Krull Dimension in Serial Rings

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The main result of this paper is that if a serial ring \( R \) has right Krull dimension \( \alpha \), then it also has left Krull dimension \( \alpha \).

In general a ring \( R \) need not have Krull dimension (as a right module over itself) and even if \( R_R \) has Krull dimension, \( R_R \) need not have the same Krull dimension; indeed, it might not have Krull dimension at all. A few rings in which the existence of Krull dimension \( R_R \) guarantees the same Krull dimension for \( R_R \) are known (cf. [4]). One purpose of this paper is to add serial rings to the list: if a serial ring has Krull dimension as a right module over itself, then it has the same Krull dimension considered as a left module over itself. In serial rings, this is a consequence of the fact that if a ring \( R \) has Krull dimension \( \alpha \), then the ideal \( J(\alpha) \) (defined below) is nilpotent [5]. The second purpose of this paper is to define and develop, for each ordinal \( \alpha \), the elementary properties of the "\( \alpha \)-cliques" of local projective modules. These are expected to play a significant role in understanding the structure of serial rings with Krull dimension (cf. [6, 10]).

A right module \( M \) over a ring \( R \) is uniserial if its submodules are linearly ordered under inclusion. A ring \( R \) is right serial if \( R \) is a direct sum of uniserial right ideals. A local module is one with a unique maximal proper submodule; a local element is any element which generates a local module. Familiarity with the definition and elementary properties of Krull dimension is assumed. See [3] for details. The Krull dimension of a module \( M \) will be denoted by \( \text{dim} M \). Throughout, \( R \) is understood to be a serial ring on both sides, with right Krull dimension. Almost without exception, we proceed by transfinite induction. Given ordinals \( \beta < \alpha \), \( (\beta, \alpha) \) denotes the set of all ordinals \( \gamma \) such that \( \beta < \gamma < \alpha \).

To begin, define inductively

\[
J(0) = J(R), \quad \text{the Jacobson radical of } R
\]

\( J(R) \) is the Jacobson radical of \( R \).

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and for an ordinal \( \alpha \),

\[
J(\alpha) = \begin{cases} 
\bigcap_{n=1}^{\infty} J(\alpha - 1)^n & \text{if } \alpha \text{ is a nonlimit ordinal} \\
\bigcap_{\beta < \alpha} J(\beta) & \text{if } \alpha \text{ is a limit ordinal.}
\end{cases}
\]

For convenience, given ideal \( I \subseteq R \), \( I^0 \) is taken to be \( R \). Choose a set of representatives of the isomorphism classes of local direct summands of \( R \): \( Q = \{ Q_1, \ldots, Q_N \} \). There exists a set of local orthogonal idempotents \( \{ e_1, \ldots, e_N \} \) such that for each \( i = 1, \ldots, N \), \( Q_i \cong e_i R \). Define inductively for each ordinal \( \alpha \) an \( \alpha \)-clique, an \( \alpha \)-successor, and an \( \alpha \)-predecessor as follows:

If \( \alpha = 0 \), the 0-cliques are the singleton sets \( \{ Q_i \} \) for \( i = 1, \ldots, N \). \( \{ Q_i \} \) is a 0-successor of \( \{ Q_j \} \) (equivalently, \( \{ Q_j \} \) is a 0-predecessor of \( \{ Q_i \} \)) if \( Q_i J(0) \) is nonzero and \( Q_j \) is a projective cover of \( Q_i J(0) \). Clearly this happens iff the simple module \( Q_j/Q_i J(0) \) is a successor of \( Q_i/Q_i J(0) \) in the sense of Warfield [8]. It is possible that a given 0-clique may fail to have a 0-successor or a 0-predecessor or both. However, when they exist, 0-successors (resp. 0-predecessors) are unique [8].

Given an ordinal \( \alpha > 0 \), consider the equivalence relation \( T(\alpha) \) generated by the following relation, \( R(\alpha) \), on \( Q \): \( (Q_i, Q_j) \in R(\alpha) \) iff for some \( \beta < \alpha \), the \( \beta \)-clique containing \( Q_i \) is a \( \beta \)-successor of the \( \beta \)-clique containing \( Q_j \). Define the \( \alpha \)-cliques to be the equivalence classes under the equivalence relation \( T(\alpha) \). The unique \( \alpha \)-clique to which a given \( Q_i \) belongs will be denoted by \( C(\alpha, Q_i) \). The \( \alpha \)-clique \( C \) is called an \( \alpha \)-successor of \( \alpha \)-clique \( C' \) (equivalently, \( C' \) is an \( \alpha \)-predecessor of \( C \)) if for some \( Q \in C \); for some \( Q' \in C' \) there exists an element \( x' \in Q' J(\alpha) \setminus Q' J(\alpha)^2 \) such that \( Q \) is a projective cover of \( x'R \). If \( Q' J(\alpha) = Q' J(\alpha)^2 \) for every \( Q' \in C' \), then \( C' \) has no successor. Note that if \( \alpha \) is a limit ordinal, \( C(\alpha, Q_i) = \bigcup_{\beta < \alpha} C(\beta, Q_i) \), \( i = 1, \ldots, N \).

We shall need the notion of an "\( \alpha \)-series" for a uniserial module \( M \). The idea is roughly analogous to that of socle series [3] or basic series [4]. Given a uniserial module \( M \), define

\[
B(\alpha, 0, M) = \left\{ \bigcup \{ L \subseteq M \mid L \text{ is } \alpha\text{-critical} \} \right. 
\quad \text{if } M \text{ has } \alpha\text{-critical submodules} \\
+ \bigcup \{ L \subseteq M \mid \text{K} \dim L < \alpha \} \quad \text{otherwise};
\]

and for \( i \geq 0 \), define

\[
B(\alpha, i + 1, M)/B(\alpha, i, M) = B(\alpha, 0, M/B(\alpha, i, M)).
\]

We call the sequence \( \{ B(\alpha, i, M) \}_{i \in \mathbb{N}} \) the \( \alpha \)-series of \( M \).
LEMMA 1. Suppose $R$ is a serial ring and $M$ is a uniserial right $R$-module such that $0 \subseteq B(\alpha, 0, M) \subseteq B(\alpha, 1, M)$ for some ordinal $\alpha$. Then either $\text{K. dim } B(\alpha, 0, M) = \alpha$ or $\text{K. dim } B(\alpha, 1, M)/B(\alpha, 0, M) = \alpha$ (or both).

Proof. Suppose $\text{K. dim } B(\alpha, 0, M) < \alpha$. Given $x \in B(\alpha, 1, M) \setminus B(\alpha, 0, M)$ either $xR/B(\alpha, 0, M)$ is $\alpha$-critical or $\text{K. dim } xR/B(\alpha, 0, M) < \alpha$. In the latter case, $\text{K. dim } xR = \max\{\text{K. dim } xR/B(\alpha, 0, M), \text{K. dim } B(\alpha, 0, M)\} < \alpha$, in which case $x \in B(\alpha, 0, M)$, a contradiction.

LEMMA 2. Suppose $R$ is a serial ring. For any ordinal $\alpha \geq 1$ and for any local element $x$,

(i) $xJ(\alpha) = \left\{ \begin{array}{ll} \bigcap_{n=1}^{\infty} xJ(\alpha-1)^n & \text{if } \alpha \text{ is a nonlimit ordinal} \\ \bigcap_{\beta < \alpha} xJ(\beta) & \text{if } \alpha \text{ is a limit ordinal} \end{array} \right.$

(ii) if $y \in xJ(\alpha)^n \setminus xJ(\alpha)^{n+1}$ for some $n \geq 0$, then $yJ(\alpha) = xJ(\alpha)^{n+1}$;

(iii) if for some ordinal $\delta \ (< \alpha) xJ(\alpha) \subseteq xJ(\delta)$, then $xJ(\delta)J(\alpha) = xJ(\alpha)$.

Proof. When $\alpha = 1$, statement (i) is Lemma 1.1 of [7], (ii) is Proposition 1.6 of [7], and (iii) follows immediately from (ii).

For any arbitrary ordinal $\alpha$ consider a projective cover $eR \rightarrow xR$ (where $e$ is a local idempotent of $R$). Clearly,

$$xJ(\alpha) \subseteq \left\{ \begin{array}{ll} \bigcap_{n=1}^{\infty} xJ(\alpha-1)^n & \text{if } \alpha \text{ is a nonlimit ordinal} \\ \bigcap_{\beta < \alpha} xJ(\beta) & \text{if } \alpha \text{ is a limit ordinal}. \end{array} \right.$$

Proper inclusion would imply

$$eJ(\alpha) \subseteq \left\{ \begin{array}{ll} \bigcap_{n=1}^{\infty} eJ(\alpha-1)^n & \text{if } \alpha \text{ is a nonlimit ordinal} \\ \bigcap_{\beta < \alpha} eJ(\beta) & \text{if } \alpha \text{ is a limit ordinal}. \end{array} \right.$$

Hence (i) holds for all $\alpha$.

We complete the proof by induction. Assuming that (ii) and (iii) hold for all $\beta < \alpha$ consider an element $y \in xJ(\alpha)^n \setminus xJ(\alpha)^{n+1}$, $n \geq 0$; we claim that for
any \( y' \in xJ(\alpha)^n \setminus yR \), \( y'J(\alpha) = yJ(\alpha) \). Otherwise, if \( \alpha \) is a nonlimit ordinal there exists \( m > 0 \) such that for all \( k \geq 0 \) \( yJ(\alpha - 1)^m \subset y'J(\alpha) \subset yJ(\alpha - 1)^k \). By the induction hypothesis and part (i), \( y \in y'J(\alpha) \subset xJ(\alpha)^{n+1} \), a contradiction. Similarly if \( \alpha \) is a limit ordinal and if \( yJ(\alpha) \subset y'J(\alpha) \), there exists \( \beta < \alpha \) such that for \( \gamma \in (\beta, \alpha) \), \( yJ(\alpha) \subset y'J(\alpha) \subset yJ(\gamma) \). By induction hypothesis, \( y \in \bigcap \gamma < \alpha y'J(\gamma) \subset y'J(\alpha) \subset xJ(\alpha)^{n+1} \), a contradiction. It follows that

\[
xJ(\alpha)^n J(\alpha) = \sum \{ y'J(\alpha) \mid y' \in xJ(\alpha)^n \}
= yJ(\alpha).
\]

Hence (ii) holds.

Given an ordinal \( \delta < \alpha \), suppose \( xJ(\alpha) \subset xJ(\delta) \). Then \( xJ(\delta) J(\alpha) = \sum \{ yJ(\alpha) \mid y \in xJ(\delta) \} = xJ(\alpha) \) by part (ii).

**PROPOSITION 3.** Over a serial ring \( R \), given any ordinal \( \alpha \), the following hold:

(i) For all \( \beta \geq \alpha \), for all local elements \( x \) such that \( xJ(\beta) J(\alpha) \subset xJ(\beta) \), the set of projective covers of f.g. submodules of \( xJ(\beta)/xJ(\beta) J(\alpha) \) is contained in some \( \alpha \)-clique.

(ii) For all \( \beta \geq \alpha \), for all local elements \( x \) such that \( 0 \subset xJ(\beta) J(\alpha) \subset xJ(\beta) \), for any local projective cover \( Q \) of a submodule \( yR/xJ(\beta) J(\alpha) \subset xJ(\beta)/xJ(\beta) J(\alpha) \), for any \( \gamma < \alpha \) such that \( QJ(\gamma)^2 \subset QJ(\gamma) \), any local projective cover of a f.g. submodule of \( QJ(\gamma)/QJ(\gamma)^2 \) is a projective cover of a submodule of \( yJ(\gamma)/yJ(\gamma)^2 \).

(iii) For all \( \beta \geq \alpha \), for all local elements \( x \) such that \( xJ(\beta) J(\alpha) \subset xJ(\beta) \), for any local projective cover \( Q \) of a submodule \( yR/xJ(\beta) J(\alpha) \subset xJ(\beta)/xJ(\beta) J(\alpha) \), if there exists \( \gamma < \alpha \) and a local projective \( Q' \) with \( Q'J(\gamma)^2 \subset Q'J(\gamma) \) such that \( Q \) is a projective cover of a submodule of \( Q'J(\gamma)/Q'J(\gamma)^2 \), then \( Q' \) is the projective cover of a submodule \( y'R \subset xJ(\beta) \) such that \( y'R^2 \subset yR \subset yJ(\gamma) \).

(iv) If \( Q \) and \( Q' \) belong to the same \( \alpha \)-clique and both \( QJ(\alpha) \) and \( Q'J(\alpha) \) are nonzero, then \( Q \) is a projective cover of some submodule of \( Q'/Q'J(\alpha) \) or vice versa.

(v) When it exists, the \( \alpha \)-successor of a given \( \alpha \)-clique is unique.

(vi) If \( C = \{ e_1 R, ..., e_m R \} \) is an \( \alpha \)-clique in \( \text{Mod}-R \) (where each \( e_i \) is a local idempotent), then \( C^* \), defined as \( \{ Re_1, ..., Re_m \} \) is an \( \alpha \)-clique in \( \text{R-Mod} \). An \( \alpha \)-clique \( C' \) in \( \text{Mod}-R \) is an \( \alpha \)-predecessor (resp. \( \alpha \)-successor)
of $C$ iff $C^{**}$ is an $x$-successor (resp. $x$-predecessor) of $C^*$ in $R$-Mod. Consequently, $x$-predecessors, when they exist, are unique.

(vii) For all $\beta > x$, for all local elements $x$ such that $xJ(\beta)J(x) \subseteq xJ(\beta)$, the $x$-clique containing all projective covers of f.g. submodules of $xJ(\beta)/xJ(\beta)J(x)$ has no $x$-predecessor.

Remarks. Conditions (i), (ii), and (iii) imply that if $\beta \geq x$, if $x$ is a local element such that $0 \subseteq xJ(\beta)J(x) \subseteq xJ(\beta)$, then the set of projective covers of f.g. submodules of $xJ(\beta)/xJ(\beta)J(x)$ is an $x$-clique. Statement (vii) assumes that (i)-(iii) have already been established.

Notation. In what follows, provided no confusion can arise, dots (.) in diagrams will denote modules whose names are irrelevant.

Proof. With the exception of parts (i) and (ii), the proof is by transfinite induction on the ordinal $x$. When $x = 0$, (i)-(iv) are trivial. Parts (v) and (vi) are due to Warfield [8]. For (vii), if $\beta > 0$ and $x$ is a local element such that $0 \subseteq xJ(\beta)J(x) \subseteq xJ(\beta)$, w.l.o.g. we may assume $x$ is a local idempotent and $xJ(\beta) = yR$ where $y = xye$ and $e$ is a local idempotent such that $eR$ is the projective cover of $yR$. Then $J(\beta)Je \subseteq J(\beta)e$. Hence $0 \subseteq J(\beta)e = Je$ and $J^2e = Je$. Hence $\{Re\}$ has no 0-successor and $\{eR\}$ no 0-predecessor.

We now prove that (i) holds for all ordinals $x \geq 1$. Given a local element $x$ and ordinals $\beta \geq x \geq 1$, suppose $Q$ and $Q'$ are projective covers of $yR$ and $y'R$, respectively, where $y, y' \in xJ(\beta) \setminus xJ(\beta)J(x)$. W.l.o.g. we may assume $y'R \subseteq yR$. It now follows from Lemma 2 that $Q$ and $Q'$ belong to the same $x$-clique. Hence (i) holds for all $x$.

(ii) Given $\beta \geq x$ and a local element $x$ such that $0 \subseteq xJ(\beta)J(x) \subseteq xJ(\beta)$, if $Q'$ is a projective cover of some submodule of $QJ(\gamma)/QJ(\gamma)^2$ ($y < x$) and $Q$ is the projective cover of $yR$ where $y \in xJ(\beta) \setminus xJ(\beta)J(x)$, we have

$$Q' \quad \downarrow$$
$$QJ(x) \subseteq QJ(\gamma)^2 \subseteq \cdots \subseteq QJ(\gamma) \subseteq Q \quad \downarrow$$
$$\downarrow$$
$$\downarrow$$
$$\downarrow$$
$$\downarrow$$
$$\downarrow$$

for some $y' \in yJ(\gamma) \setminus yJ(\gamma)^2$. The inclusion $yJ(\gamma)^2 \subseteq y'R$ is strict since $\ker \varphi \subseteq QJ(x)$. Hence (ii) holds for all $x$. 


Assume that the statements of the proposition hold true for all ordinals less than \( \alpha \).

(iii) If \( \beta \geq \alpha > \gamma \), if \( x \) is a local element, suppose \( Q \) and \( Q' \) are local projectives such that the following holds:

\[
Q'J(\gamma)^2 \subseteq \cdots \subseteq Q'J(\gamma) \subseteq Q'
\]

\[
yJ(x) = xJ(\beta)J(x) \subseteq yR \subseteq xJ(\beta).
\]

It may happen that there exists \( b \in xJ(\beta) \) such that \( y \in bJ(\gamma) \). If this is the case, choose a smallest \( \delta \leq \alpha \) such that \( bJ(\delta) \subseteq yR \). Clearly, \( \delta \) is a nonlimit ordinal, \( \delta > \gamma \). Pick a smallest \( n \ (\geq 2) \) such that \( bJ(\delta - 1)^n \subseteq yR \subseteq bJ(\delta - 1)^{n-1} \subseteq bJ(\gamma) \). If \( \delta - 1 = \gamma \), the uniqueness of \( \gamma \)-predecessors for \( \gamma < \alpha \) implies the existence of \( y' \in xJ(\beta) \) such that \( Q' \) is the projective cover of \( y'R \) and \( y \in y'J(\gamma) \setminus y'J(\gamma)^2 \). If \( \delta - 1 > \gamma \), the induction assumption applied to the factor \( bJ(\delta - 1)^{n-1}/bJ(\delta - 1)^{n} \) yields some \( y'R \subseteq bJ(\delta - 1)^{n-1} \) whose projective cover is \( Q' \) and such that \( y'J(\gamma)^2 \subseteq yR \subseteq y'J(\gamma) \).

There remains the possibility that \( y \in xJ(\beta) \setminus xJ(\beta)J(\gamma) \). But in this case, by induction hypothesis, the \( \gamma \)-clique of local projective covers of f.g. submodules of \( xJ(\beta)/xJ(\beta)J(\gamma) \) (of which \( Q \) is a member) has no \( \gamma \)-predecessor. This contradicts our assumption on \( Q \) and \( Q' \).

At this point, if it is known that the entire proposition holds for all ordinals less than \( \alpha \), then it is known that whenever \( \beta \geq \alpha \) and \( x \) is a local element such that \( 0 < xJ(\beta)J(x) \subseteq xJ(\beta) \), the set of projective covers of f.g. submodules of \( xJ(\beta)/xJ(\beta)J(\gamma) \) is an \( \alpha \)-clique.

(iv) Consider local projectives \( Q \) and \( Q' \) belonging to the same \( \alpha \)-clique, \( C \), and such that both \( QJ(\alpha) \) and \( Q'J(\alpha) \) are nonzero. If there exists \( Q'' \in C \) such that \( \forall \gamma < \alpha, \forall n > 0, Q''J(\alpha) \subseteq Q''J(\gamma)^n \), then by uniqueness of \( \gamma \)-successors (for \( \gamma < \alpha \)) and finiteness of \( C \), the desired conclusion holds. If \( \alpha \) is a limit ordinal, then \( Q \) and \( Q' \) belong to the same \( \gamma \)-clique for some \( \gamma < \alpha \); hence the desired conclusion holds by induction assumption. We have eliminated all but the case where \( \alpha \) is a nonlimit ordinal and there exists \( n > 0 \) such that for all \( P \in C \), \( PJ(\alpha - 1)^n = PJ(\alpha) \). By uniqueness of \( (\alpha - 1) \)-successors and predecessors, we may assume \( Q' \) belongs to the \( k \)th \( (\alpha - 1) \)-successor of \( C(\alpha - 1, Q) \) for some \( k \geq 1 \).

Suppose \( k = 1 \). There exist local projectives \( P \) and \( P' \) in \( C(\alpha - 1, Q) \) and \( C(\alpha - 1, Q') \) respectively such that one of the following holds (dashed...
arrows represent maps whose existence follows from that of the solid arrows):

\[
\begin{array}{c}
P' \\
/ \\
/ \\
P J(\alpha - 1)^2 \subseteq \cdots \subseteq PJ(\alpha - 1) \subseteq \cdots \subseteq P \\
/ \\
/ \\
0 \subseteq Q J(\alpha - 1)^2 \subseteq \cdots \subseteq Q J(\alpha - 1) \subseteq Q \\
or
\begin{array}{c}
P' \\
/ \\
/ \\
P J(\alpha - 1)^2 \subseteq \cdots \subseteq PJ(\alpha - 1) \subseteq P \\
/ \\
/ \\
0 \subseteq Q J(\alpha - 1)^2 \subseteq \cdots \subseteq Q J(\alpha - 1) \subseteq Q.
\end{array}
\end{array}
\]

By (i)-(iii), \( Q' \) is the projective cover of some submodule of \( Q J(\alpha - 1)/Q J(\alpha - 1)^2 \). In fact, the same diagrams show that if \( P \) is in an \((\alpha - 1)\)-successor of \( C(\alpha - 1, Q) \), then \( PJ(\alpha) \neq 0 \). It is now a simple matter of induction on \( k \) to complete the proof of (iv).

(v) Different \( \alpha \)-successors for a given \( \alpha \)-clique \( C \) would arise only if there could exist \( Q, Q' \in C \) such that \( Q J(\alpha)^2 \subsetneq Q J(\alpha) \) and \( Q' J(\alpha)^2 \subsetneq Q' J(\alpha) \), but the corresponding factors give rise to different \( \alpha \)-cliques. In light of (iv), this cannot happen.

(vi) Given local idempotents \( e_i \in R \) such that \( C = \{e_1, R, \ldots, e_m R\} \) is an \( \alpha \)-clique in \( \text{Mod}-R \), if \( \alpha \) is a limit ordinal it follows immediately from the induction hypothesis that \( C^* \) is an \( \alpha \)-clique in \( R-\text{Mod} \). Suppose that \( \alpha \) is a nonlimit ordinal. Then \( C \) is a finite disjoint union of \((\alpha - 1)\)-cliques, \( C = C_1 \cup C_2 \cup \cdots \cup C_k \) where \( C_1 \) is the \((\alpha - 1)\)-successor of \( C_{i-1} \) for \( i = 2, \ldots, k \) and either \( C_1 \) is the \((\alpha - 1)\)-successor of \( C_k \) or \( C_1 \) has no \((\alpha - 1)\)-predecessors and \( C_k \) has no \((\alpha - 1)\)-successors. (It is possible that \( k = 1 \).) By the induction assumption, \( C_1^*, \ldots, C_k^* \) are \((\alpha - 1)\)-cliques in \( R-\text{Mod} \); \( C_i^* \) is the \((\alpha - 1)\)-predecessor of \( C_{i+1}^* \) for \( i = 2, \ldots, k \), and either \( C_1^* \) is the \((\alpha - 1)\)-predecessor of \( C_k^* \) or \( C_1^* \) has no \((\alpha - 1)\)-successors and \( C_k^* \) has no \((\alpha - 1)\)-predecessors. Hence \( C^* \) is an \( \alpha \)-clique in \( R-\text{Mod} \). Also, clearly, \( C^{**} = C \). If \( C' \) is an \( \alpha \)-successor of \( C \) in \( \text{Mod}-R \), there exist \( f R \in C' \), \( e R \in C \), and a diagram
This induces a diagram in $\text{R-Mod}$:

\[
\begin{array}{c}
\text{Re} \\
\downarrow \\
J(\alpha)^2 f \subset \cdots \subset J(\alpha) f \subset Rf.
\end{array}
\]

Then $C'^*$ is an $\alpha$-predecessor of $C^*$ in $\text{R-Mod}$. Hence (vi) holds.

(vii) Let $\beta > \alpha$. By (i)--(iii), the set of local projective covers of f.g. submodules of $xJ(\beta)/xJ(\beta) J(\alpha)$ is an $\alpha$-clique if $x$ is a local element such that $0 \subset xJ(\beta) J(\alpha) \subset xJ(\beta)$. Denote this $\alpha$-clique by $C$. W.l.o.g. take $x$ to be a local idempotent of $\text{R}$. Then for all local idempotents $f$ such that $fR \notin C$, $J(\beta) J(\alpha) f \subset J(\beta) f$. By Lemma 2, $J(\beta) f = J(\alpha) f$; hence $J(\alpha)^2 f = J(\alpha) f$. Then $C^*$ has no $\alpha$-successor. By (vi), $C$ has no $\alpha$-predecessor.

**Lemma 4.** Under the same hypotheses and with the same notation as Proposition 3, if $x$ is a local element and $\alpha$ is a nonlimit ordinal such that $xR \supset xJ(\alpha - 1) \supset \cdots \supset xJ(\alpha - 1)^n = xJ(\alpha - 1)^{n+1} = \cdots = xJ(\alpha) \neq 0$, then the $(\alpha - 1)$-cliques associated with the factors $xR/xJ(\alpha - 1)$, ..., $xJ(\alpha - 1)^{n-1}/J(\alpha - 1)^n$ are pairwise disjoint.

**Proof.** Suppose that for some $0 \leq i < j < n$ the $(\alpha - 1)$-clique associated with $xJ(\alpha - 1)^j/xJ(\alpha - 1)^{j+1}$ coincides with the $(\alpha - 1)$-clique associated with $xJ(\alpha - 1)^i/xJ(\alpha - 1)^{i+1}$. By uniqueness of $(\alpha - 1)$-predecessors, we may assume $i = 0$. Let $P \rightarrow xR$ be a projective cover. Then there exist $y \in PJ(\alpha - 1)^j \setminus PJ(\alpha - 1)^{j+1}$ and $z \in xJ(\alpha - 1)^j \setminus xJ(\alpha - 1)^{j+1}$ such that $P$ is a projective cover of both $yR$ and $zR$. Indeed, there is a commutative diagram:

\[
\begin{array}{cccc}
P & \longrightarrow & xR \\
\uparrow & & \uparrow \\
PJ(\alpha - 1)^j & \longrightarrow & xJ(\alpha - 1)^j \\
\uparrow & & \uparrow \\
P & \longrightarrow & yR & \longrightarrow & zR \\
\uparrow & & \uparrow & & \uparrow \\
PJ(\alpha - 1) & \longrightarrow & PJ(\alpha - 1)^{j+1} & \longrightarrow & xJ(\alpha - 1)^{j+1}.
\end{array}
\]
Under the composition $P \to yR \to zR$, the images of $PJ(\alpha - 1)^n - j$ and $PJ(\alpha - 1)^n - j + 1$ coincide. Yet the kernel is properly contained in $PJ(\alpha - 1)^n - j + 1$. It follows that $PJ(\alpha - 1)^n - j = PJ(\alpha - 1)^n - j + 1$, hence $xJ(\alpha - 1)^n - j = xJ(\alpha - 1)^n - j + 1$, a contradiction.

**Proposition 5.** Suppose $R$ is a serial ring with $N$ nonisomorphic local projective modules and that $M$ is a uniserial right $R$-module.

(i) If $\text{K.dim} \; M < \alpha$ or if $M$ is $\alpha$-critical, then

$$B(\alpha, 0, M) \; J(\alpha - 1)^n \; J(\alpha) = 0 \quad \text{if} \; \alpha \text{ is a nonlimit ordinal; resp.}$$

$$B(\alpha, 0, M) \; J(\alpha) = 0 \quad \text{if} \; \alpha \text{ is a limit ordinal.}$$

(ii) If $0 \subset B(\alpha, 0, M) \subset B(\alpha, 1, M)$, then

$$B(\alpha, 1, M) \; J(\alpha - 1)^n \; J(\alpha) = B(\alpha, 0, M) \quad \text{if} \; \alpha \text{ is a nonlimit ordinal; resp.}$$

$$B(\alpha, 1, M) \; J(\alpha) = B(\alpha, 0, M) \quad \text{if} \; \alpha \text{ is a limit ordinal.}$$

(iii) If $\text{K.dim} \; R = \alpha$ and $M$ is a uniserial module with Krull dimension, then $B(\alpha, i, M) = M$ for some natural number $i$. In particular, $M$ is annihilated by some power of $J(\alpha)$.

**Proof.** The proof is again by transfinite induction on $\alpha$. The proposition holds for $\alpha = 0, 1$ [7, 8]. Assume the result holds for all $\beta < \alpha$.

(i) W.l.o.g. we may assume $M = B(\alpha, 0, M)$. Consider first the case where $\alpha$ is a nonlimit ordinal and $\text{K.dim} \; M < \alpha$. If $M = \bigcup \{B(\alpha - 1, i, M)\}_{i \in N}$, we appeal to the induction assumption. Suppose then that $C_0$ defined to be $\bigcup \{B(\alpha - 1, i, M)\}_{i \in N} \subset M$. Let $C_1/C_0 = B(\alpha - 1, 0, M/C_0)$. $C_2/C_1 = B(\alpha - 1, 0, M/C_0) \cdots$ by the induction assumption. For each $k \geq 1$, $C_kJ(\alpha - 2)^n \; J(\alpha - 1) \subseteq C_{k - 1}$, if $\alpha - 1$ is a nonlimit ordinal (resp. $C_kJ(\alpha - 1) \subseteq C_{k - 1}$, if $\alpha - 1$ is a limit ordinal). Moreover, $C_{k - 1} \subseteq C_k$ unless $C_k - M$. We claim that for all $k \geq 1$, $C_kJ(\alpha - 2)^n \; J(\alpha - 1) = C_{k - 1}$ if $\alpha - 1$ is a nonlimit ordinal (resp. $C_kJ(\alpha - 1) = C_{k - 1}$ if $\alpha - 1$ is a limit ordinal). Indeed, when $k = 1$ and $\alpha - 1$ is a nonlimit ordinal, suppose for a moment that $C_1J(\alpha - 2)^n \; J(\alpha - 1) \subset C_0$. Then $C_1J(\alpha - 2)^n \; J(\alpha - 1) \subset B(\alpha - 1, i, M)$ for some $i$; hence $C_1 \subseteq C_0$, a contradiction. Still assuming that $\alpha - 1$ is a nonlimit ordinal, if it is known that for $k > 1$, $C_{k - 1}J(\alpha - 2)^n \; J(\alpha - 1) = C_{k - 1}$ and if $C_kJ(\alpha - 2)^n \; J(\alpha - 1) \subset C_{k - 1}$, then $\text{K.dim} \; C_k/C_{k - 1} \leq \alpha - 1$; hence $C_k \subseteq C_{k - 1}$, a contradiction. If $\alpha - 1$ is a limit ordinal a similar proof shows that $C_kJ(\alpha - 1) = C_{k - 1}$ for all $k \geq 1$. It follows that $C_N = M$. Otherwise, given $x \in C_N \backslash C_{N - 1}$ using Lemma 2, for some $k > N$, $xJ(\alpha - 1)^k = C_0 = C_0J(\alpha - 1) = xJ(\alpha - 1)^k + 1$. By Lemma 4, the $(\alpha - 1)$-cliques associated with the factors $xR/xy(\alpha - 1), \ldots, xJ(\alpha - 1)^k/ xJ(\alpha - 1)^k + 1$ are all distinct, contradicting the hypothesis on $N$. 

Next consider the case where \( \mathrm{K.dim} \, M < \alpha \) and \( \alpha \) is a limit ordinal. Say \( \mathrm{K.dim} \, M = \beta < \alpha \). Then \( M \subseteq B(\beta + 1, 0, M) \) and \( \beta + 1 < \alpha \). Appealing to the induction assumption yields the desired result.

Suppose that \( M \) is \( \alpha \)-critical, \( \alpha > 0 \). If \( \alpha \) is a nonlimit ordinal, \( MJ(\alpha - 1)^N J(\alpha) \subseteq \bigcap \{ L \mid 0 \subseteq L \subseteq M \} \). If \( \alpha \) is a limit ordinal, \( MJ(\alpha) \subseteq \bigcap \{ L \mid 0 \subseteq L \subseteq M \} \). Clearly, \( \bigcap \{ L \mid 0 \subseteq L \subseteq M \} \) is either simple (0-critical) or zero. Since \( \alpha > 0 \), we have \( MJ(\alpha - 1)^N J(\alpha) = 0 \) if \( \alpha \) is a nonlimit ordinal (resp. \( MJ(\alpha) = 0 \) if \( \alpha \) is a limit ordinal).

(ii) For convenience, let \( I \) denote \( J(\alpha - 1)^N J(\alpha) \) if \( \alpha \) is a nonlimit ordinal (resp. \( J(\alpha) \) if \( \alpha \) is a limit ordinal), and \( B \) denote \( B(\alpha, 0, M) \). W.l.o.g. assume that \( 0 \subseteq B(\alpha, 0, M) \subseteq B(\alpha, 1, M) = M \). It is clear from (i) that \( MI \subseteq B(\alpha, 0, M) = B \). In the case where \( \alpha \) is a nonlimit ordinal, suppose for a moment that \( MJ(\alpha - 1)^N \subseteq B \). If \( M \) has \( \alpha \)-critical submodules, then given any \( 0 \subseteq C \subseteq B \), \( B/C \) is a union of submodules of Krull dimension \( \leq \alpha - 1 \); hence \( \mathrm{K.dim} \, B/C < \alpha \). It is then easy to verify that \( M \) is \( \alpha \)-critical and so \( M = B \), a contradiction. If \( M \) has no \( \alpha \)-critical submodules, then \( \mathrm{K.dim} \, M = \max \{ \mathrm{K.dim} \, M/MJ(\alpha - 1)^N, \mathrm{K.dim} \, MJ(\alpha - 1)^N \} \leq \alpha - 1 \), again a contradiction. Thus, if \( \alpha \) is a nonlimit ordinal \( B \subseteq MJ(\alpha - 1)^N \). Now given \( y \in MJ(\alpha - 1)^N \setminus B \) (resp. \( y \in M \setminus B \) if \( \alpha \) is a limit ordinal), if \( yJ(\alpha) \subseteq B \) it follows from the definition of \( J(\alpha) \) and of \( B \) that \( y \in B \), a contradiction.

(iii) Assume that \( \mathrm{K.dim} \, R = \alpha \) and \( M \) is a uniserial module with Krull dimension. Then \( \mathrm{K.dim} \, M \leq \alpha \). If \( B(\alpha, i, M) = M \) for some integer \( i \geq 0 \) we are done. If not, \( B(\alpha, i, M) \subseteq B(\alpha, i + 1, M) \) for all \( i \geq 0 \). We claim that for all local idempotents \( e \in R \), there are only finitely many indices \( i \) such that \( B(\alpha, i + 1, M)/B(\alpha, i, M) \) has a submodule with projective cover \( eR \). Once the claim is established, since there are only finitely many local projective modules, \( M \) must coincide with \( B(\alpha, i, M) \) for some \( i \).

Select a local idempotent \( e = e^2 \in R \). If the claim is false for \( e \), then for all integers \( i \geq 0 \), there exist \( j > i + 1 \) and \( x \in B(\alpha, j + 1, M) \setminus B(\alpha, j, M) \) such that \( eR \) is a projective cover of \( xR \). By part (ii), \( B(\alpha, i + 1, M)I - B(\alpha, i, M) \) for all \( i \geq 0 \). Hence there is a commutative diagram with all downward homomorphisms induced by the projective cover \( eR \to xR \):

\[
\begin{array}{ccccccc}
eI^{i+1} & \subseteq & \ldots & \subseteq & eI^{i+1} & \subseteq & \ldots & \subseteq & eI & \subseteq & eR \\
\downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
B(\alpha, 0, M) & \subseteq & \ldots & \subseteq & B(\alpha, i, M) & \subseteq & \ldots & \subseteq & xI = B(\alpha, j, M) & \subseteq & xR.
\end{array}
\]

By construction, for each \( i \geq 0 \),

(a) \( 0 \subseteq eI^{i+1} \subseteq eI^i \) and
(b) either \( \mathrm{K.dim} \, B(\alpha, i + 1, M)/B(\alpha, i, M) = \alpha \) or \( \mathrm{K.dim} \, B(\alpha, i + 2, M)/B(\alpha, i + 1, M) = \alpha \) (Lemma 1).
But then \( eR \) has the descending chain \( eR \supseteq eI \supseteq eI^2 \supseteq \cdots \) in which infinitely many factors have Krull dimension \( \alpha \), a contradiction.

As an immediate consequence of Proposition 5 we have

**Theorem 6.** If the serial ring \( R \) has Krull dimension \( \alpha \), then for some \( k \), \( J(\alpha)^k = 0 \). Consequently \( R \) also has left Krull dimension \( \alpha \).

**Proof.** It is enough to show that for each local idempotent \( e \in R \) and for each natural number \( i \), \( J(\alpha)^{i-1}e/J(\alpha)^i e \) has Krull dimension \( \leq \alpha \). This is obvious from [3, Prop. 1.2] after noting that for any proper descending chain \( J(\alpha)^{i-1}e = M_0 \supset M_1 \supset \cdots \supset J(\alpha)^i e \) and for any local \( x \in M_0 \setminus M_1 \), we have \( Rx \supseteq M_1 \supset \cdots \supset J(\alpha)x = J(\alpha)^i e \) and \( \text{K. dim } Rx/J(\alpha)x \leq \alpha \).

**Remarks.** The structure theorems now known for serial rings with Krull dimension zero or one (see [2, 5, 7, 9]) suggest that there is a unifying structure theory for serial rings with arbitrary Krull dimension. Proposition 3 and Theorem 6 are crucial steps in attempting to find such a general structure theory. It appears likely that with the aid of these results the structure of a serial ring with Krull dimension can be described in terms of matrices whose entries are from much "simpler" rings or from suitable bimodules. It will then be possible to construct nontrivial examples of serial rings with Krull dimension. One reason for studying these rings is that, like the valuation domains in the category of commutative rings, serial rings with Krull dimension provide a class of non-Noetherian rings, yet rings whose structures are still manageable, in which to test numerous conjectures or unsolved problems.

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**References**