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Non-Convex-Valued Differential Inclusions in Banach Spaces

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1. INTRODUCTION

Let $\mathbf{E}$ be a separable reflexive real Banach space. Let F be a multifunction defined on a nonempty open subset of $\mathbf{R} \times \mathbf{E}$ with values in the space of the nonempty compact convex subsets of $\mathbf{E}$.

In the present paper we are concerned with some existence and density results for the differential inclusion

$$\dot{x} \in \operatorname{ext} F(t, x), \qquad x(a) = u, \tag{1.1}$$

where $\operatorname{ext} F(t, x)$ stands for the set of the extreme points of $F(t, x)$. The case in which F satisfies assumptions which exclude compactness has been treated in [15]. The compact case is studied in the present paper.

To this end we use an improved version of the Baire category method introduced in [12–14] in order to prove the existence of solutions for non-convex-valued differential inclusions under noncompactness assumptions on F. To cover the compact case, we use here a new technique resting on the notion of a partition transversal to F and, ultimately, on an appropriate application of the nonempty intersection theorem for a decreasing sequence of nonempty compact sets. In this construction a crucial role is played by a technical approximation result (Lemma 4.2) of the type proved by Pianigiani [28], following some ideas of Ważewski [33] and Antosiewicz and Cellina [1].

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If F is compact and continuous or, more generally, if F is compact and satisfies Carathéodory hypotheses, then the set $\mathcal{M}_{\text{ext}\,F}$ of all solutions of (1.1) is proved to be nonempty. A similar result holds if F satisfies assumptions of α-Lipschitz type, where α denotes the Kuratowski measure of noncompactness. Incidentally, we note that, since $\text{ext}\, F(t, x)$ is not necessarily closed, the aforementioned existence theorems are new also in finite dimension, thus sharpening some results due to Filippov [18], Kaczynski and Olech [20], and Antosiewicz and Cellina [1].

As a matter of fact, in each of the cases considered above, the existence of solutions follows at once from corresponding density results which are of an independent interest. More specifically, suppose that F is continuous (resp. Carathéodory) and compact. Then, as shown in Theorem 4.1 (resp. Theorem 5.1), for each continuous (resp. Carathéodory) selection of F, the set $\mathcal{M}_{\text{ext}\,F}$ has nonempty intersection with P_f, where P_f stands for the solution set of the Cauchy problem $\dot{x} = f(t, x)$, $x(a) = u$.

These density results prove to be useful. Indeed, by using Theorem 5.1, it is shown (Theorem 6.1) that, under an invariance condition on the flow, the boundary value problem $\dot{x} \in \text{ext}\, F(t, x)$, $x(a) = x(b)$, admits solutions. This settles a question that, in the convex case, goes back to Cellina [7]. As another application of Theorem 5.1, it is shown (Theorem 6.2) that the solution set of a compact-valued differential inclusion is closed if and only if the right-hand side is almost everywhere convex valued (see Tolstonogov [32] and Cellina and Ornelas [9]).

The idea of using Baire category for differential inclusions in $\mathbf{R}$ appears in Cellina [8]. Subsequently, the authors [12–14] have developed a method, based on the Baire category, in order to prove the existence of solutions to the Cauchy problem for non-convex-valued differential inclusions in Banach spaces. Further contributions can be found in Bahi [4], Chuong [11], Suslov [29]. More recently, an existence theorem containing both the main result of [15] and Filippov's theorem [18] has been obtained by Bressan and Colombo [5]. In their proof a multivalued version of the Baire category theorem is used, which has furnished a hint for our present approach. For further contributions, from other view points, see Filippov [17], Mushinov [25], Tolstonogov [30], Tolstonogov and Finogenko [31], and Papageorgiou [26, 27].

The paper consists of six sections, with the Introduction. Section 2 contains notations and preliminaries, including a review of some properties of the Choquet function associated to F. In Section 3, the definition of a partition transversal to F is introduced, and some properties of such partitions are established. In Section 4, a first existence and density result (Theorem 4.1) for the Cauchy problem (1.1), under a continuity assumption on F, is proved. Two generalizations, for F satisfying Carathéodory hypotheses (Theorem 5.1), and α-Lipschitz conditions

(Theorem 5.2), are given in Section 5. Some applications are presented in Section 6, namely, an existence result for a boundary value problem (Theorem 6.1), and two characterizations of convex-valued multifunctions (Theorems 6.2 and 6.3).

2. Notations and Preliminaries

Throughout this paper **E** denotes a reflexive separable real Banach space, and $\mathbf{E}^*$ the topological dual of **E**. We denote by $\mathscr{K}(E)$ (resp. $\mathscr{C}(\mathbf{E})$) the metric space of all nonempty, compact (resp. compact convex) subsets of **E** endowed with Hausdorff metric h. For any $A \in \mathscr{C}(\mathbf{E})$, by $\operatorname{ext} A$ we mean the set of the extreme points of A. The closed convex hull of a set $A \subset \mathbf{E}$ is denoted by $\overline{\operatorname{co}}\, A$.

Let X be a metric space with distance d. For any $A \subset X$, we denote by $\bar{A}$ the closure of A and, if A is bounded, by $\alpha(A)$ the Kuratowski measure of noncompactness of A [22]. For any nonempty set $A \subset X$ and $r > 0$ we set $B_X(A, r) = \{x \in X \mid d(x, A) < r\}$ and $\tilde{B}_X(A, r) = \{x \in X \mid d(x, A) \leqslant r\}$, where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. In the sequel, when a set $A \subset X$ is considered as a metric space, it is understood that A retains the metric of X.

Let $G: X \to \mathscr{K}(\mathbf{E})$ be any multifunction. G is said to be *bounded*, *continuous* if it is so as a function from X to the metric space $\mathscr{K}(\mathbf{E})$. G is said to be *compact* if the set $\overline{G(X)}$, where $G(X) = \bigcup_{x \in X} G(x)$, is compact in **E**. Let J be a (Lebesgue) measurable subset of **R**. A multifunction $G: J \to \mathscr{K}(\mathbf{E})$ is said to be *measurable* if for every U open in **E** the set $\{t \in J \mid G(t) \cap U \neq \phi\}$ is measurable. Let $A \subset X$ be nonempty. A single-valued function $g: A \to \mathbf{E}$ such that $g(x) \in G(x)$ for every $x \in A$ is called a *selection* of G on A (a *selection* of G, if $A = X$). For any function $g: X \to \mathbf{E}$ we denote by g_A the restriction of g to A, and by graph g the *graph* of g.

If J is a nonempty compact interval of **R**, by $C(J, \mathbf{E})$ we mean the Banach space of all continuous functions $x: J \to \mathbf{E}$ with the norm of uniform convergence.

Let $J \subset \mathbf{R}$ be a left-closed, bounded, nondegenerate interval. By a *partition* of J we mean a finite family $\{J_k\}_{k=1}^n$ of left-closed, nondegenerate, pairwise disjoint intervals J_k whose union is J. The family of all partitions of J is denoted $\mathscr{F}(J)$. If the intervals of a partition are of the same length β, we say that β is the *step* of the partition.

If $J \subset \mathbf{R}$ is bounded and measurable, $m(J)$ stands for the measure of J. The length of a bounded interval $J \subset \mathbf{R}$ is denoted, for short, by $|J|$.

In the sequel the space $\mathbf{R} \times \mathbf{E}$ (resp. $\mathbf{R} \times \mathbf{E} \times \mathbf{E}$) is supposed to be endowed with the norm $\max\{|t|, \|x\|\}$ (resp. $\max\{|t|, \|x\|, \|y\|\}$), where $(t, x) \in \mathbf{R} \times \mathbf{E}$ (resp. $(t, x, y) \in \mathbf{R} \times \mathbf{E} \times \mathbf{E}$). As usual, **Z** (resp. **N**) denotes the set of the integers (resp. integers $n \geqslant 1$). Given $d \in \mathbf{N}$, we denote by $\mathbf{Z}^d$ the set of all ordered d-uples $h = (h_1, ..., h_d)$ of integers $h_i \in \mathbf{Z}$, $i = 1, ..., d$.

Now let us consider a multifunction

$$F: I \times X \to \mathscr{C}(\mathbf{E}), \tag{2.1}$$

where $I = [a, b]$, $X = \tilde{B}_E(X_0, r)$, $X_0 \in \mathscr{C}(\mathbf{E})$, and $r > 0$. We say that F satisfies (H) if:

(H_1) F is continuous on $I \times X$,

(H_2) the set $A = \overline{F(I \times X)}$ is compact in $\mathbf{E}$,

(H_3) $0 < b - a < r/M$, where $M > h(A, 0)$.

Let F satisfy (H). For $u \in X_0$, we consider the Cauchy problems

$$\dot{x} \in F(t, x), \qquad x(a) = u, \tag{2.2}$$

and

$$\dot{x} \in \operatorname{ext} F(t, x), \qquad x(a) = u. \tag{2.3}$$

By a *solution* of (2.2) (resp. (2.3)) we mean a Lipschitzean function $x: J \to \mathbf{E}$, defined on a nondegenerate interval $J \subset I$ containing a, with $x(a) = u$, satisfying the differential inclusion (2.2) (resp. (2.3)) for $t \in J$ a.e. We set

$$\mathscr{M}_F = \{x: I \to \mathbf{E} \mid x \text{ is a solution of (2.2), for some } u \in X_0\},$$

$$\mathscr{M}_{\operatorname{ext} F} = \{x: I \to \mathbf{E} \mid x \text{ is a solution of (2.3), for some } u \in X_0\}.$$

The set $\mathscr{M}_F$ is nonempty and compact in $C(I, \mathbf{E})$. Thus the space $\mathscr{M}_F$, endowed with the metric of uniform convergence, is complete. For F satisfying (H) we set

$$\mathscr{S}_F = \{f: I \times X \to \mathbf{E} \mid f \text{ is a continuous selection of } F\}.$$

By Michael's theorem [23], $\mathscr{S}_F$ is nonempty. For $f \in \mathscr{S}_F$ and $u \in X_0$, consider the Cauchy problem

$$\dot{x} = f(t, x), \qquad x(a) = u. \tag{2.4}$$

We set

$$P_f = \{x: I \to \mathbf{E} \mid x \text{ is a solution of (2.4), for some } u \in X_0\}.$$

Clearly P_f is a nonempty compact subset of $\mathscr{M}_F$.

Now, let $\{l_n\} \subset \mathbf{E}^*$, $\|l_n\| = 1$, be a sequence dense in the unit sphere of $\mathbf{E}^*$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathbf{E}^*$ and $\mathbf{E}$. For $l \in \mathbf{E}^*$ and $x \in \mathbf{E}$, we put $l(x) = \langle l, x \rangle$. Let F satisfy (H). Following Choquet [10 Vol. II, Chap. 6], we define $\varphi_F: I \times X \times \mathbf{E} \to [0, +\infty[$ by

$$\varphi_F(t, x, v) = \begin{cases} \sum_{n=1}^{+\infty} \frac{(l_n(v))^2}{2^n}, & v \in F(t, x) \\ +\infty, & v \in \mathbf{E} \setminus F(t, x). \end{cases}$$

Let $\mathscr{A}$ be the set of all continuous affine functions $a\colon \mathbf{E} \to \mathbf{R}$. Let $\hat{\varphi}_F : I \times X \times \mathbf{E} \to [-\infty, +\infty[$ be given by $\hat{\varphi}_F(t, x, v) = \inf\{a(v) \mid a \in \mathscr{A}$, and $a(z) > \varphi_F(t, x, z)$ for every $z \in F(t, x)\}$. We define $d_F\colon I \times X \times \mathbf{E} \to [-\infty, +\infty[$ by

$$d_F(t, x, v) = \hat{\varphi}_F(t, x, v) - \varphi_F(t, x, v).$$

Some properties of d_F, the Choquet function associated to F, are reviewed in the next Proposition 2.1 (see [6], [4]).

PROPOSITION 2.1. *Let F satisfy* (H). *Then we have*:

(i) *For each $(t, x) \in I \times X$ and $v \in F(t, x)$ we have $0 \leqslant d_F(t, x, v) \leqslant M^2$. Moreover, $d_F(t, x, v) = 0$ if and only if $v \in \operatorname{ext} F(t, x)$.*

(ii) *For each $(t, x) \in I \times X$, $d_F(t, x, \cdot)$ is strictly concave on $F(t, x)$, and concave on $\mathbf{E}$.*

(iii) *d_F is upper semicontinuous on $I \times X \times \mathbf{E}$.*

(iv) *For each $x \in \mathscr{M}_F$ the function $t \to d_F(t, x(t), \dot{x}(t))$ is nonnegative, bounded, and integrable on I.*

(v) *If $\{x_n\} \subset \mathscr{M}_F$ converges uniformly to $x \in \mathscr{M}_F$, then we have*

$$\int_I d_F(t, x(t), \dot{x}(t))\, dt \geqslant \limsup_{n \to +\infty} \int_I d_F(t, x_n(t), \dot{x}_n(t))\, dt.$$

3. PARTITIONS TRANSVERSAL TO F

In this Section we introduce the notion of a partition transversal to F and we establish some properties of such partitions which will be used later.

Let F satisfy (H). Set

$$C = X_0 + \bigcup_{t \in I} (t - a)\, \overline{\mathrm{co}}\, A, \tag{3.1}$$

and observe that $C \subset X$ and $C \in \mathscr{C}(\mathbf{E})$. Next, define

$$\mathscr{N} = \{x\colon I \to C \mid x \text{ is Lipschitzean with constant} \leqslant M\}. \tag{3.2}$$

Clearly $\mathscr{N}$ is a compact convex subset of $C(I, \mathbf{E})$ containing $\mathscr{M}_F$.

Now let us introduce the notion of a partition of $I \times C$ transversal to F.

DEFINITION 3.1. Let F satisfy (H). Let $l_i \in \mathbf{E}^*$, $\|l_i\| = 1$, $i = 1, ..., d$. Let $\{I_k\}_{k=1}^{k_0} \in \mathscr{F}(I)$ be a partition of I of step β, and let $\alpha > 0$. For $k = 1, ..., k_0$ and $h \in \mathbf{Z}^d$, $h = (h_1, ..., h_d)$, set

$$R_k^h = \{(t, x) \in I \times C \mid t \in I_k, h_i \alpha \leqslant l_i(x) - 2Mt < (h_i + 1)\alpha, \text{ for } i = 1, ..., d\},$$

where M is the constant in (H). The family $\mathscr{R}$ of all nonempty sets R_k^h is called a *partition* of $I \times C$ *transversal* to F (corresponding to $\{l_i\}_{i=1}^d$ and $\{I_k\}_{k=1}^{k_0}$ of *space step* α and *time step* β).

Remark 3.1. $\mathscr{R}$ is a finite partition of $I \times C$, that is $\mathscr{R}$ is a finite family of nonempty pairwise disjoint sets whose union is $I \times C$.

For any $R_k^h \in \mathscr{R}$, the corresponding interval I_k will be also called the *time interval* of R_k^h. Furthermore, by *norm* $\nu(\mathscr{R})$ of $\mathscr{R}$ we mean the largest of the diameters of the sets R_k^h, when R_k^h ranges over $\mathscr{R}$.

LEMMA 3.1. *Let F satisfy* (H). *Let* $\lambda > 0$. *Then there exists a partition* $\mathscr{R}$ *of* $I \times C$ *transversal to F with norm* $\nu(\mathscr{R}) < \lambda$.

Proof. Let $\mathscr{L} = \{l_i\} \subset \mathbf{E}^*$, $\|l_i\| = 1$, be a sequence dense in the unit sphere of $\mathbf{E}^*$. For each $n \in \mathbf{N}$, let $\mathscr{R}_n = \{R_k^h(n)\}$ denote a partition of $I \times C$ transversal to F, corresponding to $\{l_i\}_{i=1}^n$ and $\{I_k^n\}_{k=1}^{e_n}$ of space step α_n and time step β_n, where $\alpha_n \to 0$ and $\beta_n \to 0$ as $n \to +\infty$. We claim that there exists $n_0 \in \mathbf{N}$ such that for every $n \geqslant n_0$ we have $\nu(\mathscr{R}_n) < \lambda$. Suppose the contrary. Then there is a strictly increasing sequence $\{n_j\} \subset \mathbf{N}$ such that for every $j \in \mathbf{N}$ there exist a set $R_k^h(n_j) \in \mathscr{R}_{n_j}$, and two points $(s_{n_j}, x_{n_j}), (t_{n_j}, y_{n_j}) \in R_k^h(n_j)$, such that

$$\max\{|s_{n_j} - t_{n_j}|, \|x_{n_j} - y_{n_j}\|\} > \lambda/2. \tag{3.3}$$

On the other hand, from the definition of $R_k^h(n_j)$, we have

$$|l_i(x_{n_j} - y_{n_j})| < \alpha_{n_j} + 2M\,|s_{n_j} - t_{n_j}| \leqslant \alpha_{n_j} + 2M\beta_{n_j}, \qquad i = 1, ..., n_j.$$

Since $\{x_{n_j}\}$, $\{y_{n_j}\}$ are contained in C, a compact set, passing to subsequences (without change of notations) we have that $x_{n_j} \to x$ and $y_{n_j} \to y$ as $j \to +\infty$, for some $x, y \in C$. Now let $l_i \in \mathscr{L}$ be arbitrary. For every $n_j \geqslant i$ we have

$$\begin{aligned} |l_i(x - y)| &\leqslant |l_i(x_{n_j} - y_{n_j})| + |l_i((x - x_{n_j}) - (y - y_{n_j}))| \\ &< \alpha_{n_j} + 2M\beta_{n_j} + \|x - x_{n_j}\| + \|y - y_{n_j}\| \end{aligned}$$

from which, letting $j \to +\infty$, it follows that $l_i(x - y) = 0$. As $l_i \in \mathscr{L}$ is arbitrary and $\mathscr{L}$ is dense in the unit sphere of $\mathbf{E}^*$, we have $x = y$. This contradicts (3.3), because $|s_{n_j} - t_{n_j}| \to 0$ as $j \to +\infty$, and so the proof is complete.

Remark 3.2. Under the hypotheses of Lemma 3.1, there exists a partition $\mathscr{R}_0$ of $I \times C$ transversal to F, corresponding to $\{l_i\}_{i=1}^{d}$ and $\{J_r\}_{r=1}^{m}$ of space step α and time step β_0, where $0 < \beta_0 < \alpha/(3M)$, with norm $\nu(\mathscr{R}_0) < \lambda$.

LEMMA 3.2. *Let F satisfy* (H). *Let $\varepsilon > 0$ and $\lambda > 0$. Then there exists a partition $\mathscr{R}$ of $I \times C$ transversal to F corresponding to $\{l_i\}_{i=1}^{d}$ and $\{I_k\}_{k=1}^{k_0}$ of space step α and time step β, $0 < \beta < \min\{\varepsilon\,|I|, \varepsilon/(3M)\}$, with norm $\nu(\mathscr{R}) < \lambda$, such that we have:*

$$m\left(\bigcup_{k \in K \setminus K_x} I_k\right) < \varepsilon\,|I| \qquad \textit{for every} \quad x \in \mathscr{N}, \tag{3.4}$$

where $K = \{1, ..., k_0\}$, and

$$K_x = \{k \in K \mid \textit{there exists } h \in \mathbf{Z}^d \textit{ such that graph } x_{I_k} \subset R_k^h,\ R_k^h \in \mathscr{R}\}.$$

Proof. Let $\varepsilon > 0$ and $\lambda > 0$. Let $\mathscr{R}_0$ be as in Remark 3.2. Fix $n \in \mathbf{N}$ such that $n > \max\{\beta_0/(\varepsilon\,|I|), d/\varepsilon\}$. Denote by $\mathscr{R}$ a partition of $I \times C$ transversal to F, corresponding to $\{l_i\}_{i=1}^{d}$ and $\mathscr{I} = \{I_k\}_{k=1}^{k_0}$, $k_0 = mn$, of space step α and time step $\beta = \beta_0/n$. Clearly $\nu(\mathscr{R}) < \lambda$ and $0 < \beta < \min\{\varepsilon\,|I|, \alpha/(3M)\}$. It remains to show that, for such $\mathscr{R}$, (3.4) is satisfied.

Indeed, let $x \in \mathscr{N}$ be any. Let $J_r \in \mathscr{I}_0$, $\mathscr{I}_0 = \{J_r\}_{r=1}^{m}$, and denote by t_r, t_{r+1} $(t_r < t_{r+1})$ the end points of J_r. For $i = 1, ..., d$, set $\psi_i(t) = l_i(x(t)) - 2Mt$, $t \in I$, and observe that $-M > \dot{\psi}_i(t) > -3M$, $t \in I$ a.e. For some $R_k^h \in \mathscr{R}$ we have $(t_r, x(t_r)) \in R_k^h$, thus

$$h_i\alpha \leqslant \psi_i(t_r) < (h_i + 1)\alpha, \qquad i = 1, ..., d. \tag{3.5}$$

Since $\dot{\psi}_i(t) > -3M$ a.e. in J_r, $\psi_i(t_r) \geqslant h_i\alpha$, and $\beta_0 < \alpha/(3M)$, we have

$$\psi_i(t) \geqslant \psi_i(t_r) - 3M(t - t_r) \geqslant h_i\alpha - 3M\beta_0 > (h_i - 1)\alpha, \qquad t \in J_r, \quad i = 1, ..., d. \tag{3.6}$$

As ψ_i is continuous and strictly decreasing on J_r, with $\psi_i(t_r) \geqslant h_i\alpha$, for each $i = 1, ..., d$ there exists at most one point $\tau_i \in J_r$ such that $\psi_i(\tau_i) = h_i\alpha$. Suppose that τ_i is in the interior of J_r (the argument is similar if τ_i is an end point of J_r). Then, by virtue of (3.5) and (3.6), for $i = 1, ..., d$ we have

$$h_i\alpha < \psi_i(t) < (h_i + 1)\alpha, \qquad \text{for each} \quad t < \tau_i, \quad t \in J_r$$

and

$$(h_i - 1)\alpha < \psi_i(t) < h_i\alpha, \qquad \text{for each} \quad t > \tau_i, \quad t \in J_r.$$

From these inequalities it follows that if an interval $I_k \in \mathscr{I}$ contains none of the points τ_i then, for some $h' \in \mathbf{Z}^d$, we have

$$\operatorname{graph} x_{I_k} \subset R_k^{h'}. \tag{3.7}$$

As the intervals $I_k \in \mathscr{I}$, $I_k \subset J_r$, containing some point τ_i are at most d, it follows that in J_r there are at most d intervals $I_k \in \mathscr{I}$ for which (3.7) fails. Since the intervals $J_r \in \mathscr{I}_0$ are m, there are at most md intervals $I_k \in \mathscr{I}$ for which (3.7) fails. Hence

$$m\left(\bigcup_{k \in K \setminus K_x} I_x\right) \leqslant md\,|I_k| = \frac{d}{n}\,k_0\beta < \varepsilon\,|I|,$$

for $d/n < \varepsilon$ and $k_0\beta = |I|$. As $x \in \mathscr{N}$ is arbitrary, (3.4) is satisfied. This completes the proof.

Remark 3.3. From (3.4) it follows that K_x is nonempty, if $0 < \varepsilon < 1$.

Let $\mathscr{R} = \{R_k^h\}$ be a partition of $I \times C$ transversal to F, corresponding to $\{l_i\}_{i=1}^d$ and $\{I_k\}_{k=1}^{k_0}$ of space step α and time step β, satisfying the properties stated in Lemma 3.2. For each $R_k^h \in \mathscr{R}$ consider a partition $\{J_j\}_{j=1}^p \in \mathscr{F}(I_k)$ of I_k, where $p = p(R_k^h)$, and I_k is the time interval of R_k^h. Set

$$R_{k,j}^h = R_k^h \cap (J_j \times \mathbf{E}), \qquad j = 1, ..., p. \tag{3.8}$$

We agree to call J_j the *time interval* of $R_{k,j}^h$. Let $\mathscr{R}'$ be the family of all nonempty sets $R_{k,j}^h$ given by (3.8), when R_k^h ranges over $\mathscr{R}$. Clearly $\mathscr{R}'$ is a finite partition of $I \times C$, and $\mathscr{R}'$ is a refinement of $\mathscr{R}$. Now, set

$$\mu_0 = \min\left\{\frac{c_0}{2}, \frac{\alpha}{2M}, \frac{\varepsilon\,|I|}{2k_0\,p_0}\right\}, \tag{3.9}$$

where $p_0 = \max\{p(R_k^h) \mid R_k^h \in \mathscr{R}\}$, and $c_0 = \min\{|J_j| \mid R_{k,j}^h \in \mathscr{R}'\}$ (J_j the time interval of $R_{k,j}^h$). Let $0 < \mu < \mu_0$. For each $R_{k,j}^h \in \mathscr{R}'$, define

$$\begin{aligned} R_{k,j}^h(\mu) = \{(t, x) \in R_{k,j}^h \mid t \in J_j(\mu),\ & h_i\alpha + \mu M \leqslant l_i(x) - 2Mt \\ & \leqslant (h_i + 1)\alpha - \mu M,\ i = 1, ..., d\}, \end{aligned} \tag{3.10}$$

where $J_j(\mu) = [t_j + \mu, t_{j+1} - \mu]$, and t_j, t_{j+1} ($t_j < t_{j+1}$) are the end points of the time interval J_j of $R_{k,j}^h$. As $0 < \mu < \min\{c_0/2, \alpha/(2M)\}$, the definition of $R_{k,j}^h(\mu)$ makes sense. Denote by $\mathscr{R}'_\mu$ the family of all nonempty sets $R_{k,j}^h(\mu)$ given by (3.10), when $R_{k,j}^h$ ranges over $\mathscr{R}'$. Set

$$A_\mu = \bigcup_{R_{k,j}^h(\mu) \in \mathscr{R}'_\mu} R_{k,j}^h(\mu). \tag{3.11}$$

LEMMA 3.3. *Let F satisfy* (H). *Let $0<\varepsilon<1$ and $\lambda>0$. Let $\mathscr{R}$ be a partition of $I\times C$ transversal to F with the properties stated in Lemma* 3.2. *Let $\mathscr{R}'$, $\mathscr{R}'_\mu$ and A_μ, with $0<\mu<\mu_0$, be as above. Then we have*:

$$m(I\setminus I_x)<2\varepsilon\,|I|, \qquad \textit{for every} \quad x\in\mathscr{N}, \tag{3.12}$$

where $I_x=\{t\in I\mid(t,x(t))\in A_\mu\}$. Moreover, A_μ is a nonempty compact subset of $I\times C$.

Proof. Let $x\in\mathscr{N}$ be any. By Remark 3.3, the set K_x is nonempty. Let $k\in K_x$, thus there exists an $R^h_k\in\mathscr{R}$ such that graph $x_{I_k}\subset R^h_k$, where I_k is the time interval of R^h_k. Let $R^h_{k,j}\subset R^h_k$, $R^h_{k,j}\in\mathscr{R}'$, be any, and let $J_j=[t_j,t_{j+1}]$ be the time interval of $R^h_{k,j}$. Since $(t_j,x(t_j))\in R^h_k$, for each $i=1,...,d$ we have $h_i\alpha\leqslant\psi_i(t_j)<(h_i+1)\alpha$, where $\psi_i(t)=l_i(x(t))-2Mt$, $t\in I$. We claim that

$$\text{graph } x_{J_j(\mu)}\subset R^h_{k,j}(\mu), \tag{3.13}$$

where $R^h_{k,j}(\mu)$ is given by (3.10), and $J_j(\mu)=[t_j+\mu,t_{j+1}-\mu]$. In fact, since $\dot\psi_i(t)<-M$, $t\in I$ *a.e.*, $i=1,...,d$, for each $t\in J_j(\mu)$ we have

$$\psi_i(t)<\psi_i(t_j)-M(t-t_j)<(h_i+1)\alpha-\mu M, \qquad i=1,...,d. \tag{3.14}$$

Similarly, for each $t\in J_j(\mu)$ we have $\psi_i(t)>h_i\alpha+\mu M$, $i=1,...,d$. From these inequalities and from (3.14), the claim (3.13) follows at once. As graph $x_{I_k}\subset R^h_k$ and $R^h_k=\bigcup_{j=1}^p R^h_{k,j}$, by virtue of (3.13) we have

$$m(\{t\in I_k\mid(t,x(t))\notin A_\mu\})\leqslant m\left(\bigcup_{j=1}^{p}(J_j\setminus J_j(\mu))\right)\leqslant 2\mu p\leqslant 2\mu p_0,$$

and so

$$m\left(\bigcup_{k\in K_x}\{t\in I_k\mid(t,x(t))\notin A_\mu\}\right)\leqslant 2\mu p_0k_0<\varepsilon\,|I|, \tag{3.15}$$

for $\mu<\varepsilon\,|I|/(2p_0k_0)$. From (3.15) and Lemma 3.2 we have

$$\begin{aligned} m(I\setminus I_x)&=m\left(\bigcup_{k\in K_x}\{t\in I_k\mid(t,x(t))\notin A_\mu\}\right)\\ &\quad+m\left(\bigcup_{k\in K\setminus K_x}\{t\in I_k\mid(t,x(t))\notin A_\mu\}\right)\\ &<\varepsilon\,|I|+\varepsilon\,|I|=2\varepsilon\,|I|, \end{aligned}$$

from which (3.12) follows, as $x\in\mathscr{N}$ is arbitrary. The last statement of the lemma is evident. This completes the proof.

4. Main Results

In this Section we prove an existence and density theorem for the Cauchy problem (2.3). Our approach is based on the method of the Baire category. Here a fundamental role is played by the approximation Lemma 4.2.

Let F satisfy (H). For $\theta > 0$, set

$$\mathcal{M}_\theta = \left\{ x \in \mathcal{M}_F \,\middle|\, \int_I d_F(t, x(t), \dot{x}(t))\, dt < \theta \right\}.$$

Lemma 4.1. *Let F satisfy* (H). *Then for every $\theta > 0$ the set $\mathcal{M}_\theta$ is open in $\mathcal{M}_F$.*

Proof. Let $\{x_n\} \subset \mathcal{M}_F \setminus \mathcal{M}_\theta$ be any sequence which converges uniformly to an x in $\mathcal{M}_F$. By Proposition 2.1 (v), we have

$$\int_I d_F(t, x(t), \dot{x}(t))\, dt \geqslant \limsup_{n \to +\infty} \int_I d_F(t, x_n(t), \dot{x}_n(t))\, dt \geqslant \theta,$$

and thus $x \in \mathcal{M}_F \setminus \mathcal{M}_\theta$. This shows that $\mathcal{M}_F \setminus \mathcal{M}_\theta$ is closed in $\mathcal{M}_F$, completing the proof.

Lemma 4.2. *Let F satisfy* (H). *Let $f \in \mathcal{S}_F$, and let $\theta > 0$ and $\delta > 0$. Then there exists a function $g \in \mathcal{S}_F$ such that, for every $x \in \mathcal{N}$, we have*

$$\int_I d_F(t, x(t), g(t, x(t)))\, dt < \theta, \tag{4.1}$$

and

$$\sup_{t \in I} \left\| \int_a^t [g(s, x(s)) - f(s, x(s))]\, ds \right\| < \delta. \tag{4.2}$$

Proof. Let $f \in \mathcal{S}_F$, $\theta > 0$, and $\delta > 0$. Let ε be such that

$$0 < \varepsilon < \min \left\{ \frac{\theta}{4(1 + M^2)\,|I|}, \frac{\delta}{2(1 + 4M)\,|I|}, 1 \right\},$$

where M is the constant occurring in the assumption (H). Moreover, set $Z = I \times X$.

Step 1. (Local approximation of f by functions taking values near the extreme points of F).

Let $(s, u) \in I \times C$, where C is given by (3.1). Since $f(s, u) \in F(s, u)$ by Krein–Mil'man's theorem there exist an integer $p_{s,u} \in \mathbf{N}$, points $v^j_{s,u} \in \operatorname{ext} F(s, u)$, $j = 1, ..., p_{s,u}$, and numbers $\lambda^j_{s,u}$, $0 < \lambda^j_{s,u} \leqslant 1$, $j = 1, ..., p_{s,u}$, with $\lambda^1_{s,u} + \cdots + \lambda^{p_{s,u}}_{s,u} = 1$, such that

$$\left\| \sum_{j=1}^{p_{s,u}} \lambda^j_{s,u} v^j_{s,u} - f(s, u) \right\| < \frac{\varepsilon}{3}. \tag{4.3}$$

By Proposition 2.1 (i), (iii), d_F is upper semicontinuous and vanishes at $(s, u, v^j_{s,u})$, thus there exists $0 < \rho^0_{s,u} < \varepsilon/3$ such that, for every $(t, x) \in B_Z((s, u), \rho^0_{s,u})$ and every $v \in \tilde{B}_E(v^j_{s,u}, \rho^0_{s,u})$, $j = 1, ..., p_{s,u}$, we have

$$d_F(t, x, v) < \varepsilon. \tag{4.4}$$

As F is continuous at (s, u) and $v^j_{s,u} \in F(s, u)$, there exists $0 < \rho^1_{s,u} < \rho^0_{s,u}$ such that for every $(t, x) \in B_Z((s, u), \rho^1_{s,u})$ we have

$$F(t, x) \cap B_E(v^j_{s,u}, \rho^0_{s,u}) \neq \phi, \qquad j = 1, ..., p_{s,u}.$$

By Michael's theorem [23], there exist $p_{s,u}$ continuous functions $z^j_{s,u}\colon B_z((s, u), \rho^1_{s,u}) \to \mathbf{E}$ such that for every $(t, x) \in B_Z((s, u), \rho^1_{s,u})$ we have

$$z^j_{s,u}(t, x) \in F(t, x) \cap \tilde{B}_E(v^j_{s,u}, \rho^0_{s,u}), \qquad j = 1, ..., p_{s,u}. \tag{4.5}$$

By the continuity of f at (s, u), and by virtue of (4.5) and (4.4), there exists $0 < \rho_{s,u} < \rho^1_{s,u}$ such that for every $(t, x) \in B_Z((s, u), \rho_{s,u})$ we have:

$$\|f(t, x) - f(s, u)\| < \frac{\varepsilon}{3}, \tag{4.6}$$

$$\|z^j_{s,u}(t, x) - v^j_{s,u}\| < \frac{\varepsilon}{3}, \qquad j = 1, ..., p_{s,u}. \tag{4.7}$$

$$d_F(t, x, z^j_{s,u}(t, x)) < \varepsilon, \qquad j = 1, ..., p_{s,u}. \tag{4.8}$$

By construction, the functions $z^j_{s,u}$ assume values near the extreme points of $F(s, u)$ (by virtue of (4.8)), and approximate f (by virtue of (4.3), (4.6), and (4.7)).

Step 2. (Construction of γ, a discontinuous selection of F on $I \times C$).

The family $\{z((s, u), \rho_{s,u}) \mid (s, u) \in I \times C\}$ is an open covering of $I \times C$, a compact set, and so it contains a finite subcovering, say

$$\{B_Z((s_n, u_n), \rho_{s_n, u_n})\}_{n=1}^{n_0}. \tag{4.9}$$

Let $\lambda > 0$ be a Lebesgue number of this subcovering. By Lemma 3.2, for the given ε and λ, there exists a partition $\mathscr{R} = \{R^h_k\}$ of $I \times C$ transversal to F, corresponding to $\{l_i\}_{i=1}^d$ and $\{I_k\}_{k=1}^{k_0}$ of space step α and time step β, $0 < \beta < \min\{\varepsilon\, |I|, \alpha/(3M)\}$, with $v(\mathscr{R}) < \lambda$, such that (3.4) is satisfied. Clearly each $R^h_k \in \mathscr{R}$ is contained in at least one ball of the family (4.9) for, by construction, R^h_k has diameter strictly smaller than λ. Now let $\Phi\colon \mathscr{R} \to \mathbf{N}$

be a function which assigns to each $R_k^h \in \mathscr{R}$ one and only one integer, fixed in an arbitrary way among the integers n, $1 \leqslant n \leqslant n_0$, such that

$$R_k^h \subset B_Z((s_n, u_n), \rho_n), \qquad \text{where} \quad \rho_n = \rho_{s_n, u_n}. \tag{4.10}$$

Take $R_k^h \in \mathscr{R}$ and suppose $\Phi(R_k^h) = n$. Then (4.10) is fulfilled and, from the construction in Step 1, there exist a $p_n = p_{s_n, u_n} \in \mathbf{N}$, p_n numbers $\lambda_n^j = \lambda_{s_n, u_n}^j$ with $0 < \lambda_n^j \leqslant 1$, and $\lambda_n^1 + \cdots + \lambda_n^{p_n} = 1$, p_n points $v_n^j = v_{s_n, u_n}^j \in \operatorname{ext} F(s_n, u_n)$, and p_n continuous selections $z_n^j = z_{s_n, u_n}^j$ of F, defined on $B_Z((s_n, u_n), \rho_n)$, such that we have

$$\left\| \sum_{j=1}^{p_n} \lambda_n^j v_n^j - f(s_n, u_n) \right\| < \frac{\varepsilon}{3} \tag{4.11}$$

and, for every $(t, x) \in B_Z((s_n, u_n), \rho_n)$,

$$\|f(t, x) - f(s_n, u_n)\| < \frac{\varepsilon}{3}, \tag{4.12}$$

$$\|z_n^j(t, x) - v_n^j\| < \frac{\varepsilon}{3}, \qquad j = 1, ..., p_n, \tag{4.13}$$

$$d_F(t, x, z_n^j(t, x)) < \varepsilon, \qquad j = 1, ..., p_n. \tag{4.14}$$

Now we are ready to construct γ, a perhaps discontinuous selection of F on $I \times C$. To this end, let $R_k^h \in \mathscr{R}$ be any and let $\Phi(R_k^h) = n$. Let p_n, λ_n^j, v_n^j, and z_n^j, $j = 1, ..., p_n$, correspond accordingly so that (4.11)–(4.14) are satisfied. Denoting I_k the time interval of R_k^h, construct the partition $\{J_j\}_{j=1}^{p_n} \in \mathscr{F}(I_k)$ of I_k in p_n intervals J_j of length

$$|J_j| = \lambda_n^j \, |I_k|, \qquad j = 1, ..., p_n, \tag{4.15}$$

and set

$$R_{k,j}^h = R_k^h \cap (J_j \times \mathbf{E}), \qquad j = 1, ..., p_n. \tag{4.16}$$

As R_k^h ranges over $\mathscr{R}$, the family $\mathscr{R}'$ of those sets $R_{k,j}^h$, given by (4.16), which are nonempty, is a finite partition of $I \times C$. Define $\gamma \colon I \times C \to \mathbf{E}$ by

$$\gamma(t, x) = z_n^j(t, x), \qquad \text{if} \quad (t, x) \in R_{k,j}^h. \tag{4.17}$$

This definition is unambiguous. Moreover, as the functions z_n^j, $j = 1, ..., p_n$ are continuous selections of F on $B_Z((s_n, u_n), \rho_n) \supset R_k^h$, it follows that γ is a perhaps discontinuous selection of F on $I \times C$ whose restriction to each set $R_{k,j}^h$ is continuous.

Step 3. (Properties of γ).
For every $x \in \mathcal{N}$ we have

$$\int_I d_F(t, x(t), \gamma(t, x(t)))\, dt < \frac{\theta}{2}, \tag{4.18}$$

$$\sup_{t \in I} \left\| \int_a^t [\gamma(s, x(s)) - f(s, x(s))]\, ds \right\| < \frac{\delta}{2}. \tag{4.19}$$

Indeed, let $x \in \mathcal{N}$ be any. By construction $\mathcal{R}$ satisfies the properties stated in Lemma 3.2, so we have

$$m\left(\bigcup_{k \in K \setminus K_x} I_k \right) < \varepsilon\, |I|, \tag{4.20}$$

where $K = \{1, ..., k_0\}$, and

$$K_x = \{k \in K \mid \text{there exists } h \in \mathbf{Z}^d \text{ such that graph } x_{I_k} \subset R_k^h,\ R_k^h \in \mathcal{R}\}.$$

As $0 < \varepsilon < 1$, the set K_x is nonempty.

Let $k \in K_x$. Then there exists an $R_k^h \in \mathcal{R}$ such that graph $x_{I_k} \subset R_k^h$ (I_k the time interval of R_k^h) and, for some $1 \leqslant n \leqslant n_0$, (4.10) is satisfied. Moreover, with the notations of Step 2, we have

$$\text{graph } x_{J_j} \subset R_{k,j}^h \subset B_Z((s_n, u_n), \rho_n), \qquad j = 1, ..., p_n, \tag{4.21}$$

where $\{J_j\}_{j=1}^{p_n} \in \mathcal{F}(I_k)$ satisfies (4.15), and the sets $R_{k,j}^h$ are given by (4.16). As consequence of (4.21) and (4.17) we have

$$\gamma(t, x(t)) = z_n^j(t, x(t)), \qquad t \in J_j, \quad j = 1, ..., p_n. \tag{4.22}$$

Now we prove the following inequalities:

$$\int_{I_k} d_F(t, x(t), \gamma(t, x(t)))\, dt < \varepsilon\, |I_k|, \qquad k \in K_x, \tag{4.23}$$

$$\left\| \int_{I_k} [\gamma(t, x(t)) - f(t, x(t))]\, dt \right\| < \varepsilon\, |I_k|, \qquad k \in K_x. \tag{4.24}$$

Let $k \in K_x$. By virtue of (4.22), (4.21), (4.14), and (4.15) we have

$$\begin{aligned}\int_{I_k} d_F(t, x(t), \gamma(t, x(t)))\, dt &= \sum_{j=1}^{p_n} \int_{J_j} d_F(t, x(t), z_n^j(t, x(t)))\, dt \\ &< \varepsilon \sum_{j=1}^{p_n} |J_j| = \varepsilon\, |I_k|,\end{aligned}$$

and so (4.23) is true. It remains (4.24). By virtue of (4.22) we have

$$\left\| \int_{I_k} [\gamma(t, x(t)) - f(t, x(t))] \, dt \right\| = \left\| \sum_{j=1}^{p_n} \int_{J_j} [z_n^j(t, x(t)) - f(t, x(t))] \, dt \right\|$$

$$\leqslant \sum_{j=1}^{p_n} \int_{J_j} [\|z_n^j(t, x(t)) - v_n^j\| + \|f(t, x(t)) - f(s_n, u_n)\|] \, dt$$

$$+ \left\| \sum_{j=1}^{p_n} \int_{J_j} [v_n^j - f(s_n, u_n)] \, dt \right\|.$$

Hence, by using (4.21), (4.13), (4.12), (4.15), and (4.11) we have

$$\left\| \int_{I_k} [\gamma(t, x(t)) - f(t, x(t))] \, dt \right\| < \sum_{j=1}^{p_n} \left(\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \right) |J_j|$$

$$+ \left\| \sum_{j=1}^{p_n} (v_n^j - f(s_n, u_n)) \, |J_j| \right\| = \frac{2}{3} \varepsilon \, |I_k| + \left\| \sum_{j=1}^{p_n} \lambda_n^j v_n^j - f(s_n, u_n) \right\| \, |I_k| < \varepsilon \, |I_k|,$$

and so (4.24) is also satisfied.

Now we are ready to prove (4.18) and (4.19). Let $x \in \mathcal{N}$. By virtue of (4.23) and (4.20) we have

$$\int_I d_F(t, x(t), \gamma(t, x(t))) \, dt = \sum_{k \in K_x} \int_{I_k} d_F(t, x(t), \gamma(t, x(t))) \, dt$$

$$+ \sum_{k \in K \setminus K_x} \int_{I_k} d_F(t, x(t), \gamma(t, x(t))) \, dt$$

$$< \sum_{k \in K_x} \varepsilon \, |I_k| + \sum_{k \in K \setminus K_x} M^2 \, |I_k| < \varepsilon \, |I| + \varepsilon M^2 \, |I| < \frac{\theta}{4},$$

for $\varepsilon < \theta/(4(1 + M^2) \, |I|)$, and so (4.18) is true. It remains (4.19). Let $t \in I$ be any, thus $t \in I_{\bar{k}}$ for some $\bar{k} \in K$. We have

$$\left\| \int_a^t [\gamma(s, x(s)) - f(s, x(s))] \, ds \right\| \leqslant \sum_{k \in K_x} \left\| \int_{I_k} [\gamma(s, x(s)) - f(s, x(s))] \, ds \right\|$$

$$+ \sum_{k \in K \setminus K_x} \left\| \int_{I_k} [\gamma(s, x(s)) - f(s, x(s))] \, ds \right\| + \int_{I_{\bar{k}}} \|\gamma(s, x(s)) - f(s, x(s))\| \, ds.$$

From this, by virtue of (4.24) and (4.20), recalling that $|I_{\bar{k}}| = \beta < \varepsilon \, |I|$, we have

$$\begin{aligned} &\left\| \int_a^t [\gamma(s, x(s)) - f(s, x(s))] \, ds \right\| \\ &\quad < \sum_{k \in K_x} \varepsilon \, |I_k| + \sum_{k \in K \setminus K_x} 2M \, |I_k| + 2M\varepsilon \, |I| \\ &\quad < \varepsilon \, |I| + 2\varepsilon M \, |I| + 2\varepsilon M \, |I| = \varepsilon(1 + 4M) \, |I|, \end{aligned}$$

from which (4.19) follows at once, because $t \in I$ is arbitrary and $\varepsilon < \delta/(2(1 + 4M) \, |I|)$.

Step 4. (Construction of a continuous selection g of F approximating γ).

By restricting γ to an appropriate compact set $A_\mu \subset I \times C$ we make γ a continuous selection of F on A_μ. By Michael's theorem [23], γ admits a continuous extension g, say, which is also a selection of F. It is shown that such g satisfies the properties stated in Lemma 4.2.

We retain the notations of Step 2. Let $\mathscr{R}$ and $\mathscr{R}'$ be as in Step 2. Let

$$\mu_0 = \min\{c_0/2, \alpha/(2M), \varepsilon \, |I|/(2k_0 \, p_0)\},$$

where $c_0 = \min\{|J_j| \mid R^h_{k,j} \in \mathscr{R}'\}$ (J_j the time interval of $R^h_{k,j}$), and $p_0 = \max\{p_n \mid 1 \leqslant n \leqslant n_0\}$. Fix $0 < \mu < \mu_0$. Denote by $\mathscr{R}'_\mu$ the family of all nonempty sets $R^h_{k,j}(\mu)$, given by (3.10), when $R^h_{k,j}$ ranges over $\mathscr{R}'$ and let A_μ be defined by (3.11). By Lemma 3.3, A_μ is a nonempty compact subset of $I \times C$. As γ, restricted to A_μ, is a continuous selection of F, by Michael's theorem [23] there exists a continuous selection g of F such that

$$g(t, x) = \gamma(t, x), \qquad \text{for every} \quad (t, x) \in A_\mu. \tag{4.25}$$

Now let $x \in \mathscr{N}$ be any. Set $I_x = \{t \in I \mid (t, x(t)) \in A_\mu\}$, and observe that by Lemma 3.3 we have

$$m(I \setminus I_x) < 2\varepsilon \, |I|. \tag{4.26}$$

By virtue of (4.25), we have

$$\int_I d_F(t, x(t), g(t, x(t)))\, dt \leqslant \int_I d_F(t, x(t), \gamma(t, x(t)))\, dt$$

$$+ \int_{I \setminus I_x} |d_F(t, x(t), g(t, x(t))) - d_F(t, x(t), \gamma(t, x(t)))|\, dt.$$

From this, in view of (4.18), Proposition 2.1(i), and (4.26) we have

$$\int_I d_F(t, x(t), g(t, x(t)))\, dt < \frac{\theta}{2} + M^2 m(I \setminus I_x) < \frac{\theta}{2} + 2\varepsilon M^2 \, |I| < \theta,$$

where the last inequality holds since $\varepsilon < \theta/(4(1+M^2)\,|I|)$. Hence (4.1) is satisfied. It remains (4.2). For each $t \in I$ we have

$$\left\| \int_a^t [g(s, x(s)) - f(s, x(s))]\, ds \right\| \leqslant \int_I \|g(s, x(s)) - \gamma(s, x(s))\|\, ds$$

$$+ \left\| \int_a^t [\gamma(s, x(s)) - f(s, x(s))]\, ds \right\|.$$

From this, by virtue of (4.25), (4.19), and (4.26) we have

$$\left\| \int_a^t [g(s, x(s)) - f(s, x(s))]\, ds \right\| < \int_{I \setminus I_x} \|g(s, x(s)) - \gamma(s, x(s))\|\, ds + \frac{\delta}{2}$$

$$< 2Mm(I \setminus I_x) + \frac{\delta}{2} < 4\varepsilon M\, |I| + \frac{\delta}{2} < \delta,$$

where the last inequality holds since $\varepsilon < \delta/(2(1+4M)\,|I|)$. Hence also (4.2) is satisfied. As $g \in \mathscr{S}_F$ and, for $x \in \mathscr{N}$ arbitrary, (4.1) and (4.2) are satisfied, Lemma 4.2 is proved.

LEMMA 4.3. *Let F satisfy* (H). *Let $f \in \mathscr{S}_F$, and let $\varepsilon > 0$. Then there exists $\delta = \delta_f(\varepsilon) > 0$ such that, for $x \in \mathscr{M}_F$ any,*

$$\sup_{t \in I} \left\| \int_a^t [\dot{x}(s) - f(s, x(s))]\, ds \right\| < \delta \qquad \textit{implies} \quad x \in B_{\mathscr{M}_F}(P_f, \varepsilon). \tag{4.27}$$

Proof. If the statement is not true, there exist $f \in \mathscr{S}_F$, $\varepsilon > 0$, and a sequence $\{x_n\} \subset \mathscr{M}_F \setminus B_{\mathscr{M}_F}(P_f, \varepsilon)$ satisfying

$$\sup_{t \in I} \left\| \int_a^t [\dot{x}_n(s) - f(s, x_n(s))]\, ds \right\| < \frac{1}{n}, \qquad n \in \mathbf{N}.$$

As $\mathcal{M}_F$ is compact, a subsequence of $\{x_n\}$ converges uniformly to a point $x \in P_f$, and so for n large enough we have $x_n \in B_{\mathcal{M}_F}(P_f, \varepsilon)$, a contradiction. This completes the proof.

LEMMA 4.4. *Let F satisfy* (H). *Let $f \in \mathcal{S}_F$, and let $\varepsilon > 0$ and $\theta > 0$. Then there exists a function $g \in \mathcal{S}_F$ such that $P_g \subset \mathcal{M}_\theta \cap B_{\mathcal{M}_F}(P_f, \varepsilon)$.*

Proof. Let $f \in \mathcal{S}_F$, $\varepsilon > 0$, and $\theta > 0$. By Lemma 4.3 there exists $\delta > 0$ such that (4.27) holds, for $x \in \mathcal{M}_F$. By Lemma 4.2, there exists $g \in \mathcal{S}_F$ (corresponding to f, θ, and δ) such that, for every $x \in \mathcal{N}$, we have

$$\int_I d_F(t, x(t), g(t, x(t)))\, dt < \theta, \tag{4.28}$$

and

$$\sup_{t \in I} \left\| \int_a^t [g(s, x(s)) - f(s, x(s))]\, ds \right\| < \delta. \tag{4.29}$$

Now let $x \in P_g$ be arbitrary. As $\dot{x}(t) = g(t, x(t))$, $t \in I$, from (4.29) and (4.27) it follows that $x \in B_{\mathcal{M}_F}(P_f, \varepsilon)$. Moreover, (4.28) implies that $x \in \mathcal{M}_\theta$. Hence $x \in \mathcal{M}_\theta \cap B_{\mathcal{M}_F}(P_f, \varepsilon)$ and so $P_g \subset \mathcal{M}_\theta \cap B_{\mathcal{M}_F}(P_f, \varepsilon)$, as $x \in P_g$ is arbitrary. This completes the proof.

THEOREM 4.1. *Let F satisfy* (H). *Let $f \in \mathcal{S}_F$ and $\varepsilon > 0$. Then $\mathcal{M}_{\text{ext}\, F} \cap B_{\mathcal{M}_F}(P_f, \varepsilon) \neq \phi$ and, in particular, the Cauchy problem* (2.3) *has solutions.*

Proof. Set $\theta_n = 1/n$, $n \in \mathbf{N}$. By Lemma 4.4, there exists $g_1 \in \mathcal{S}_F$ such that $P_{g_1} \subset B_{\mathcal{M}_F}(P_f, \varepsilon)$. Hence, as P_{g_1} is compact, there exists $0 < \eta_1 < \theta_1$ such that

$$\tilde{B}_{\mathcal{M}_F}(P_{g_1}, \eta_1) \subset B_{\mathcal{M}_F}(P_f, \varepsilon). \tag{4.30}$$

Similarly, there exists $g_2 \in \mathcal{S}_F$ such that $P_{g_2} \subset \mathcal{M}_{\theta_1} \cap B_{\mathcal{M}_F}(P_{g_1}, \eta_1)$. This set is open in $\mathcal{M}_F$, for $\mathcal{M}_{\theta_1}$ is so by Lemma 4.1. As P_{g_2} is compact, there exists $0 < \eta_2 < \theta_2$ such that

$$\tilde{B}_{\mathcal{M}_F}(P_{g_2}, \eta_2) \subset \mathcal{M}_{\theta_1} \cap B_{\mathcal{M}_F}(P_{g_1}, \eta_1).$$

Continuing in this way gives a decreasing sequence of nonempty compact subsets $\tilde{B}_{\mathcal{M}_F}(P_{g_n}, \eta_n)$ of $\mathcal{M}_F$, with $g_n \in \mathcal{S}_F$ and $0 < \eta_n < \theta_n$, satisfying

$$\tilde{B}_{\mathcal{M}_F}(P_{g_{n+1}}, \eta_{n+1}) \subset \mathcal{M}_{\theta_n} \cap B_{\mathcal{M}_F}(P_{g_n}, \eta_n), \qquad n \in \mathbf{N}.$$

Let $x \in \mathscr{M}_F$ be a point belonging to each set $\tilde{B}_{\mathscr{M}_F}(P_{g_n}, \eta_n)$, $n \in \mathbf{N}$. By (4.30), we have $x \in B_{\mathscr{M}_F}(P_f, \varepsilon)$. Moreover, since $x \in \mathscr{M}_{\theta_n}$ for every $n \in \mathbf{N}$, we have

$$\int_I d_F(t, x(t), \dot{x}(t))\, dt = 0,$$

and thus, by Proposition 2.1 (i), $\dot{x}(t) \in \operatorname{ext} F(t, x(t))$, $t \in I$ *a.e.*. Hence $x \in \mathscr{M}_{\operatorname{ext} F} \cap B_{\mathscr{M}_F}(P_f, \varepsilon)$. The last statement of Theorem 4.1 is evident. This completes the proof.

5. Some Extensions

The existence and density results proved in Section 4 under a continuity assumption on F are extended in this Section to the case in which F is Carathéodory.

Let F be the multifunction given by (2.1). We say that F satisfies (H′) if:

(H'_1) for each $t \in I$ the multifunction $x \to F(t, x)$ is continuous on X, and for each $x \in X$ the multifunction $t \to F(t, x)$ is measurable on I,

(H'_2) the set $A = \overline{F(I \times X)}$ is compact in $\mathbf{E}$,

(H'_3) $0 < b - a < r/M$, where $M > h(A, 0)$.

Let F satisfy (H′), and let $\mathscr{M}_F$, $\mathscr{M}_{\operatorname{ext} F}$, $\mathscr{M}_\theta$, and $\mathscr{N}$ be defined accordingly. The set $\mathscr{M}_F$ is nonempty and compact in $C(I, \mathbf{E})$. Hence $\mathscr{M}_F$, endowed with the metric of uniform convergence, is complete.

A selection f of F is said to be a *Carathéodory selection* of F if for each $t \in I$ the function $x \to f(t, x)$ is continuous on X, and for each $x \in X$ the function $t \to f(t, x)$ is Bochner measurable on I. For F satisfying (H′) we set

$$\mathscr{S}'_F = \{f\colon I \times X \to \mathbf{E} \mid f \text{ is a Carathéodory selection of } F\}.$$

By virtue of the theorems of Scorza Dragoni [19] and Michael [23], the set $\mathscr{S}'_F$ is nonempty.

Remark 5.1. If F satisfies (H′), then the properties (i), (ii), (iv), (v) in Proposition 2.1 are satisfied.

The following Lemma 5.1 can be proved as Lemma 4.1.

LEMMA 5.1. *Let F satisfy* (H′). *Then for every $\theta > 0$ the set $\mathscr{M}_\theta$ is open in $\mathscr{M}_F$.*

LEMMA 5.2. *Let F satisfy* (H′). *Let $f \in \mathscr{S}'_F$, and let $\theta > 0$ and $\delta > 0$. Then there exists a function $g \in \mathscr{S}'_F$ such that, for every $x \in \mathscr{N}$, we have*

$$\int_I d_F(t, x(t), g(t, x(t)))\, dt < \theta, \tag{5.1}$$

and

$$\sup_{t \in I} \left\| \int_a^t [g(s, x(s)) - f(s, x(s))] \, ds \right\| < \delta. \tag{5.2}$$

Proof. Let $f \in \mathscr{S}'_F$, $\theta > 0$, $\delta > 0$. By Scorza Dragoni's theorem [19], there exists a nonempty compact set $J \subset I$, with $m(I \setminus J) < \min\{\theta/(2M^2), \delta/(8M)\}$, such that F and f restricted to the set $J \times X$ are continuous. By a continuous extension theorem for multifunctions [2], there exists a continuous compact multifunction $\tilde{F}: I \times X \to \mathscr{C}(\mathbf{E})$, with values contained in $\overline{\text{co}}\, A$, such that $\tilde{F}(t, x) = F(t, x)$ for every $(t, x) \in J \times X$. By Michael's theorem [23], there exists a continuous selection $\tilde{f}$ of $\tilde{F}$ such that $\tilde{f}(t, x) = f(t, x)$ for every $(t, x) \in J \times X$. Clearly $\mathscr{M}_{\tilde{F}} \subset \mathscr{N}$.

As $\tilde{F}$ satisfies (H) and $\tilde{f} \in \mathscr{S}_{\tilde{F}}$, by Lemma 4.2 there exists $\tilde{g} \in \mathscr{S}_{\tilde{F}}$ such that for every $x \in \mathscr{N}$ we have

$$\int_I d_{\tilde{F}}(t, x(t), \tilde{g}(t, x(t))) \, dt < \frac{\theta}{2}, \tag{5.3}$$

and

$$\sup_{t \in I} \left\| \int_a^t [\tilde{g}(s, x(s)) - \tilde{f}(s, x(s))] \, ds \right\| < \frac{\delta}{2}. \tag{5.4}$$

Since $\tilde{g} \in \mathscr{S}_{\tilde{F}}$, and $\tilde{F}(t, x) = F(t, x)$ for every $(t, x) \in J \times X$, there exists a Carathéodory selection g of F such that $g(t, x) = \tilde{g}(t, x)$ for every $(t, x) \in J \times X$. As $g \in \mathscr{S}'_F$, to complete the proof it remains to show that for this g, and for $x \in \mathscr{N}$ arbitrary, (5.1) and (5.2) are satisfied.

Indeed, let $x \in \mathscr{N}$. For every $t \in J$ we have $F(t, x(t)) = \tilde{F}(t, x(t))$ and $g(t, x(t)) = \tilde{g}(t, x(t))$. In view of these relations, of (5.3) and of Proposition 2.1 (i), we have

$$\begin{aligned} \int_I d_F(t, x(t), g(t, x(t))) \, dt &\leqslant \int_I d_{\tilde{F}}(t, x(t), \tilde{g}(t, x(t))) \, dt \\ &\quad + \int_{I \setminus J} |d_F(t, x(t), g(t, x(t))) - d_{\tilde{F}}(t, x(t), \tilde{g}(t, x(t)))| \, dt \\ &< \frac{\theta}{2} + M^2 m(I \setminus J) < \theta, \end{aligned}$$

for $m(I \setminus J) < \theta/(2M^2)$. Hence (5.1) is satisfied. To prove (5.2), observe that $g(t, x(t)) = \tilde{g}(t, x(t))$ and $f(t, x(t)) = \tilde{f}(t, x(t))$, for every $t \in J$. In view of these equalities and of (5.4), we have

$$\sup_{t \in I} \left\| \int_a^t [g(s, x(s)) - f(s, x(s))] \, ds \right\| \leqslant \sup_{t \in I} \left\| \int_a^t [\tilde{g}(s, x(s)) - \tilde{f}(s, x(s))] \, ds \right\|$$

$$+ \int_{I \setminus J} [\| g(s, x(s)) - \tilde{g}(s, x(s)) \| + \| \tilde{f}(s, x(s)) - f(s, x(s)) \|] \, ds$$

$$< \frac{\delta}{2} + 4Mm(I \setminus J) < \delta,$$

for $m(I \setminus J) < \delta/(8M)$, and so also (5.2) is satisfied. As $x \in \mathcal{N}$ is arbitrary, the proof is complete.

The following Lemma 5.3 and Lemma 5.4 can be proved as Lemma 4.3 and Lemma 4.4, respectively.

LEMMA 5.3. *Let F satisfy* (H′). *Let $f \in \mathcal{S}'_F$, and let $\varepsilon > 0$. Then there exists $\delta = \delta_f(\varepsilon) > 0$ such that, for $x \in \mathcal{M}_F$ any, the implication* (4.27) *is satisfied.*

LEMMA 5.4. *Let F satisfy* (H′). *Let $f \in \mathcal{S}'_F$, and let $\varepsilon > 0$ and $\theta > 0$. Then there exists a function $g \in \mathcal{S}'_F$ such that $P_g \subset \mathcal{M}_\theta \cap B_{\mathcal{M}_F}(P_f, \varepsilon)$.*

By virtue of Lemma 5.4, Lemma 5.1, using the same argument of Theorem 4.1, one can prove the following Theorem 5.1.

THEOREM 5.1. *Let F satisfy* (H′). *Let $f \in \mathcal{S}'_F$ and $\varepsilon > 0$. Then $\mathcal{M}_{\text{ext}\,F} \cap B_{\mathcal{M}_F}(P_f, \varepsilon) \neq \phi$ and, in particular, the Cauchy problem* (2.3) *has solutions.*

We say that the multifunction F, given by (2.1), satisfies (K) if:

(K_1) F is continuous on $I \times X$,

(K_2) the set $A = F(I \times X)$ is bounded, that is $h(A, 0) < M$, and there exists a constant $L > 0$ such that $\alpha(F(I \times Y)) \leqslant L\alpha(Y)$ for every $Y \subset X$,

(K_3) $0 < b - a < \min\{r/M, 1/L\}$.

Suppose that F satisfies (K), and let $\mathcal{M}_F$, $\mathcal{M}_{\text{ext}\,F}$, $\mathcal{S}_F$ be defined accordingly. It is routine to see that $\mathcal{M}_F$ is nonempty and compact in $C(I, \mathbf{E})$. Hence $\mathcal{M}_F$, endowed with the metric of uniform convergence, is complete. Moreover for each $f \in \mathcal{S}_F$, P_f is a nonempty compact subset of $\mathcal{M}_F$. As F satisfies (K), there exists a set $C \in \mathscr{C}(\mathbf{E})$, $C \subset X$, such that, for every $x \in \mathcal{M}_F$ and $t \in I$, we have $x(t) \in C$. With this choice of C, let $\mathcal{N}$ be given by (3.2).

Remark 5.2. Lemmas 4.1, 4.2, 4.3, and 4.4 remain valid with assumption (K) in the place of (H).

By virtue of Remark 5.2, using the same argument of Theorem 4.1, one can prove the following Theorem 5.2.

THEOREM 5.2. *Let* F *satisfy* (K). *Let* $f \in \mathscr{S}_F$ *and* $\varepsilon > 0$. *Then* $\mathscr{M}_{\text{ext } F} \cap B_{\mathscr{M}_F}(P_f, \varepsilon) \neq \phi$ *and, in particular, the Cauchy problem* (2.3) *has solutions.*

6. SOME APPLICATIONS

In this Section, as applications of the previous results, we present an existence theorem for a boundary value problem and two characterizations of convex valued multifunctions.

Let F be given by (2.1). For $u \in X_0$ and $t \in I$, we set $\mathscr{A}_F(u; t) = \{x(t) \mid x: I \to \mathbf{E}$ is a solution of (2.2)$\}$ and $\mathscr{A}_{\text{ext } F}(u; t) = \{x(t) \mid x: I \to \mathbf{E}$ is a soluton of (2.3)$\}$.

THEOREM 6.1. *Let* F *satisfy* (H′). *If there exists* a_1, *with* $a < a_1 \leqslant b$, *such that* $\mathscr{A}_F(u; a_1) \subset X_0$ *for every* $u \in X_0$, *then the boundary value problem*

$$\dot{x} \in \text{ext } F(t, x), \qquad x(a) = x(a_1) \tag{6.1}$$

has at least one solution.

Proof. Let $f \in \mathscr{S}'_F$. We show first that the boundary value problem

$$\dot{x} = f(t, x), \qquad x(a) = x(a_1) \tag{6.2}$$

has solutions. Indeed, by Scorza Dragoni's theorem [19], there is a sequence $\{I_n\}$ of nonempty compact sets $I_n \subset I$, $I_n \subset I_{n+1}$, $n \in \mathbf{N}$, with $m(I \setminus I_n) \to 0$ as $n \to +\infty$, such that the restriction of f to $I_n \times X$ is continuous. For each $n \in \mathbf{N}$, let $\varphi_n : I \times X \to \mathbf{E}$ be a locally Lipschitzean function, with values contained in $\overline{\text{co}}\, A$, such that

$$\sup_{(t, x) \in I_n \times X} \|\varphi_n(t, x) - f(t, x)\| < \frac{1}{n}.$$

By adapting an argument from Cellina [7], it can be proved that, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbf{N}$ such that $\mathscr{A}_{\varphi_n}(X_0; a_1) \subset B_E(X_0, \varepsilon)$ for every $n \geqslant n_0$. Using this, one can construct a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that $\mathscr{A}_{\varphi_{n_k}}(X_0; a_1) \subset B_E(X_0, 1/k)$, $k \in \mathbf{N}$. By the theorem of Kakutani [21] and Fan [16] for each $k \in \mathbf{N}$ the multifunction $u \to \tilde{B}_E(\mathscr{A}_{\varphi_{n_k}}(u; a_1), 1/k) \cap X_0$, from X_0 to the nonempty compact convex subsets of X_0, has a fixed point u_k, say. Hence, for each $k \in \mathbf{N}$, there exists a solution x_k of the Cauchy problem $\dot{x} = \varphi_{n_k}(t, x)$, $x_k(a) = u_k$, such that $\|x_k(a) - x_k(a_1)\| \leqslant 1/k$. Since $\{x_k\} \subset \mathscr{M}_F$ is compact, a subsequence, say $\{x_k\}$, converges uniformly to some $x \in \mathcal{M}_F$. Clearly $x(a) = x(a_1) \in X_0$. Moreover, x is a solution of (6.2), because for every $t \in I$ we have

$$\left\| x(t) - x(a) - \int_a^t f(s, x(s))\, ds \right\| \leqslant c_k + \int_I \|f(s, x(s)) - \varphi_{n_k}(s, x_k(s))\|\, ds, \tag{6.3}$$

where $c_k = \sup_{t \in I} \|x(t) - x_k(t)\| + \|x(a) - x_k(a)\|$, and the right-hand side of the inequality (6.3) vanishes as $k \to +\infty$.

By virtue of Lemma 5.4, Lemma 5.1, using the same argument of Theorem 4.1, one can construct a decreasing sequence of nonempty compact subsets $\tilde{B}_{\mathcal{M}_F}(P_{g_n}, \eta_n)$ of $\mathcal{M}_F$, with $g_n \in \mathcal{S}'_F$ and $0 < \eta_n < \theta_n = 1/n$, satisfying

$$\tilde{B}_{\mathcal{M}_F}(P_{g_{n+1}}, \eta_{n+1}) \subset \mathcal{M}_{\theta_n} \cap B_{\mathcal{M}_F}(P_{g_n}, \eta_n), \qquad n \in \mathbf{N}.$$

For each $n \in \mathbf{N}$, let x_n be a solution of the boundary value problem $\dot{x} = g_n(t, x)$, $x(a) = x(a_1)$. Since $\{x_n\}$ is compact, a subsequence converges uniformly to some $x \in \mathcal{M}_F$. Clearly, $x(a) = x(a_1)$. Moreover $x \in \mathcal{M}_{\text{ext}\, F}$, because x lies in each set $\mathcal{M}_{\theta_n}$, $n \in \mathbf{N}$. Hence x is a solution of the boundary value problem (6.1). This completes the proof.

Remark 6.1. The set $\mathcal{A}_{\text{ext}\, F}(u; a_1)$ is, in general, neither closed nor convex. Nevertheless, by Theorem 6.1, the multifunction $u \to \mathcal{A}_{\text{ext}\, F}(u; a_1)$, from X_0 to the nonempty subsets of X_0, has at least one fixed point. For the multifunction $u \to \mathcal{A}_F(u; a_1)$, where F is continuous (or, more generally, upper semicontinuous) with values in $\mathscr{C}(\mathbf{E})$, the existence of a fixed point has been proved by Cellina [7] by an approximation method which is not applicable under the assumptions of Theorem 6.1. It is worth noting that Theorem 6.1 is no longer true if in (H′) the assumption (H'_1) is replaced by "F is upper semicontinuous on $I \times X$."

Given a multifunction $G\colon I \times X \to \mathcal{K}(\mathbf{E})$, where $I = [a, b]$ and $X = \tilde{B}_{\mathbf{E}}(u, r)$, $u \in \mathbf{E}$, $r > 0$, consider the Cauchy problem

$$\dot{x} \in G(t, x), \qquad x(a) = u. \tag{6.4}$$

We set $\mathcal{M}_G = \{x\colon I \to \mathbf{E} \mid x \text{ is a solution of (6.4)}\}$, and $\mathcal{A}_G(u; t) = \{x(t) \mid x\colon I \to \mathbf{E} \text{ is a solution of (6.4)}\}$, $t \in I$. By $\overline{\text{co}}\, G\colon I \times X \to \mathscr{C}(\mathbf{E})$ we denote the multifunction defined by $(\overline{\text{co}}\, G)(t, x) = \overline{\text{co}}\, G(t, x)$, $(t, x) \in I \times X$.

Remark 6.2. If G satisfies (H′), then $\mathcal{M}_{\text{ext}\overline{\text{co}}\, G} \subset \mathcal{M}_G$. Indeed, since $\overline{\text{co}}\, G$ satisfies (H′), by Theorem 5.1 the set $\mathcal{M}_{\text{ext}\overline{\text{co}}\, G}$ is nonempty. Moreover, by Mil'man's theorem [24], for each $(t, x) \in I \times X$ we have $\text{ext}\overline{\text{co}}\, G(t, x) \subset G(t, x)$, and so $\mathcal{M}_{\text{ext}\overline{\text{co}}\, G} \subset \mathcal{M}_G$.

THEOREM 6.2. *Let* $G: I \times X \to \mathscr{K}(\mathbf{E})$ *satisfy* (H′). *Then the following statements are equivalent*:

(i) $\mathscr{M}_G$ *is closed in* $C(I, \mathbf{E})$,

(ii) *there exists a measurable set* $I_0 \subset I$, *with* $m(I \setminus I_0) = 0$, *such that on the set* $Z = \{(t, x) \mid t \in I_0,\ x \in \mathscr{A}_G(u; t)\}$ *the multifunction* G *is convex valued.*

Proof. It suffices to show that (i) implies (ii), the reverse implication being known. So let $G: I \times X \to \mathscr{K}(\mathbf{E})$ satisfy (H′), and suppose that $\mathscr{M}_G$ is closed in $C(I, \mathbf{E})$. By Scorza Dragoni's theorem [19], there exists a sequence $\{I_n\}$ of nonempty pairwise disjoint measurable sets I_n satisfying the properties: (j) $I_n \subset]a, b[$; (jj) $m(I \setminus \bigcup_n I_n) = 0$; (jjj) each point of I_n is a density point; (jv) the multifunction G restricted to $I_n \times X$ is continuous. Set $I_0 = \bigcup_n I_n$.

We claim that, with such I_0 in (ii), the multifunction G is convex valued on Z. Indeed, suppose the contrary. Then there exist $\nu \in \mathbf{N}$, $\tau \in I_\nu$ and $\xi \in \mathscr{A}_G(u; \tau)$ such that $G(\tau, \xi)$ is not convex. Let $v \in (\overline{\text{co}}\, G(\tau, \xi)) \setminus G(\tau, \xi)$. Let G_ν denote the restriction of G to $I_\nu \times X$. By Michael's theorem [23], there exists a continuous selection $g_\nu: I_\nu \times X \to \mathbf{E}$ of $\overline{\text{co}}\, G_\nu$ such that $g_\nu(\tau, \xi) = v$. Since $d(g_\nu(\tau, \xi), G_\nu(\tau, \xi)) > 0$, and g_ν, G_ν are continuous at (τ, ξ), there exists a $\tau' \in I_\nu$, $\tau' > \tau$, such that

$$d(g_\nu(t, z), G_\nu(t, z)) > 0, \text{ for each } t \in \tilde{J} \text{ and } z \in \tilde{B}(\xi, (\tau' - \tau) M), \tag{6.5}$$

where $\tilde{J} = I_\nu \cap J$ and $J = [\tau, \tau']$. Denote by G_J the restriction of G to $J \times X$, and let $g_J: J \times X \to \mathbf{E}$ be a Carathéodory selection of $\overline{\text{co}}\, G_J$ such that $g_J(t, z) = g_\nu(t, z)$ for every $(t, z) \in \tilde{J} \times X$. Furthermore, denote by $\mathscr{M}_{G_J}$ (resp. P_{g_J}) the set of all solutions $z: J \to \mathbf{E}$ of the Cauchy problem $\dot{z} \in G_J(t, z)$, $z(\tau) = \xi$ (resp. $\dot{z} = g_J(t, z)$, $z(\tau) = \xi$). Clearly, $\mathscr{M}_{G_J}$ and P_{g_J} are nonempty. Let $z \in P_{g_J}$ be any. Since for each $t \in \tilde{J}$ we have $g(t, z(t)) = g_\nu(t, z(t))$, $G_J(t, z(t)) = G_\nu(t, z(t))$, and $\|z(t) - \xi\| \leqslant (\tau' - \tau) M$, from (6.5) we have

$$\int_J d(\dot{z}(t), G_J(t, z(t)))\, dt \geqslant \int_{\tilde{J}} d(g_J(t, z(t)), G_J(t, z(t)))\, dt$$

$$= \int_{\tilde{J}} d(g_\nu(t, z(t)), G_\nu(t, z(t)))\, dt > 0,$$

for $\tilde{J}$ has measure $m(\tilde{J}) > 0$. As $z \in P_{g_J}$ is arbitrary, it follows that

$$P_{g_J} \cap \mathscr{M}_{G_J} = \phi. \tag{6.6}$$

On the other hand, by virtue of Theorem 5.1 and Remark 6.2, one can construct a sequence $\{z_n\} \subset \mathcal{M}_{G_J}$ such that

$$z_n \in \mathcal{M}_{G_J} \cap B_{\mathcal{M}_{\overline{co}\, G_J}}(P_{g_J}, 1/n), \qquad n \in \mathbf{N}.$$

As $\{z_n\}$ is compact, there exists a subsequence, say $\{z_n\}$, which converges uniformly to some $z \in P_{g_J}$. From (6.6), one has $z \notin \mathcal{M}_{G_J}$. Now $\xi \in \mathcal{A}_G(u; \tau)$, thus there exists a $y \in \mathcal{M}_G$ such that $y(\tau) = \xi$. For every $n \in \mathbf{N}$, define $x_n : I \to \mathbf{E}$ by

$$x_n(t) = \begin{cases} y(t), & t \in [a, \tau] \\ z_n(t), & t \in [\tau, \tau'] \\ w_n(t), & t \in [\tau', b], \end{cases}$$

where w_n: $[\tau', b] \to \mathbf{E}$ is any solution of the Cauchy problem $w' \in G(t, w)$, $w(\tau') = z_n(\tau')$. Clearly $\{x_n\} \subset \mathcal{M}_G$. As $\{x_n\}$ is compact, there is a subsequence, say $\{x_n\}$, which converges uniformly to a function $x \in \mathcal{M}_{\overline{co}\, G}$. Since $x(t) = z(t)$ for every $t \in J$, it follows that $x \notin \mathcal{M}_G$. Hence $\mathcal{M}_G$ is not closed in $C(I, \mathbf{E})$, a contradiction. This completes the proof.

Let $I = [a, b[$, $a < b$, and let $X = \tilde{B}_{\mathbf{E}}(D, r)$, where D is a nonempty subset of $\mathbf{E}$, and $r > 0$. Let G: $I \times X \to \mathcal{K}(\mathbf{E})$ be given. For $\tau \in I$ and $\xi \in D$, consider the Cauchy problem

$$\dot{x} \in G(t, x), \qquad x(\tau) = \xi. \tag{6.7}$$

Remark 6.3. Let G satisfy (H'). Then it is easy to see that for each $\tau \in I$ there exists a $\delta_\tau > 0$ such that for every $\xi \in D$ the Cauchy problem (6.7) has solutions x: $I_\tau \to \mathbf{E}$ defined on $I_\tau = [\tau, \tau + \delta_\tau]$.

For $(\tau, \xi) \in I \times D$, set $\mathcal{M}_G^{\tau;\xi} = \{x: I_\tau \to \mathbf{E} \mid x \text{ is a solution of } (6.7)\}$ and observe that, if G satisfies (H'), then $\mathcal{M}_G^{\tau;\xi}$ is a nonempty subset of $C(I_\tau, \mathbf{E})$.

The following theorem, of the type obtained by Tolstonogov [32] and Cellina and Ornelas [9], can be proved as Theorem 6.2.

THEOREM 6.3. *Let G satisfy* (H'), *with I and X as above. Then the following statements are equivalent*:

(i) *there exists a measurable set $J_0 \subset I$, with $m(I \setminus J_0) = 0$, such that for every $(\tau, \xi) \in J_0 \times D$ the set $\mathcal{M}_G^{\tau;\xi}$ is closed in $C(I_\tau, \mathbf{E})$*;

(ii) *there exists a measurable set $I_0 \subset I$, with $m(I \setminus I_0) = 0$, such that on the set $I_0 \times D$ the multifunction G is convex valued.*

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