Relatively Regular Operators and a Spectral Mapping Theorem

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Let T be a bounded linear operator on a complex Banach space X. The following essential spectrum of T is introduced:

$$\sigma_{rr}(T)=\left\{\lambda\in\mathbb{C} : T-\lambda I \text{ is not relatively regular or } N(T-\lambda I)\not\subseteq\bigcap_{n\geq 1}(T-\lambda I)^n(X)\right\}.$$

In this note, for a function f admissible in the analytic calculus, we show that $\sigma_{rr}(f(T))=f(\sigma_{rr}(T))$.

1. TERMINOLOGY AND INTRODUCTION

Let X be a complex Banach space and $L(X)$ the Banach algebra of all bounded linear operators on X. We denote by $N(T)$ the kernel and by $T(X)$ the range of $T\in L(X)$. The spectrum of T is denoted by $\sigma(T)$. The resolvent set $\rho(T)$ of T is the complement of $\sigma(T)$ in the complex plane $\mathbb{C}$.

In [4, Theorem 3] T. Kato showed that for an operator $T\in L(X)$ the set

$$\rho_K(T)=\left\{\lambda\in\mathbb{C} : (T-\lambda I)(X) \text{ is closed and } N(T-\lambda I)\subseteq\bigcap_{n\geq 1}(T-\lambda I)^n(X)\right\}$$

is an open subset of $\mathbb{C}$. Since $\rho(T)\subseteq\rho_K(T)$, it follows that the complement $\sigma_K(T)=\mathbb{C}\setminus\rho_K(T)$ is a compact subset of $\sigma(T)$. We showed in [7, Satz 2] that $\partial\sigma(T)\subseteq\sigma_K(T)$, hence $\sigma_K(T)\neq\emptyset$.

The set of all complex valued functions which are analytic in some neighbourhood of $\sigma(T)$ is denoted by $\mathscr{H}(T)$. For $f\in\mathscr{H}(T)$, the operator $f(T)$ is defined by the well known analytic calculus.

In [7, Satz 6] we proved the following spectral mapping theorem for $\sigma_K(T)$.

THEOREM 1. *If $T\in L(X)$ and $f\in\mathscr{H}(T)$ then*

$$\sigma_K(f(T))=f(\sigma_K(T)).$$

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An operator $T \in L(X)$ is called relatively regular, if the equation $TST = T$ is satisfied for some operator $S \in L(X)$. It is easy to see that if $TST = T$, then the operator $S_0 = STS$ satisfies the equations

$$TS_0T = T \qquad \text{and} \qquad S_0TS_0 = S_0.$$

It is well known that T is relatively regular if and only if $N(T)$ and $T(X)$ are closed, complemented subspaces of X [3, Satz 74.2].

DEFINITION. For $T \in L(X)$ we denote by $\rho_{rr}(T)$ the set

$$\rho_{rr}(T) = \Big\{ \lambda \in \mathbb{C} : T - \lambda I \text{ is relatively regular and } N(T - \lambda I) \subseteq \bigcap_{n \geq 1} (T - \lambda I)^n (X) \Big\}.$$

The complement of $\rho_{rr}(T)$ in $\mathbb{C}$ is denoted by $\sigma_{rr}(T)$.

The next theorem shows that the points in $\rho_{rr}(T)$ are in a certain sense "good" points of T (for a proof see [5, Théorème 2.6] or [8, Theorem 1.4]).

THEOREM 2. *Let $T \in L(X)$. Then $\lambda_0 \in \rho_{rr}(T)$ if and only if there is a neighbourhood U of λ_0 and a holomorphic function $F: U \to L(X)$ such that*

$$(T - \lambda I)\, F(\lambda)(T - \lambda I) = T - \lambda I \qquad \text{for all} \quad \lambda \in U.$$

The aim of this paper is to show that $f(\sigma_{rr}(T)) = \sigma_{rr}(f(T))$ for all $f \in \mathscr{H}(T)$. This is done in Section 3.

2. Preliminary Results

In this section we collect some properties of the sets $\rho_K(T)$ and $\rho_{rr}(T)$.

Notation. The conjugate space of the Banach space X is denoted by X^* and the adjoint of a linear operator T in $L(X)$ by T^*.

PROPOSITION 1. *Let $T \in L(X)$.*

(a) *The functions*

$$\lambda \mapsto \overline{\bigcup_{n=1}^{\infty} N((T - \lambda I)^n)} \qquad \text{and} \qquad \lambda \mapsto \bigcap_{n=1}^{\infty} (T - \lambda I)^n (X)$$

are constant on connected components of $\rho_K(T)$;

(b) $\rho_K(T) = \rho_K(T^*)$;

(c) $\rho_{rr}(T)$ *is open*;

(d) $\sigma_K(T) \subseteq \sigma_{rr}(T) \subseteq \sigma(T)$ *and* $\sigma_{rr}(T) \neq \emptyset$;

(e) $\sigma_{rr}(T^*) \subseteq \sigma_{rr}(T)$.

Proof. (a) [2, Theorem 3]. (b) It suffices to show that $0 \in \rho_K(T)$ if and only if $0 \in \rho_K(T^*)$. Let $0 \in \rho_K(T)$ and $n \in \mathbb{N}$. By Theorem 1, $0 \in \rho_K(T^n)$, hence $T^n(X)$ is closed. Therefore $(T^*)^n(X^*)$ is closed [3, Satz 55.7]. Since $N(T) \subseteq T^n(X)$, we derive

$$N((T^*)^n) = T^n(X)^\perp \subseteq N(T)^\perp = T^*(X^*).$$

Since $n \in \mathbb{N}$ was arbitrary, we conclude that

$$\bigcup_{n=1}^{\infty} N((T^*)^n) \subseteq T^*(X^*).$$

Lemma 511 in [4] asserts now that

$$N(T^*) \subseteq \bigcap_{n=1}^{\infty} (T^*)^n (X^*).$$

This proves $0 \in \rho_K(T^*)$. Similar arguments show that $0 \in \rho_K(T^*)$ implies $0 \in \rho_K(T)$. (c) follows from Theorem 2. (d) is clear. (e) Let $\lambda_0 \in \rho_{rr}(T)$. Then $\lambda_0 \in \rho_K(T)$ and $T - \lambda_0 I$ is relatively regular. (b) shows that $\lambda_0 \in \rho_K(T^*)$. The relative regularity of $T^* - \lambda_0 I^* = (T - \lambda_0 I)^*$ is obvious. ∎

The following example shows that in general $\sigma_{rr}(T^*) \neq \sigma_{rr}(T)$ and $\sigma_K(T) \neq \sigma_{rr}(T)$.

EXAMPLE. Let l^∞ denote the Banach space of all complex bounded sequences (x_n) with norm $\|(x_n)\| = \sup_{n=1}^{\infty} |x_n|$. The closed subspace of l^∞ consisting of all sequences (x_n) with $\lim_{n\to\infty} x_n = 0$ is denoted by c_0. Put $X = c_0 \times l^\infty$ and consider the linear operator $T: X \to X$ given by

$$T((x_n), (y_n)) = ((0, 0, 0, ...), (x_1, y_1, x_2, y_2, ...)).$$

T has the following properties (for proofs see [1, p. 15]): $N(T) = \{0\}$, $T(X)$ is not complemented, and T^* is relatively regular. This gives

$$0 \in \sigma_{rr}(T), \qquad 0 \notin \sigma_K(T), \quad \text{and} \quad 0 \notin \sigma_{rr}(T^*).$$

PROPOSITION 2. *Let* $0 \in \rho_{rr}(T)$ *and* $TST = T$ *for some* $S \in L(X)$. *Then* $0 \in \rho_{rr}(T^n)$ *and* $T^n S^n T^n = T^n$ *for all* $n \in \mathbb{N}$.

Proof. It is clear that $0 \in \rho_K(T)$, thus $0 \in \rho_K(T^n)$ for all $n \in \mathbb{N}$ (Theorem 1). It remains to show that $T^nS^nT^n = T^n$ for all $n \in \mathbb{N}$. This is clear for $n = 1$. Now suppose that $T^nS^nT^n = T^n$ for some $n \in \mathbb{N}$. Put $P = T^nS^n$ and $Q = I - ST$; then $P^2 = P$, $Q^2 = Q$, $P(X) = T^n(X)$, and $Q(X) = N(T)$. Since $0 \in \rho_K(T)$, we have $N(T) \subseteq T^n(X)$, thus $Q(X) \subseteq P(X)$. This gives

$$I - ST = Q = PQ = T^nS^n - T^nS^{n+1}T,$$

hence $T^nS^{n+1}T = T^nS^n - I + ST$. We conclude that

$$\begin{aligned} T^{n+1}S^{n+1}T^{n+1} &= T(T^nS^n - I + ST)T^n \\ &= T(T^nS^nT^n) - T^{n+1} + TSTT^n = T^{n+1}, \end{aligned}$$

and the result follows. ∎

We close this section with two remarks.

1. If X is a Hilbert space, then $T \in L(X)$ is relatively regular if and only if $T(X)$ is closed [1, p. 12]. Thus $\sigma_K(T) = \sigma_{rr}(T)$.

2. Let X be a reflexive Banach space. In this case Proposition 1(e) shows that $\sigma_{rr}(T) = \sigma_{rr}(T^*)$.

3. The Spectral Mapping Theorem for $\sigma_{rr}(T)$

It reads as follows:

THEOREM 3. *If $T \in L(X)$ and $f \in \mathscr{H}(T)$ then*

$$f(\sigma_{rr}(T)) = \sigma_{rr}(f(T)).$$

Proof. 1. We first show that $\sigma_{rr}(f(T)) \subseteq f(\sigma_{rr}(T))$. Let $\lambda_0 \in \sigma_{rr}(f(T))$ and assume that $\lambda_0 \notin f(\sigma_{rr}(T))$. Then $\lambda_0 \notin f(\sigma_K(T)) = \sigma_K(f(T))$ (Theorem 1). Thus we have

$$\lambda_0 \in \rho_K(f(T)). \tag{1}$$

Define the function g by $g(\lambda) = f(\lambda) - \lambda_0$. It follows that

$$g(\lambda) \neq 0 \quad \text{for all} \quad \lambda \in \sigma_{rr}(T), \tag{2}$$

hence

$$g(\lambda) \neq 0 \quad \text{for all} \quad \lambda \in \sigma_K(T). \tag{3}$$

Case 1. g has no zeros in $\sigma(T)$. Then $g(T)=f(T)-\lambda_0 I$ is invertible in $L(X)$, consequently $\lambda_0 \in \rho(f(T)) \subseteq \rho_{rr}(f(T))$, which is impossible, since $\lambda_0 \in \sigma_{rr}(f(T))$.

Case 2. g has zeros in $\sigma(T)$. Since (3) holds, [7, Satz 3] asserts now that g has only a finite number of zeros in $\sigma(T)$. Let $\mu_1, ..., \mu_k$ be these zeros ($\mu_i \neq \mu_j$ for $i \neq j$) and $n_1, ..., n_k$ their respective orders. By (2), $\mu_j \in \rho_{rr}(T)$ for $j=1, ..., k$. Now Proposition 2 can be applied. It shows that $(T-\mu_j I)^{n_j}$ is relatively regular ($j=1, ..., k$). Using the main result in [6], one obtains the relative regularity of $g(T)=f(T)-\lambda_0 I$. From (1) it follows now that $\lambda_0 \in \rho_{rr}(f(T))$ in contradiction to the hypothesis $\lambda_0 \in \sigma_{rr}(f(T))$.

Therefore we have $\sigma_{rr}(f(T)) \subseteq f(\sigma_{rr}(T))$.

2. We prove the containment $f(\sigma_{rr}(T)) \subseteq \sigma_{rr}(f(T))$. Let $\lambda_0 \in f(\sigma_{rr}(T))$, i.e., $\lambda_0 = f(\mu_0)$ for some $\mu_0 \in \sigma_{rr}(T)$. We define the function g by $g(\lambda)=f(\lambda)-\lambda_0$. Since $g(\mu_0)=0$, there exists $h \in \mathscr{H}(T)$ such that $g(\lambda)=(\lambda-\mu_0)\,h(\lambda)$. Consequently $g(T)=(T-\mu_0 I)\,h(T)$. Let us assume that $\lambda_0 \in \rho_{rr}(f(T))$. Then

$$0 \in \rho_{rr}(g(T)) = \rho_{rr}((T-\mu_0 I)\,h(T)) \subseteq \rho_K((T-\mu_0 I)\,h(T)).$$

Use [7, Satz 5] to derive $0 \in \rho_K(T-\mu_0 I)$. Therefore

$$\mu_0 \in \rho_K(T). \tag{4}$$

Since $g(T)$ is relatively regular, there is an operator $R \in L(X)$ with $g(T)\,Rg(T)=g(T)$ and $Rg(T)R=R$. Now we choose a complex number η such that

$$0<|\eta|<(\|T-\mu_0 I\|\ \|R\|)^{-1} \tag{5}$$

and

$$h(\mu_0) \neq \eta. \tag{6}$$

Let $U \in L(X)$ be defined by $U=\eta(T-\mu_0 I)$. Then we have, by (5), $\|U\|<\|R\|^{-1}$. Since $Ug(T)=g(T)U$, it follows that $\bigcap_{n \geqslant 1} g(T)^n(X)$ is an invariant subspace of U. Theorem 9 in [1, Sect. 5.2] shows now that $g(T)-U$ is relatively regular. Obviously $g(T)-U=(T-\mu_0 I)(h(T)-\eta I)=\varphi(T)$ with $\varphi(\lambda)=(\lambda-\mu_0)(h(\lambda)-\eta)$. It follows from (6) that μ_0 is an isolated, simple zero of φ. Since $\varphi(T)$ is relatively regular, the main result in [6] implies that $T-\mu_0 I$ is relatively regular. With the aid of (4) we derive $\mu_0 \in \rho_{rr}(T)$. This contradiction shows that $\lambda_0 \in \sigma_{rr}(f(T))$. Thus we have indeed $f(\sigma_{rr}(T)) \subseteq \sigma_{rr}(f(T))$.

The proof is now complete. ∎

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