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Geometrical Constructions of Flock Generalized Quadrangles

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cal construction of $\mathcal{S}(F)$ which works for any q. Here we show how, for q odd, one can derive Knarr's construction from Thas' one. To that end we describe an interesting representation of the point-plane flags of PG$(3, q)$, which can be generalized to any dimension and which can be useful for other purposes. Applying this representation onto Thas' model of $\mathcal{S}(F)$, another interesting model of $\mathcal{S}(F)$ on a hyperbolic cone in PG$(6, q)$ is obtained. In a final section we show how subquadrangles and ovoids of $\mathcal{S}(F)$ can be obtained via the description in PG$(6, q)$.

1. INTRODUCTION

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines* respectively, and for which I is a symmetric point-line *incidence relation* satisfying the following axioms.

(i) Each point is incident with $1 + t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \,\mathrm{I}\, M \,\mathrm{I}\, y \,\mathrm{I}\, L$.

Generalized quadrangles were introduced by Tits [18] in his celebrated work on triality.

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The integers s and t are the *parameters* of the generalized quadrangle and $\mathcal{S}$ is said to have *order* (s, t); if $s = t$, $\mathcal{S}$ is said to have *order* s. There is a point-line duality for GQ (of order (s, t)) for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a (finite) GQ of order (s, t). Then $\mathcal{S}$ has $v = |P| = (1+s)(1+st)$ points and $b = |B| = (1+t)(1+st)$ lines; see 1.2.1 of Payne and Thas [11]. Also, $s+t$ divides $st(1+s)(1+t)$, and, for $s \neq 1 \neq t$, we have $t \leqslant s^2$ and, dually, $s \leqslant t^2$; see Payne and Thas [11, 1.2.2 and 1.2.3].

2. FLOCKS, BLT-SETS, AND FLOCK GENERALIZED QUADRANGLES

Let F be a *flock* of the quadratic cone K with vertex x of PG$(3, q)$, that is, a partition of $K - \{x\}$ into q disjoint irreducible conics. Then, by Thas [13] with F there corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q); in fact it was shown that with F there corresponds a q-clan and then by work of Payne [7, 8] and Kantor [3, 4] with F there corresponds a GQ of order (q^2, q). Also, independently, Walker [19] and Thas discovered that with each flock of an irreducible quadric of PG$(3, q)$ there corresponds a translation plane of order q^2; see also Fisher and Thas [2] and Thas [13].

Let $F = \{C_1, C_2, ..., C_q\}$ be a flock of the quadratic cone K with vertex x of PG$(3, q)$, with q odd. The plane of C_i is denoted by π_i, $i = 1, 2, ..., q$. Let K be embedded in the nonsingular quadric Q of PG$(4, q)$. Let the polar line of π_i with respect to Q be denoted by L_i and let $L_i \cap Q = \{x, x_i\}$, $i = 1, 2, ..., q$. If H_i is the tangent hyperplane of Q at x_i, then put $H_i \cap Q = K_i$, $H_i \cap H_j \cap Q = K_i \cap H_j = C_{ij}$ and $C_{ii} = C_i$, with $i, j = 1, 2, ..., q$ and $i \neq j$. Then Bader, Lunardon and Thas [1] prove that $F_i = \{C_{i1}, C_{i2}, ..., C_{iq}\}$ is a flock of K_i, $i = 1, 2, ..., q$. We say that the flocks $F_1, F_2, ..., F_q$ are *derived* from the given flock F. In many cases this process of derivation produces new flocks and new planes, but Payne and Rogers [10] prove that the GQ $\mathcal{S}(F), \mathcal{S}(F_1), ..., \mathcal{S}(F_q)$ are always isomorphic.

The main result of Bader *et al.* [1] amounts to proving that in the GQ $Q(4, q)$ arising from the quadric Q, the set $\mathcal{V} = \{x_0, x_1, ..., x_q\}$, with $x = x_0$, has the property that for any three distinct points x_i, x_j, x_k of $\mathcal{V}$ there is no point on $Q(4, q)$ collinear with all of them. Let $W(q)$ be the classical GQ arising from a symplectic polarity of PG$(3, q)$. Then $W(q)$ is isomorphic to the dual of $Q(4, q)$; see Payne and Thas [11, 3.2.1]. With $\mathcal{V}$ there corresponds a set $\mathcal{W}$ of $q+1$ lines $L_0, L_1, ..., L_q$ of $W(q)$, having the property that for any three distinct lines L_i, L_j, L_k of $\mathcal{W}$ there is no line in $W(q)$ concurrent with all of them. Such a set of $q+1$ lines in $W(q)$, q odd, was called a *BLT-set* by Kantor [5]. Hence any given flock F defines just one BLT-set, and any BLT-set produces $q+1$ flocks (possibly non-isomorphic) but just one GQ.

3. THE CONSTRUCTION OF KNARR

Start with a symplectic polarity θ of $\mathrm{PG}(5, q)$, q odd. Let $p \in \mathrm{PG}(5, q)$ and let $\mathrm{PG}(3, q)$ be a 3-dimensional subspace of $\mathrm{PG}(5, q)$ for which $p \notin \mathrm{PG}(3, q) \subset p^{\theta}$. In $\mathrm{PG}(3, q)$ θ induces a symplectic polarity θ', and hence a GQ $W(q)$. Let $\mathscr{W}$ be a BLT-set of the GQ $W(q)$ and construct a geometry $\mathscr{S} = (P, B, \mathrm{I})$ as follows.

Points are of three types:

(i) the q^5 points of $\mathrm{PG}(5, q)$ not in p^{θ};

(ii) the $q^3 + q^2$ lines of $\mathrm{PG}(5, q)$ not containing p but contained in one of the planes $\pi_i = pL_i$, with L_i a line of the BLT-set $\mathscr{W}$;

(iii) p.

Lines are of two types:

(a) the $q^4 + q^3$ totally isotropic planes of θ not contained in p^{θ} and meeting some π_i in a line (not through p);

(b) the $q+1$ planes $\pi_i = pL_i$, with $L_i \in \mathscr{W}$.

The incidence relation I is just the natural incidence inherited from $\mathrm{PG}(5, q)$.

Then Knarr [6] proves that $\mathscr{S}$ is a GQ of order (q^2, q) isomorphic to $\mathscr{S}(F)$, with F any flock arising from the BLT-set $\mathscr{W}$. We emphasize that in this construction q must be odd.

4. THE CONSTRUCTION OF THAS

Let K be a quadratic cone with vertex x of $\mathrm{PG}(3, q)$. Further, let y be a point of $K - \{x\}$ and let ζ be a plane of $\mathrm{PG}(3, q)$ not containing y. Now we project $K - \{y\}$ from y onto ζ. Let τ be the tangent plane of K at the line xy and let $\tau \cap \zeta = T$. Then with the q^2 points of $K - xy$ there correspond the q^2 points of the affine plane $\zeta - T = \zeta'$, with any point of $xy - \{y\}$ there corresponds the intersection ∞ of xy and ζ, with the generators of K distinct from xy there correspond the lines of ζ distinct from T containing ∞, with the (nonsingular) conics on K passing through y there correspond the affine parts of the q^2 lines of ζ not passing through ∞, and with the (nonsingular) conics on K not passing through y there correspond the $q^2(q-1)$ (nonsingular) conics of ζ which are tangent to T at ∞.

Let $F=\{C_1^*, C_2^*, ..., C_q^*\}$ be a flock of the cone K. Now consider the set $\tilde{F}=\{C_1, C_2, ..., C_{q-1}, N\}$ consisting of the $q-1$ nonsingular conics $C_1, C_2, ..., C_{q-1}$ and the line N of ζ, which is obtained by projecting the elements of F from y onto ζ. So $C_1, C_2, ..., C_{q-1}$ are conics which are mutually tangent at ∞ (with common tangent line T) and N is a line of ζ not containing ∞.

Now we consider planes $\pi_\infty \neq \zeta$ and $\mu \neq \zeta$ of PG(3, q), respectively containing T and N; in μ we consider a point r, with $r \notin \zeta \cup \pi_\infty$. Next, let O_i be the nonsingular quadric which contains C_i, which is tangent to π_∞ at ∞ and which is tangent to μ at r, with $i=1, 2, ..., q-1$. As $C_i \cap N = \emptyset$, the quadric O_i is elliptic, $i=1, 2, ..., q-1$.

Next, let $\mathcal{S}$ be the following incidence structure.

Points of $\mathcal{S}$

(a) The $q^3(q-1)$ nonsingular elliptic quadrics O containing $O_i \cap \pi_\infty = L_\infty^{(i)} \cup M_\infty^{(i)}$ (over GF(q^2)) such that the intersection multiplicity of O_i and O at ∞ is at least three (that are O_i, the nonsingular elliptic quadrics $O \neq O_i$ containing $L_\infty^{(i)} \cup M_\infty^{(i)}$ (over GF(q^2)) and intersecting O_i over GF(q) in a nonsingular conic containing ∞, and the nonsingular elliptic quadrics $O \neq O_i$ for which $O \cap O_i$ over GF(q^2) is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice), with $i=1, 2, ..., q-1$.

(b) The q^3 points of PG(3, q) $- \pi_\infty$.

(c) The q^3 planes of PG(3, q) not containing ∞.

(d) The $q-1$ sets $\mathcal{O}_i$, where $\mathcal{O}_i$ consists of the q^3 quadrics O of type (a) corresponding with O_i, $i=1, 2, ..., q-1$.

(e) The plane π_∞.

(f) The point ∞.

Lines of $\mathcal{S}$

(i) Let (w, γ) be a point-plane flag of PG(3, q), with $w \notin \pi_\infty$ and $\infty \notin \gamma$. Then all quadrics O of type (a) which are tangent to γ at w, together with w and γ, form a line of type (i). Any two distinct quadrics of such a line have exactly two points (∞ and w) in common. The total number of lines of type (i) is q^5.

(ii) Let O be a point of type (a) which corresponds to the quadric O_i, $i \in \{1, 2, ..., q-1\}$. If $O \cap \pi_\infty = O_i \cap \pi_\infty = L_\infty^{(i)} \cup M_\infty^{(i)}$ (over $\mathrm{GF}(q^2)$), then all points O' of type (a) for which $O' \cap O$ over $\mathrm{GF}(q^2)$ is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice, together with O and O_i, form a line of type (ii). There are $q^2(q-1)$ lines of type (ii).

(iii) A set of q parallel planes of $\mathrm{AG}(3, q) = \mathrm{PG}(3, q) - \pi_\infty$, where the line at infinity does not contain ∞, together with the plane π_∞, is a line of type (iii).

(iv) Lines of type (iv) are the lines of $\mathrm{PG}(3, q)$, not in π_∞, containing ∞.

(v) $\{\infty, \pi_\infty, \mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_{q-1}\}$ is the unique line of type (v).

Incidence of $\mathcal{S}$

Incidence is containment.

Then it is proved in Thas [15] that $\mathcal{S}$ is a GQ isomorphic to the point-line dual of the flock GQ $\mathcal{S}(F)$. We emphasize that this construction works for any prime power q.

5. AN INTERESTING REPRESENTATION OF POINT-PLANE FLAGS OF PG(3, q)

Consider the point-plane flag $((0, 0, 0, 1), X_0 = 0)$ of $\mathrm{PG}(3, q)$. Now let (w, γ) be any point-plane flag of $\mathrm{PG}(3, q)$, with $(0, 0, 0, 1) \notin \gamma$ and w not in $X_0 = 0$. Let $w(1, x, y, z)$ and γ: $aX_0 + bX_1 + cX_2 + X_3 = 0$. Then we put

$$(w, \gamma)^\eta = (x, y, z, a, b, c) \in \mathrm{AG}(6, q).$$

All images are points of the quadric Φ of $\mathrm{PG}(6, q)$ with equation

$$X_1X_5 + X_2X_6 + X_3X_0 + X_4X_0 = 0.$$

This quadric Φ is a hyperbolic cone with vertex $s(0, 0, 0, 1, -1, 0, 0)$. The hyperplane Π with equation $X_0 = 0$ of $\mathrm{PG}(6, q)$ is tangent to Φ along the line joining s to $(0, 0, 0, 0, 1, 0, 0)$. Clearly η is a bijection from the set of all flags (w, γ), with $(0, 0, 0, 1) \notin \gamma$ and w not in $X_0 = 0$, onto $\Phi - \Pi$.

If O is a nonsingular quadric of $\mathrm{PG}(3, q)$ which is tangent to $X_0 = 0$ at $(0, 0, 0, 1)$, then the flags (w, γ), with $w \in O - \{(0, 0, 0, 1)\}$ and γ the tangent plane of O at w are mapped by η onto the points not in $X_0 = 0$ of a nonsingular quadric $O' \subset \Phi$, contained in some 3-dimensional subspace δ of $\mathrm{PG}(6, q)$ and having the same character as O. Also, O' contains the point $h(0, 0, 0, 1, 1, 0, 0)$ of Φ.

If O has equation

$$\sum_{\substack{i,j=0\\ i\leqslant j}}^{3} a_{ij}X_iX_j=0, \qquad \text{with} \quad a_{33}=a_{13}=a_{23}=0,$$

then O is nonsingular if and only if $a_{03}\neq 0\neq 4a_{11}a_{22}-a_{12}^2$. So we may assume $a_{03}=1$. It is easily checked that the space δ containing O' is represented by

$$\begin{cases} X_4=2a_{00}X_0+a_{01}X_1+a_{02}X_2+X_3,\\ X_5=a_{01}X_0+2a_{11}X_1+a_{12}X_2,\\ X_6=a_{02}X_0+2a_{22}X_2+a_{12}X_1.\end{cases}$$

Also, if $w(1, x, y, z)\in O$ and if γ is the tangent plane of O at w, then

$$(w,\gamma)^{\eta}=(1, x, y, z, 2a_{00}+a_{01}x+a_{02}y+z,$$
$$a_{01}+2a_{11}x+a_{12}y, a_{02}+2a_{22}y+a_{12}x).$$

Let Φ^+ be the hyperbolic quadric $X_3=X_1X_5+X_2X_6+X_4X_0=0$ (so Φ^+ is the base of the cone Φ). If q is odd, then the quadric O' is tangent to $\Pi\cap\delta$ at h. Now let q be even. Then $s=h$. In such a case δ belongs to Φ. Hence $\delta\cap\Phi^+$ is a plane δ' of Φ^+, and in this way there arise $q^2(q-1)$ planes δ' of Φ^+; the planes δ' all belong to a same family $\mathscr{A}$ of generators of Φ^+. The space $X_3=X_0=0$ is tangent to Φ^+ at $e_4(0, 0, 0, 0, 1, 0, 0)$, and as $e_4\notin\delta'$ the plane δ' does not belong to Π. Further, δ' has no point in common with the planes $X_0=X_3=X_5=X_6=0$ and $X_0=X_3=X_1=X_2=0$ of Φ^+. Finally, for a fixed δ' the q^3 corresponding quadrics O' belong to a common linear 3-dimensional system of quadrics in the 3-dimensional space $s\delta'=\delta$.

Assume again that q is odd. Then we consider the projection ξ from h onto the hyperplane PG(5, q) with equation $X_3=0$. We have $(\delta-\{h\})^{\xi}=\tilde{\delta}$, with $\tilde{\delta}$ the plane having equations

$$\begin{cases} X_3=0,\\ X_4=2a_{00}X_0+a_{01}X_1+a_{02}X_2,\\ X_5=a_{01}X_0+2a_{11}X_1+a_{12}X_2,\\ X_6=a_{02}X_0+a_{12}X_1+2a_{22}X_2.\end{cases}$$

Then

$$(\tilde{\delta})^{\theta} = \tilde{\delta},$$

with θ the symplectic polarity of PG(5, q) represented by the bilinear form

$$X_0 Y_4 - X_4 Y_0 + X_1 Y_5 - X_5 Y_1 + X_2 Y_6 - X_6 Y_2.$$

Hence $\tilde{\delta}$ is a totally isotropic plane of the polarity θ. It is readily checked that $\tilde{\delta} \not\subset \Pi$, that $\tilde{\delta} \cap \Pi = R$ is not a line of the quadric H with equations $X_3 = X_0 = X_1 X_5 + X_2 X_6 = 0$ (that is, $H = \Phi^+ \cap \Pi$), and that R is not tangent to H. The number of totally isotropic planes of θ in Π is equal to $(q+1)(q^2+1)$, the number of totally isotropic planes intersecting Π in a line of H is $(q+3)q^3$ ($(q+1)q^3$ of these planes intersect a plane of one system of generators of H, the remaining $2q^3$ planes intersect a plane of the second system of generators of H), and the number of totally isotropic planes intersecting Π in a tangent of H, not contained in H, is equal to $2(q^2-1)q^3$. Any of the remaining $q^5(q-1)$ totally isotropic planes of θ corresponds to a nonsingular quadric O of PG(3, q).

6. FROM THE CONSTRUCTION OF THAS TO THE CONSTRUCTION OF KNARR

We start from the construction of Thas of the dual of a flock GQ $\mathcal{S}(F)$ of order (q^2, q), q odd. We will use the notations of Section 4.

A point of type (a) is an elliptic quadric of PG(3, q) touching a fixed plane π_∞ at a fixed point ∞. We identify the point-plane flag (∞, π_∞) with the point-plane flag ((0, 0, 0, 1), $X_0 = 0$) of Section 5. So with O there corresponds an elliptic quadric O' on the cone Φ, and a totally isotropic plane $\tilde{\delta}$ of the symplectic polarity θ of PG(5, q). Suppose that with O_i there corresponds the totally isotropic plane $\tilde{\delta}_i$ of θ, and that $\tilde{\delta}_i \cap \Pi = R_i$, with $i = 1, ..., q-1$. One can show that with the $q-1$ nonsingular elliptic quadrics O of PG(3, q) for which $O \cap O_i$ over GF(q^2) is $L_\infty^{(i)} \cup M_\infty^{(i)}$ counted twice, there correspond the $q-1$ totally isotropic planes distinct from $\tilde{\delta}_i$ and not contained in Π, which contain the line R_i, $i \in \{1, 2, ..., q-1\}$. With the $q^3 - q$ elliptic quadrics O containing $L_\infty^{(i)} \cup M_\infty^{(i)}$ (over GF(q^2)) and intersecting O_i over GF(q) in a nonsingular conic containing ∞, there correspond the totally isotropic planes not in Π intersecting the totally isotropic plane $e_4 R_i$ in a line distinct from R_i, $i \in \{1, 2, ..., q-1\}$.

Let w be a point of type (b). With the flags (w, γ), $\infty \notin \gamma$, there correspond q^2 points of a totally isotropic plane of θ, not in Π and containing a line of the totally isotropic plane $X_0 = X_1 = X_2 = X_3 = 0$ (on H).

Let γ be a point of type (c). With the flags $(w, \gamma), w \notin \pi_\infty$, there correspond q^2 points of a totally isotropic plane of θ, not in Π and containing a line of the totally isotropic plane $X_0 = X_3 = X_5 = X_6 = 0$ (on H).

With the $q-1$ points of type (d) we let correspond the $q-1$ totally isotropic planes $e_4 R_i$ of θ, $i = 1, 2, ..., q-1$.

With the unique point π_∞ of type (e) we let correspond the plane $X_0 = X_3 = X_5 = X_6 = 0$, and with the unique point ∞ of type (f) we let correspond the plane $X_0 = X_1 = X_2 = X_3 = 0$.

Consider the line of type (i) defined by the point-plane flag (w, γ) of PG(3, q), with $w \notin \pi_\infty$ and $\infty \notin \gamma$. With this line we let correspond the point $(w, \gamma)^{\eta\xi}$ of PG(5, q) ($(w, \gamma)^{\eta\xi} \notin \Pi$).

Now we consider the line of type (ii) defined by the elliptic quadric O, where O corresponds to O_i. With O there corresponds a line U of $e_4 R_i$, with $e_4 \notin U$ (with the point O of type (a) corresponds a totally isotropic plane $\tilde{\delta}$ of θ, and $U = \tilde{\delta} \cap \Pi$). With the given line of type (ii) we let correspond the line U of Π.

A line of type (iii) consists of q parallel planes of PG(3, q) $- \pi_\infty$, where the line at infinity does not contain ∞, together with the plane π_∞. With these q parallel planes correspond q totally isotropic planes of θ containing a common line U of $X_0 = X_3 = X_5 = X_6 = 0$. With the given line of type (iii) we let correspond the line U of Π.

A line of type (iv) is a line of PG(3, q), not in π_∞, containing ∞. With the q points not in π_∞ of that line correspond q totally isotropic planes of θ containing a common line U of $X_0 = X_1 = X_2 = X_3 = 0$. With the given line of type (iv) we let correspond the line U of Π.

With the unique line of type (v) we let correspond the point e_4.

Hence in PG(5, q) we have an incidence structure $\mathcal{S}'$ with six types of points and five types of lines. With the natural incidence this structure $\mathcal{S}'$ is a GQ isomorphic to the dual of $\mathcal{S}(F)$.

The elliptic quadrics $O_1, O_2, ..., O_{q-1}$ are tangent to the plane μ at the point r, hence the totally isotropic planes $\tilde{\delta}_1, \tilde{\delta}_2, ..., \tilde{\delta}_{q-1}$ have a point g in common, with $g \notin \Pi$. Hence these planes are contained in a 4-dimensional space β. Hence $R_1, R_2, ..., R_{q-1}$ are contained in the 3-dimensional space $\beta \cap \Pi = \alpha \subset e_4^\theta$, with $e_4 \notin \alpha$. Let R_q be the intersection of α with $X_0 = X_1 = X_2 = X_3 = 0$, and let R_{q+1} be the intersection of α with $X_0 = X_3 = X_5 = X_6 = 0$. Then $R_1, R_2, ..., R_{q+1}$ are totally isotropic for the symplectic polarity θ' induced by θ in α, that is, $R_1, R_2, ..., R_{q+1}$ are lines of the GQ $W(q)$ determined by θ'.

Consequently, the points of $\mathcal{S}'$ are:

(a)′ the $q^4 + q^3$ totally isotropic planes of θ not contained in e_4^θ and meeting some plane $e_4 R_i$ in a line (not through p), $i \in \{1, 2, ..., q+1\}$.

(b)′ the $q+1$ planes $e_4 R_i$, $i = 1, 2, ..., q+1$.

The lines of $\mathcal{S}'$ are:

(i)′ the q^5 points PG(5, q) not in e_4^θ;

(ii)′ the q^3+q^2 lines of PG(5, q) not containing e_4 but contained in one of the planes $e_4 R_i$, $i \in \{1, 2, ..., q+1\}$.

(iii)′ e_4.

Now we show that no two of the lines $R_1, R_2, ..., R_{q+1}$ are concurrent in $W(q)$. Assume, by way of contradiction, that r' is a common point of R_i and R_j in $W(q)$, $i \neq j$. Then there exist totally isotropic planes δ'_i and δ'_j of θ which respectively contain R_i and R_j, and which intersect in a line not contained in e_4^θ. It follows that the points δ'_i and δ'_j of $\mathcal{S}'$ are incident with more than one line of $\mathcal{S}'$, clearly a contradiction. Hence no two of the lines $R_1, R_2, ..., R_{q+1}$ are concurrent in $W(q)$.

Assume, by way of contradiction, that $W(q)$ contains a line R which is concurrent with three distinct lines of $\{R_1, R_2, ..., R_{q+1}\} = \mathcal{W}'$, say with R_i, R_j, R_k. Let r_l be the common point of R and R_l in $W(q)$, $l = i, j, k$. Further, let ρ be a totally isotropic plane of θ which contains R but is not contained in e_4^θ, and let R'_k be a line of $e_4 R_k$ distinct from $e_4 r_k$ and R_k. Also, let $\rho \cap R_i^\theta = R''_i$, $\rho \cap R_j^\theta = R''_j$ and $\rho \cap R_k'^\theta = R''_k$. Then clearly $R''_l \neq R$, $l = i, j, k$, and the lines R''_i, R''_j, R''_k are not concurrent. If we now consider the totally isotropic planes $\delta''_i = R''_i R_i$, $\delta''_j = R''_j R_j$, $\delta''_k = R''_k R'_k$ of θ, then δ''_i, δ''_j, δ''_k are points of $\mathcal{S}'$ which form a triangle, a contradiction. Consequently $\mathcal{W}'$ has the property that no three of its lines are concurrent with a common line of $W(q)$.

From the preceding it follows that $\mathcal{W}'$ is a BLT-set of $W(q)$. The construction of Knarr [6] applied to the BLT-set $\mathcal{W}'$ yields the dual of the GQ $\mathcal{S}'$. Let $\mathcal{W}$ be the BLT-set of $W(q)$ defined by the flock F. As $\mathcal{S}'$ is isomorphic to the dual of $\mathcal{S}(F)$, then, by the proof of Theorem IV.1 in Payne and Thas [12], $\mathcal{W}$ and $\mathcal{W}'$ are equivalent with respect to the group $P\Gamma Sp_4(q)$. Now we show that $\mathcal{W}$ and $\mathcal{W}'$ are even equivalent with respect to the group $PSp_4(q)$.

In the construction of Thas, Let F be the flock of the quadratic cone K with equation $X_0 X_1 = X_2^2$, let the respective planes π_i of the elements of F have equation

$$l_i X_0 + m_i X_1 + n_i X_2 + X_3 = 0, \qquad \text{with } i = 0, 1, ..., q-1,$$

and let $l_0 = m_0 = n_0 = 0$. Now we project $K - \{(0, 1, 0, 0)\}$ from $(0, 1, 0, 0)$ onto the plane ζ with equation $X_1 = 0$. The plane π_0 contains $(0, 1, 0, 0)$. The projection from $(0, 1, 0, 0)$ onto ζ of the conic $\pi_i \cap K$, with $i \neq 0$, is the conic C_i with equations

$$l_i X_0^2 + m_i X_2^2 + n_i X_0 X_2 + X_0 X_3 = 0 = X_1.$$

Also, we have $\pi_0 \cap \zeta = N : X_1 = X_3 = 0$. Further, let μ have equation $X_3 = 0$ and let $r(1, 1, 0, 0)$. Then

$$O_i\colon l_i X_0^2 + l_i X_1^2 + m_i X_2^2 - 2l_i X_0 X_1 + n_i X_0 X_2 + X_0 X_3 - n_i X_1 X_2 = 0,$$

with $i = 1, 2, ..., q-1$. With O_i there corresponds an elliptic quadric O'_i on the cone Φ, where O'_i is contained in the 3-dimensional space δ_i with equations

$$X_4 = 2l_i X_0 - 2l_i X_1 + n_i X_2 + X_3,$$
$$X_5 = -2l_i X_0 + 2l_i X_1 - n_i X_2,$$
$$X_6 = n_i X_0 - n_i X_1 + 2m_i X_2.$$

Hence the line R_i has equations

$$\begin{cases} X_0 = X_3 = X_4 + X_5 = 0, \\ X_4 = -2l_i X_1 + n_i X_2, \\ X_6 = -n_i X_1 + 2m_i X_2, \end{cases}$$

with $i = 1, 2, ..., q-1$. Further we have

$$R_q\colon X_0 = X_1 = X_2 = X_3 = X_4 + X_5 = 0,$$

and

$$R_{q+1}\colon X_0 = X_3 = X_4 = X_5 = X_6 = 0.$$

In the space α: $X_0 = X_3 = X_4 + X_5 = 0$ the symplectic polarity θ' can be represented by (in Grassmann coordinates)

$$p_{15} + p_{26} = 0.$$

As F is a flock, we have that

$$(n_i - n_j)^2 - 4(l_i - l_j)(m_i - m_j) \text{ is a nonsquare,}$$

for all $i, j \in \{0, 1, ..., q-1\}$. Now it is easy to check that the BLT-set $\mathscr{W}$ defined by the flock F is equivalent for the group $PSp_4(q)$ with the BLT-set $\mathscr{W}' = \{R_1, R_2, ..., R_{q+1}\}$.

We conclude that the above constructed model of the dual of $\mathscr{S}(F)$ in the projective space PG(5, q), is exactly the model constructed by Knarr starting from the BLT-set $\mathscr{W}$ defined by F.

7. SUBQUADRANGLES AND OVOIDS

7.1. *Subquadrangles in the Even Case*

Let $\mathcal{S}(F)$ be the GQ of order (q^2, q) arising from the flock F of the quadratic cone K of PG$(3, q)$, q even. Then $\mathcal{S}(F)$ has at least q^3+q^2 subquadrangles of order q; see Thas [16]. By Payne [8] any of these subquadrangles $\mathcal{S}'$ is a $T_2(O)$ of Tits, with O an oval of PG$(2, q)$. Similarly as in the odd case (see Sections 5 and 6) $\mathcal{S}(F)$ can be represented on the hyperbolic cone Φ with vertex s in PG$(6, q)$. Let φ be a plane on the hyperbolic quadric Φ^+ such that φ and the planes $X_0=X_3=X_5=X_6=0$ and $X_0=X_3=X_1=X_2=0$ belong to a common system of generators of Φ^+, but with φ not belonging to Π (that is, not containing $e_4(0, 0, 0, 0, 1, 0, 0)$). There are exactly q^3+q^2 such planes φ. Consider the 3-dimensional space $s\varphi=\Delta$. Then with the q^3 points of Δ not in Π correspond the q^3 points of a subquadrangle $\mathcal{S}'$ of order q of $\mathcal{S}(F)$ which are not collinear with the point ∞ of $\mathcal{S}(F)$. In this way we find all q^3+q^2 subquadrangles $\mathcal{S}'$ of order q containing the sets of type $\{L, L_1, L_2\}^{\perp} \cup \{L, L_1, L_2\}^{\perp\perp}$ with L, L_1, L_2 pairwise nonconcurrent, with $\infty \mathrm{I} L$, and with $\infty \mathrm{I} M$ and $M \in \{L, L_1, L_2\}^{\perp}$.

One can also show that with q^3-q^2 lines of $\mathcal{S}'$ there correspond q^3-q^2 conics in Δ through s. Each plane ω through s but neither containing the intersection of φ and $X_0=X_3=X_5=X_6=0$, nor the intersection of φ and $X_0=X_3=X_1=X_2=0$, contains exactly q of these conics. Then q conics are tangent to $\omega \cap \Pi$ at s, and define a partition of $\omega - \Pi$.

7.2. *Ovoids in the Odd Case*

Consider again $\mathcal{S}(F)$ and its representation on the cone Φ. Assume that q is odd. Let β be a plane of Φ^+ which is disjoint from the planes $X_0=X_1=X_2=X_3=0$ and $X_0=X_3=X_5=X_6=0$; such a plane β does not belong to Π. Now we consider the 3-dimensional space $s\beta=\Delta'$, with $s(0, 0, 0, 1, -1, 0, 0)$. Then it can be shown that the q^3 points of $\Delta'-\Pi$ together with the point $e_4(0, 0, 0, 0, 1, 0, 0)$ correspond to the points of an ovoid of $\mathcal{S}(F)$. These ovoids are of the type described in Thas and Payne [17, 6.2].

7.3. *Ovoids in the Even Case*

Consider again $\mathcal{S}(F)$ and its representation on the cone Φ. Let O' be an ovoid of Φ^+ which contains e_4. Then it can be shown that the q^3 points of $sO'-\Pi$ together with the point e_4 correspond to the points of an ovoid of $\mathcal{S}(F)$. These ovoids are exactly the ovoids described in Thas [14, 7.3 (a)].

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