_{X_T} \leqslant cT^{1-\frac{2}{p}}\gamma\big(1+\mathcal{E}(u_0)+\mathcal{E}(\tilde{u}_0)+\|w\|_{X_T}+\|\tilde{w}\|_{X_T}\big)^2, \tag{45}$$

where we denoted by $\gamma = \delta_E(u_0,\tilde{u}_0) + \|w-\tilde{w}\|_{X_T}$.

As a consequence, we can prove the global wellposedness result from Theorem 1.2 on Gross–Pitaevskii equation (2).

Proof. We fix $u_0 \in B \subset E$. From estimate (44) we deduce that there exist $T, R > 0$, depending only on $B \subset E$ ($u_0 \in B$), such that, for $w \in X_T$ with $\|w\|_{X_T} \leqslant R$, we have $\|\Phi(w)\|_{X_T} < R$.

For $\tilde{u}_0 = u_0$ estimate (45) reads:

$$\left\|\Phi(w) - \Phi(\tilde{w})\right\|_{X_T} \leqslant cT^{1-\frac{2}{p}}\left(1 + \|w\|_{X_T} + \|\tilde{w}\|_{X_T}\right)^2 \|w - \tilde{w}\|_{X_T}.$$

As $2 < p$, choosing T eventually smaller ensures that Φ is a contraction on the ball $B(0,R) \subset X_T$, $B(0,R) = \{w \in X_T,\ \|w\|_{X_T} < R\}$. Consequently, there exists a fixed point of Φ in $B(0,R)$, which is therefore solution to (2).

For the Lipschitz property of the flow let us consider $u, \tilde{u} \in B(0,R) \subset X_T$ two solutions of $\Phi(u - u_L) = u - u_L$, therefore of (2), with initial data, respectively, $u_0, \tilde{u}_0 \in B$.

From (45) we have, for $w = u - u_L$ and $\tilde{w} = \tilde{u} - \tilde{u}_L$,

$$\|w - \tilde{w}\|_{X_T} \leqslant cT^{1-\frac{2}{p}}\left(\delta_E(u_0, \tilde{u}_0) + \|w - \tilde{w}\|_{X_T}\right)\left(1 + \mathcal{E}(u_0) + \mathcal{E}(\tilde{u}_0) + \|w\|_{X_T} + \|\tilde{w}\|_{X_T}\right)^2.$$

For $T, R > 0$ chosen before we have $cT^{1-\frac{2}{p}}(1 + \mathcal{E}(u_0) + \mathcal{E}(\tilde{u}_0) + \|w\|_{X_T} + \|\tilde{w}\|_{X_T})^2 < 1$ and therefore, $\exists \tilde{c} > 0$ such that

$$\|w - \tilde{w}\|_{H^1} \leqslant \|w - \tilde{w}\|_{X_T} \leqslant \tilde{c}\delta_E(u_0, \tilde{u}_0).$$

From (27) we have $\delta_E(u(t), \tilde{u}(t)) \leqslant C(R,B)(\delta_E(u_0, \tilde{u}_0) + \|w - \tilde{w}\|_{L_T^\infty H^1})$. Consequently, there exists $C > 0$ such that $\delta_E(u(t), \tilde{u}(t)) \leqslant c\delta_E(u_0, \tilde{u}_0)$, for all $|t| \leqslant T$. We conclude that the flow $u_0 \mapsto u(t)$ is Lipschitz on $B \subset E$.

The proof of the propagation of regularity from Section 3.3 of [16] adapts to the framework of exterior domains using techniques similar to those of Section 4.2. Those techniques combined with the stability of E by summation with an H^1 element (see Lemma 4.2) enables us to show that $u_0 \in E$ can be approached, in δ_E distance, by $u_0^\varepsilon \in E$ such that $\triangle u_0^\varepsilon \in L^2(\Omega)$. As one can prove conservation of energy $\mathcal{E}$ for initial data $f \in E$ such that $\triangle f \in L^2(\Omega)$, from (29) we deduce that conservation of energy holds for $u_0 \in E$: $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$.

Notice that T, the existence time for which we applied the fixed point method, depends on $\mathcal{E}(u_0)$ and on R. From the conservation of energy for the solutions of (2) we have $\mathcal{E}(u_0) = \mathcal{E}(u(t))$ for all $|t| \leqslant T$. Consequently, we can apply a bootstrap argument and conclude to the extension globally in time of $u \in C(\mathbb{R}, E)$, solution of (2). □

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