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TOPOLOGICAL SPACES CONTAINING COMPACT PERFECT SETS AND PROHOROV SPACES

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Let X be a Suslin–Borel set in a compact space. It is proved that X is either σ-scattered or contains a compact perfect set. If X is first countable, the result remains valid when X is a Suslin–Borel set in a Prohorov space. It is also proved that every first countable Prohorov space is a Baire space.

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Čech-complete space	Prohorov space	σ-scattered set
Suslin–Borel set	τ-additive measure	Radon measure

0. Introduction

As it is well known, every analytic subset X of a Polish (separable complete metric) space is either countable or contains a copy of the Cantor set [12, p. 479]. This theorem has been generalized (with 'σ-discrete' replacing 'countable') to Borel and Suslin subsets of arbitrary complete metric spaces ([18] and [2], respectively). Further, completeness can be replaced by the Prohorov property [10], a measure-theoretic property enjoyed by complete metric spaces.

In this paper we generalize the above results when X is a Suslin–Borel subset of a Čech-complete space or a first countable Suslin–Borel subset of a Prohorov space. Here the conclusion states that X is either σ-scattered or contains a compact perfect set. This is proved in Theorems 3.1 and 4.2 in view of the remarks of Section 2. From Theorem 4.2 we deduce that the Sorgenfrey line is not Prohorov, answering a question of Mosiman and Wheeler ([13] and [22]). It is also proved that every first countable Prohorov space is a Baire space (Theorem 4.4). Other results concerning the Prohorov property for first countable spaces are proved in Theorems 5.6 and 5.7.

I would like to thank Professor D.H. Fremlin for providing a copy of [4].

1. Preliminaries

Throughout this paper all topological spaces are assumed to be regular and Hausdorff. A non-negative Borel measure μ on a space X is called (a) *τ-additive* if for every increasing net $\{G_\alpha\}$ of open sets in X, $\lim_\alpha \mu(G_\alpha)=\mu(\bigcup_\alpha G_\alpha)$, and (b) *Radon* (or *tight*) if μ is inner regular with respect to compact sets. It is well known that every Radon measure is τ-additive and that every τ-additive measure is regular with respect to closed sets.

The support $S(\mu)$ of a τ-additive measure μ on X is defined by

$$S(\mu)=\bigcap\{F\colon F \text{ closed in } X,\ \mu(F)=\mu(X)\}$$

and is the least closed subset of X with full measure. If μ vanishes on singletons, $S(\mu)$ is dense in itself.

A space X is called scattered if no nonempty subset of X is dense in itself. It is called σ-scattered if it is a countable union of scattered subsets. A nonempty, closed, dense in itself subset of X is called perfect. We have that X contains a compact perfect set if and only if X admits a non-zero Radon measure, vanishing on singletons, i.e. non-atomic ([8] and [17]).

We denote by $M_t^+(X)$ the space of non-negative Radon measures on X, endowed with the weak topology. That is, for a net $\{\mu_\alpha\}$ in $M_t^+(X)$, $\mu_\alpha\to\mu$ if and only if $\int f\,d\mu_\alpha\to\int f\,d\mu$ for all bounded continuous real-valued functions f on X. We say that X is a *Prohorov space* if every compact set H in $M_t^+(X)$ is uniformly tight, that is, for every $\varepsilon>0$ there exists a compact set K in X such that $\mu(X\setminus K)<\varepsilon$ for all $\mu\in H$.

Concerning the Prohorov property, one of our main tools will be the deep result of Preiss [15] that the space of rational numbers is not Prohorov. We shall also use the fact that the Prohorov property is preserved by countable products, countable intersections, closed subspaces and open subspaces (see [13] and [20]). Since every compact space is trivially Prohorov, it follows that every Čech-complete space (i.e. homeomorphic to a G_δ-subspace of a compact space) is Prohorov. The class of Čech-complete spaces has the above stability properties of Prohorov spaces (see [3, Section 3.9]).

We are primarily concerned with the following classes of spaces:

Definition (i) A completely regular space X is called *Čech-analytic* if there exists a Čech-complete space $R\subset X\times\mathbb{N}^{\mathbb{N}}$ such that $X=\pi_1(R)$, where $\pi_1\colon X\times\mathbb{N}^{\mathbb{N}}\to X$ denotes the projection.

(ii) A space X is called *Prohorov-analytic* if there exists a Prohorov space $R\subset X\times\mathbb{N}^{\mathbb{N}}$ such that $X=\pi_1(R)$.

Čech-analytic spaces were introduced by Fremlin in [4] in an equivalent form as a common generalization of K-analytic spaces and absolutely analytic metric spaces.

2. Suslin–Borel sets

In this section we show that a wide class of subsets of a Čech-complete (resp. Prohorov) space consists of Čech-analytic (resp. Prohorov-analytic) spaces.

For every $\sigma \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $\sigma|n = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ and set $\mathbb{N}^{(\mathbb{N})} = \{\sigma|n: \sigma \in \mathbb{N}^{\mathbb{N}}, n \in \mathbb{N}\}$. A subset A of a topological space X is called *Suslin–Borel* if A is expressible as

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} A_{\sigma|n}, \tag{$*$}$$

where each $A_{\sigma|n}$ is a Borel set in X. If the sets $A_{\sigma|n}$ can be chosen to be closed, A is called a *Suslin* set. When X is metrizable, every Suslin–Borel set is Suslin.

If a subset A of a space X is the intersection of a closed with an open set, A is called a D-set. A countable intersection (resp. countable union) of D-sets is called a D_δ-set (resp. D_σ-set).

The next proposition follows from the arguments of the proof of Theorem 4(b) in [4] and we sketch the proof for completeness.

Proposition 2.1. *A subset A of a topological space X is Suslin–Borel if and only if there exists a D_δ-subset R of $X \times \mathbb{N}^{\mathbb{N}}$ such that $A = \pi_1(R)$.*

Proof. The 'if' part follows from the fact that for every Borel set $R \subset X \times \mathbb{N}^{\mathbb{N}}$, $\pi_1(R)$ is Suslin–Borel in X (cf. [16, Theorem 2.6.5]).

If A is open or closed in X, then $A = \pi_1(R)$, where $R = A \times \{\alpha\}$ (for any $\alpha \in \mathbb{N}^{\mathbb{N}}$) is D_δ in $X \times \mathbb{N}^{\mathbb{N}}$. Now let A be as in ($*$) above, where for every $s \in \mathbb{N}^{(\mathbb{N})}$, $A_s = \pi_1(R_s)$ for some D_δ-set $R_s \subset X \times \mathbb{N}^{\mathbb{N}}$. Then $A = \pi_1(R)$, where R is the subset of $X \times \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{(\mathbb{N})}}$ given by

$$R = \bigcap_{n=1}^{\infty} \{(x, \alpha, (\beta_\sigma)_{\sigma \in \mathbb{N}^{(\mathbb{N})}}: (x, \beta_{\alpha|n}) \in R_{\alpha|n}\}.$$

Finally, we see that R is a D_δ-set and as $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}^{(\mathbb{N})}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, this completes the proof. □

Remarks. As every D_δ-subspace of a Prohorov space is Prohorov, it follows from Proposition 2.1 that every Suslin–Borel set in a Prohorov space is Prohorov-analytic.

For subsets of Čech-complete spaces we have an equivalence: a subset of a Čech-complete space is Čech-analytic if and only if it is Suslin–Borel [4, Theorem 5]. The 'only if' part follows from the fact that every Čech-complete subspace of a completely regular space is a D_δ-subset, being G_δ in its closure (see [3, Exercise 3.9.A]). Thus, Čech-analytic spaces are those completely regular spaces that are Suslin–Borel in some (or any) of their compactifications.

3. Čech-analytic spaces

In this section we prove:

Theorem 3.1. *For a Čech-analytic space X, exactly one of the following alternatives holds: either* (i) *X is σ-scattered, or* (ii) *X contains a compact perfect set.*

For the proof of Theorem 3.1 we shall need some lemmas; Lemmas 3.2, 3.3 and 3.4 are concerned with σ-scattered spaces; Lemma 3.5 is based on the Baire category method for the space of Radon measures used in [14, Theorem 6.1] and [8, Theorem 1].

Lemma 3.2. *If X is a σ-scattered space, then X does not contain any compact perfect set.*

Proof. Suppose that $X=\bigcup_{n=1}^{\infty} X_n$, where each X_n is scattered, and that X contains a compact perfect set. Then there is a non-zero Radon measure μ on X vanishing on singletons. Fix an n such that $\mu^*(X_n)>0$. Then μ^* induces a non-zero τ-additive measure ν on X_n, vanishing on singletons. But the support of ν, as a dense in itself subset of X_n, is empty and so $\nu=0$, a contradiction. □

Lemma 3.3. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{C}$ a countable family of subsets of X and $\tilde{\mathcal{T}}$ the topology on X generated by $\mathcal{T}$ and $\mathcal{C}$. If $(X, \tilde{\mathcal{T}})$ is scattered then $(X, \mathcal{T})$ is σ-scattered.*

Proof. Consider the sets G_ξ and F_ξ given by:

(i) $G_0=\emptyset$, $F_0=X$;

(ii) $G_{\xi+1}=\{x\in F_\xi\colon x \text{ is } \tilde{\mathcal{T}}\text{-isolated in } F_\xi\}$, $F_{\xi+1}=F_\xi\setminus G_{\xi+1}$; and

(iii) $G_\xi=\emptyset$ and $F_\xi=\bigcap_{\zeta<\xi} F_\zeta$ for limit ordinals $\xi>0$.

Since $(X, \tilde{\mathcal{T}})$ is scattered, there is an ordinal κ such that $F_\kappa=\emptyset$, so $X=\bigcup_{\xi\leqslant\kappa} G_\xi$.

Clearly we can assume that $\mathcal{C}$ is closed under finite intersections. Let $\mathcal{C}=\{C_n\colon n=1,2,\dots\}$ and define $G_{\xi,n}$, for $\xi\leqslant\kappa$ and $n=1,2,\dots$, by:

(a) $G_{\xi+1,n}=\{x\in F_\xi\colon \text{there exists a } \mathcal{T}\text{-open set } V \text{ such that } V\cap C_n\cap F_\xi=\{x\}\}$; and

(b) $G_{\xi,n}=\emptyset$ if ξ is limit.

Then we have $G_\xi=\bigcup_{n=1}^{\infty} G_{\xi,n}$ and $G_{\xi+1,n}\subset C_n\cap F_\xi$. Finally, we set $X_n=\bigcup_{\xi\leqslant\kappa} G_{\xi,n}$ for $n=1,2,\dots$ and we observe that $X_n\subset C_n$ and $X=\bigcup_{n=1}^{\infty} X_n$.

We claim that if $x\in G_{\xi,n}$ then x is $\mathcal{T}$-isolated in $\bigcup_{\rho\geqslant\xi} G_{\rho,n}$. If this holds then each X_n is $\mathcal{T}$-scattered (because $X_n=\bigcup_{\xi\leqslant\kappa} G_{\xi,n}$) and so $(X, \mathcal{T})$ is σ-scattered. To prove the claim, assume that $x\in G_{\xi,n}$, so that $\xi=\zeta+1$ and there is a $\mathcal{T}$-open set V such that $V\cap C_n\cap F_\zeta=\{x\}$. Since $x\in X_n\subset C_n$, we have $V\cap X_n\cap F_\zeta=\{x\}$, that is, x is $\mathcal{T}$-isolated in $X_n\cap F_\zeta$. Since $X_n\cap F_\zeta\supset\bigcup_{\rho\geqslant\xi} G_{\rho,n}$, x is $\mathcal{T}$-isolated in $\bigcup_{\rho\geqslant\xi} G_{\rho,n}$. □

Lemma 3.4. *If $R \subset X \times \mathbb{N}^{\mathbb{N}}$ is σ-scattered, then $\pi_1(R)$ is σ-scattered.*

Proof. Clearly, we can assume that R is scattered. Moreover, since for every $x \in X$,

$$R_x = \{\alpha \in \mathbb{N}^{\mathbb{N}}: (x, \alpha) \in R\}$$

is countable, as a scattered subset of $\mathbb{N}^{\mathbb{N}}$, we can also assume that each R_x is at most a singleton. Thus, there is a function $f: \pi_1(R) \to \mathbb{N}^{\mathbb{N}}$ such that $R = \mathrm{Gr}(f)$, the graph of f.

Now let $\mathscr{B}$ be a countable base for the topology of $\mathbb{N}^{\mathbb{N}}$ and set $\mathscr{C} = \{f^{-1}(B): B \in \mathscr{B}\}$. If $\mathscr{T}$ denotes the relative topology of $\pi_1(R)$ and $\tilde{\mathscr{T}}$ the topology on $\pi_1(R)$ generated by $\mathscr{T}$ and $\mathscr{C}$, then $(\pi_1(R), \tilde{\mathscr{T}})$ is homeomorphic to R and so is scattered. It now follows from Lemma 3.3 that $\pi_1(R)$ is σ-scattered. □

For the next lemma we shall use the following notation. Given a family $\mathscr{E}$ of subsets of a space Y we denote by $K(\mathscr{E}, Y)$ the largest subset Z of Y with the property that no nonempty open set in Z is contained in any member of $\mathscr{E}$; $K(\mathscr{E}, Y)$ is closed in Y and is called the 'non-locally $\mathscr{E}$ kernel of Y' (see [18, Theorem 1] for the existence of this kernel).

If $f: Y \to X$ is a continuous function and μ is a Borel measure on Y, we denote by $f(\mu)$ the image measure on X defined by $f(\mu)(B) = \mu(f^{-1}(B))$ for all Borel sets B in X.

Lemma 3.5. *Let Y be a Čech-complete space, $f: Y \to X$ a continuous function and set $\mathscr{E} = \{f^{-1}(\{x\}): x \in X\}$. Then the following are equivalent:*

(i) $K(\mathscr{E}, Y) \neq \emptyset$;

(ii) *there exists a compact perfect set $K \subset Y$ and a Radon probability measure μ on K such that $S(\mu) = K$ and $\mu(E \cap K) = 0$ for all $E \in \mathscr{E}$.*

Proof. (ii)⇒(i). It is obvious that $K \subset K(\mathscr{E}, Y)$, so $K(\mathscr{E}, Y) \neq \emptyset$.

(i)⇒(ii). Without loss of generality we assume that $K(\mathscr{E}, Y) = Y$. (Otherwise, consider the restriction of f to $K(\mathscr{E}, Y)$ which is Čech-complete as a closed subset of Y.)

For every $\varepsilon > 0$ we set

$$C_\varepsilon = \{\mu \in M_t^+(Y): \mu(E) \geq \varepsilon \text{ for some } E \in \mathscr{E}\}.$$

We show that each C_ε is closed and has empty interior in $M_t^+(Y)$.

Let $\{\mu_\alpha\}_{\alpha \in \mathscr{A}}$ be a net in C_ε with $\mu_\alpha \to \mu \in M_t^+(Y)$. For every $\alpha \in \mathscr{A}$, choose $E_\alpha \in \mathscr{E}$ with $\mu_\alpha(E_\alpha) \geq \varepsilon$ and set

$$F = \bigcap\nolimits_{\alpha \in \mathscr{A}} \overline{\bigcup_{\beta \geq \alpha} E_\beta}.$$

If $\alpha \geq \gamma$, then

$$\mu_\alpha\left(\overline{\bigcup_{\beta \geq \gamma} E_\beta}\right) \geq \mu_\alpha(E_\alpha) \geq \varepsilon.$$

So, by [21, Part II, Theorem 2],

$$\mu\left(\overline{\bigcup_{\beta\geq\gamma} E_\beta}\right)\geq \lim_\alpha \sup \mu_\alpha\left(\overline{\bigcup_{\beta\geq\gamma} E_\beta}\right)\geq\varepsilon$$

for every γ. By the τ-additivity of μ, it follows that $\mu(F)\geq\varepsilon$. So there exists some $x\in f(F)$. Also, by the definition of $\mathscr{E}$, each $f(E_\alpha)$ is a singleton, say $f(E_\alpha)=\{x_\alpha\}$. We claim that there is a subnet $\{x_\beta\}_{\beta\in\mathscr{B}}$ of $\{x_\alpha\}_{\alpha\in\mathscr{A}}$ converging to x. Indeed, if V is an open neighborhood of x then $f^{-1}(V)\cap F\neq\emptyset$ and $f^{-1}(V)$ is open. Thus, for every $\alpha\in\mathscr{A}$ there exists $\beta\geq\alpha$ such that $f^{-1}(V)\cap E_\beta\neq\emptyset$, that is, $x_\beta\in V$. This proves the claim.

Now, let K be a closed neighborhood of x. We have that $x_\beta\in K$ for large enough β and so $f(\mu_\beta)(K)\geq f(\mu_\beta)(\{x_\beta\})=\mu_\beta(E_\beta)\geq\varepsilon$. Since the net $\{f(\mu_\beta)\}_{\beta\in\mathscr{B}}$ converges to $f(\mu)$, $f(\mu)(K)\geq\lim_\beta\sup f(\mu_\beta)(K)\geq\varepsilon$. By the regularity of $f(\mu)$, it follows that $f(\mu)(\{x\})\geq\varepsilon$. Thus, $\mu(f^{-1}(\{x\}))\geq\varepsilon$ and we have shown that C_ε is closed in $M_t^+(Y)$.

We now show that the interior of C_ε is empty. Let $\mu=\sum_{i=1}^n c_i\delta_{y_i}$, where δ_{y_i} denotes the Dirac measure at $y_i\in Y$, $c_i\geq 0$ and $n\in\mathbb{N}$. Since the set of all measures of this form is dense in $M_t^+(X)$ (see [21, Part II, Theorem 10]) it suffices to show that μ is in the closure of $M_t^+(Y)\setminus C_\varepsilon$. If U is a nonempty open set in Y, then U meets uncountably many members of $\mathscr{E}$, because $K(\mathscr{E}, Y)=Y$, as a Čech-complete space, is a Baire space (see [3, Theorem 3.9.3]). Thus, for every open neighborhood U_i of y_i, we can distribute the mass c_i among points of U_i belonging to distinct elements of $\mathscr{E}$ such that no point has mass $>\varepsilon/2n$. Let μ_{U_i} be the resulting measure. It is easy to see that $\mu_{U_i}\to c_i\delta_{y_i}$, for $i=1,\dots,n$, where the family of the neighborhoods U_i of y_i is directed in the obvious way. Therefore, μ is the limit of a net of measures in $M_t^+(Y)\setminus C_\varepsilon$ and we have shown that C_ε has empty interior.

By the above, the set $\{0\}\cup\bigcup_{n=1}^\infty C_{1/n}$ is of the first category in $M_t^+(Y)$. Since Y is Čech-complete, so is $M_t^+(Y)$ (cf. [21, Part II, Theorem 17]) and by the Baire Category Theorem for $M_t^+(Y)$ there exists a nonzero measure $\nu\in M_t^+(Y)\setminus\bigcup_{n=1}^\infty C_{1/n}$. Then $\nu(E)=0$ for all $E\in\mathscr{E}$. Choose a compact set K in Y such that $\nu(K)>0$ and K is the support of the restriction $\nu_{|K}$ of ν to the Borel sets in K. It is clear that K and $\mu=(1/\nu(K))\nu_{|K}$ have the desired properties. □

Proof of Theorem 3.1. By Lemma 3.2, (i) and (ii) cannot hold simultaneously. So we prove only that ¬(i) implies (ii).

Let R be a Čech-complete subspace of $X\times\mathbb{N}^\mathbb{N}$ such that $X=\pi_1(R)$. We set $f=\pi_{1|R}:R\to X$, $\mathscr{E}=\{f^{-1}(\{x\}): x\in X\}$ and choose a subset U of $X\times\mathbb{N}^\mathbb{N}$ containing exactly one point from each member of $\mathscr{E}$. As $\pi_1(U)=X$ and X is not σ-scattered, it follows from Lemma 3.4 that U is not scattered. Thus, $K(\mathscr{E}, R)\neq\emptyset$ and by Lemma 3.5 there exists a compact set $K\subset R$ and a probability Radon measure μ on K such that $S(\mu)=K$ and $\mu(K\cap E)=0$ for all $E\in\mathscr{E}$. Setting $L=\pi_1(K)$ and $\nu=(\pi_{1|K})(\mu)$, we have that ν is a probability Radon measure on L, vanishing on singletons, such that $S(\nu)=L$. Thus, L is a compact perfect subset of X. □

4. Prohorov-analytic spaces

We shall prove that the analogue of Theorem 3.1 holds for first countable Prohorov-analytic spaces. The proof is similar except that Lemma 3.5 is replaced by the next lemma which is based on an idea using Preiss' Theorem and contained in [10].

Given a family $\mathscr{E}$ of subsets of a space X, we say that a set $Z \subset X$ is $\mathscr{E}$-discrete if for every $x \in Z$ there exists $E \in \mathscr{E}$ such that $\{x\} = E \cap Z$.

Lemma 4.1. *Let X be a first countable Prohorov space and $\mathscr{E}$ a partition of X into D-sets. Then the following are equivalent*:

(i) $K(\mathscr{E}, X) \neq \emptyset$;

(ii) *there exists an $\mathscr{E}$-discrete set Q such that Q is homeomorphic to the rational numbers and* $\mathrm{cl}_X(Q)$ *is a compact perfect set.*

Proof. (ii)$\Rightarrow$(i) is trivial because $Q \subset K(\mathscr{E}, X)$.

(i)$\Rightarrow$(ii). Let $\{V_x^n, n = 1, 2, \ldots\}$ be a decreasing neighborhood base for each $x \in X$. Without loss of generality we assume that $K(\mathscr{E}, X) = X$. (Otherwise, consider $Z = K(\mathscr{E}, X)$ and $\mathscr{K} = \{Z \cap E \colon E \in \mathscr{E}\}$ which is a partition of Z to D-sets such that $K(\mathscr{K}, Z) = Z$.) If $E \in \mathscr{E}$, then $\mathrm{int}(E) = \emptyset$ and $E = F \cap G$, where F is closed and G is open in X. Therefore, $\mathrm{int}(F) \cap G = \emptyset$ and so $F \cap G \subset F \backslash \mathrm{int}(F)$. This implies that every $E \in \mathscr{E}$, and therefore every finite union of members of $\mathscr{E}$, is nowhere dense. Using this fact we can easily construct by induction points $x(s) \in X$ for every $s \in \mathbb{N}^{(\mathbb{N})}$ such that the set P of all $x(s)$ is $\mathscr{E}$-discrete and $x(s, n) \in V_{x(s)}^n \backslash \{x(s)\}$, where $(s, n) = (s_1, s_2, \ldots, s_m, n)$ if $s = (s_1, s_2, \ldots, s_m)$ and $n \in \mathbb{N}$. It is clear that P is countable dense in itself and metrizable, so by a well-known theorem of Sierpinski (see [12, p. 287]) P is homeomorphic to the rationals.

By Preiss' Theorem [15], P is not Prohorov and so there is a compact set H of probability measures in $M_t^+(P)$ such that H is not uniformly tight. Fix an $\varepsilon > 0$ such that for every compact set $K \subset P$ there is $\mu \in H$ with $\mu(K) \leq 1 - \varepsilon$. Now we consider H as a compact set of Radon measures on the Prohorov space X. Thus we find a compact set $L \subset X$ such that $\mu(P \cap L) > 1 - \varepsilon/2$ for all $\mu \in H$.

To complete the proof it is enough to show that $P \cap L$ contains a copy Q of the rationals. Suppose that this is false. Then $P \cap L$ is a countable scattered metrizable space and so a Polish space. Since the set $\{\mu|_{P \cap L} \colon \mu \in H\}$ is relatively compact in $M_t^+(P \cap L)$ (see [20, Lemma 5.1]), by the Prohorov property of $P \cap L$, there exists a compact set $K \subset P \cap L$ such that $\mu(P \cap L \backslash K) < \varepsilon/2$ for all $\mu \in H$. But then $\mu(K) > 1 - \varepsilon$ for all $\mu \in H$, contradicting the choice of ε and completing the proof. □

Theorem 4.2. *For a first countable Prohorov-analytic space X, exactly one of the following alternatives holds: either* (i) *X is σ-scattered, or* (ii) *X contains a compact perfect set.*

Proof. As in Theorem 3.1 we prove only that $\neg$(i) implies (ii). Let R be a Prohorov subspace of $X \times \mathbb{N}^{\mathbb{N}}$ such that $X = \pi_1(R)$ and set $\mathscr{E} = \{R_x: x \in X\}$. As X is not σ-scattered, by the arguments used in Theorem 3.1, we see that $K(\mathscr{E}, R) \neq \emptyset$. Applying Lemma 4.1, we find an $\mathscr{E}$-discrete set Q homeomorphic to the rationals such that $\mathrm{cl}_R(Q)$ is compact and perfect. Setting $L = \pi_1(\mathrm{cl}_R(Q))$, we have that $\pi_1(Q)$ is dense in L and dense in itself. Thus L is a compact perfect subset of X. □

We shall now give some other consequences of Lemma 4.1.

Corollary 4.3. (a) *Every first countable Prohorov space is either scattered or contains a compact perfect set.* (b) *Every σ-scattered first countable Prohorov space is scattered.*

Proof. (a) follows directly from Lemma 4.1 when $\mathscr{E}$ is the partition of the space into singletons and (b) follows from (a) and Lemma 3.2. □

Of course Corollary 4.3 remains valid if we replace 'first countable Prohorov' by 'Čech-complete' (we use Lemma 3.5 instead of Lemma 4.1). In this case part (a) is essentially [8, Theorem 3(i)].

A useful observation about Lemma 4.1 is that the compact set $K = \mathrm{cl}_X(Q)$ in (ii) meets uncountably many members of $\mathscr{E}$. This is because $K \cap E$ is nowhere dense for every $E \in \mathscr{E}$ and K is a Baire space. Using this fact we prove the next theorem which was first proved by Preiss [15, Theorem 5] for metric Prohorov spaces.

Theorem 4.4. *Every first countable Prohorov space is a Baire space.*

Proof. Let X be a first countable Prohorov space and assume that X is of the first category in itself, that is, $X = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed nowhere dense. Set $E_n = F_n \setminus \bigcup_{i<n} F_i$, $n = 1, 2, \ldots$, and $\mathscr{E} = \{E_n: n = 1, 2, \ldots\}$. Then $\mathscr{E}$ is a partition of X to D-sets with empty interior, so $K(\mathscr{E}, X) = X$. Using Lemma 4.1, we find a compact set meeting uncountably many members of $\mathscr{E}$. Since $\mathscr{E}$ is countable, this is a contradiction and we have shown that X is of the second category in itself. As the same argument applies for every nonempty open set in X, we conclude that X is a Baire space. □

I do not know whether the above theorem can be generalized to the space M_t^+; that is, if X is first countable and Prohorov, must $M_t^+(X)$ be a Baire space? If the answer is 'yes', then we can prove that Lemma 3.5 holds for first countable Prohorov spaces Y using the same method.

In view of Theorem 4.4 we have the following extension of Lemma 4.1.

Corollary 4.5. *Lemma* 4.1 *remains valid when $\mathscr{E}$ is a partition of X to D_σ-sets.*

Proof. For the essential direction (i)$\Rightarrow$(ii), we assume as in Lemma 4.1 that $K(\mathscr{E}, X) = X$. Then each member of $\mathscr{E}$ is a D_σ-set with empty interior, so a set of the first category in X. But X is by Theorem 4.4 a Baire space and so every countable union of members of $\mathscr{C}$ has empty interior. Now we construct the set P as in Lemma 4.1 and proceed in the same way. □

Remarks. (1) Since every complete metric space is Čech-complete, either of Theorems 3.1 and 4.2 applies when X is an absolutely analytic metric space (i.e. homeomorphic to a Suslin subspace of a complete metric space). But every metrizable (or, more generally, paracompact and perfectly normal) σ-scattered space is σ-discrete (cf. [18, Theorem 4′]). Thus, we obtain the results of [2] and [18] mentioned in the introduction.

(2) Let S be the Sorgenfrey line, i.e. the real line with the topology of the right half-open intervals. S is a first countable, dense in itself, regular space and every compact subset of S is countable. Thus, by Corollary 4.3(a), S is not Prohorov; this solves [22, Problem 12.15], first raised by Mosiman and Wheeler in [13].

Moreover, because S is hereditarily Lindelöf and so paracompact and perfectly normal, it follows from (1) that S is not σ-scattered. (This can also be deduced from Lemma 5.1 below, because S is a dense in itself Baire space.) Therefore, by Theorem 4.2, S is not even Prohorov-analytic.

We conclude this section by showing that none of the above results holds if 'first countable' is dropped.

Haydon [6, Theorem 2.4 and p. 9] proved that every subspace Z of $\beta\mathbb{N}$ (the Stone–Čech compactification of $\mathbb{N}$) whose compact sets are finite is Prohorov. Moreover, he proved that there exists such a space Z admitting a τ-additive non-Radon measure [6, Example 2.5]. In particular, there exists a τ-additive non-zero measure on Z vanishing on singletons and, as in the proof of Lemma 3.2, we conclude that Z is not σ-scattered. Thus, Z is a counterexample for Theorem 4.2.

Next we give a common counterexample for Lemma 4.1, Theorem 4.4 and Corollaries 4.3 and 4.5. Another common feature of these results is that each of them implies Preiss' Theorem that the rationals are not Prohorov.

Example 4.6. There exists a finer topology $\mathscr{T}$ on the space Q of rational numbers such that $(Q, \mathscr{T})$ is a regular, dense in itself, Prohorov space whose compact sets are finite. (In particular, $(Q, \mathscr{T})$ is σ-scattered, but not a Baire space.)

Let $f: \beta\mathbb{N} \to [0, 1]$ be a continuous surjection and let K be the intersection of all compact subsets F of $\beta\mathbb{N}$ such that $f(F) = [0, 1]$. Choose a countable dense subset D of $[0, 1]$ and a subset E of K such that $f|_E: E \to D$ is one-to-one and onto. Because D is homeomorphic to the rationals, it suffices to show that E is a dense in itself Prohorov space whose compact sets are finite.

It is clear that K is dense in itself and that E is dense in K. So, E is dense in itself. Because E is countable and $\beta\mathbb{N}$ does not contain nontrivial convergent sequences, every compact subset of E is finite. Thus, by Haydon's result mentioned above, E is Prohorov.

5. Scattered and σ-scattered spaces

In this section we characterize those first countable spaces that are scattered (resp. σ-scattered) in terms of the Prohorov (resp. Prohorov-analytic) property (Theorems 5.6 and 5.7). First we shall prove some lemmas, beginning with the following extension of Lemma 3.2.

Lemma 5.1. *If X is a σ-scattered space, then X does not contain any dense in itself nonempty Baire subspace.*

Proof. Suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where the X_n's are scattered and pairwise disjoint, and that X contains a nonempty dense in itself Baire space Y. Define $f: Y \to \mathbb{R}$ with $f|_{Y \cap X_n} = n$ (if $Y \cap X_n \neq \emptyset$) for all n. It is clear that the graph $\mathrm{Gr}(f)$ of f is scattered. Since Y is a Baire space, by a special form of Blumberg's Theorem (see [23, Corollary 1.5]), there is a dense subset Z of Y such that $f|_Z$ is continuous. But then $\mathrm{Gr}(f|_Z)$ is a dense in itself subset of $\mathrm{Gr}(f)$, a contradiction. □

The next lemma is proved in [9, Theorem 4.15] and in [11, Theorem 2.3]; it is a special case of a result of Graf [5, Theorem 5]. We refer to [11, Section 2] and [7] for the definition of a strong lifting ρ for a Radon measure on a space Y and the fact that the sets $\rho(A)$, where A is Borel in Y, form a base for a topology $\mathcal{T}_\rho$ on Y.

Lemma 5.2. *Let X and Y be compact spaces and ν a Radon measure on Y admitting a strong lifting ρ. Then every continuous surjection $f: X \to Y$ has a $\mathcal{T}_\rho$-continuous right inverse.*

Using this result, we prove the next lemma on the existence of spaces for which none of the alternatives of Theorems 3.1 and 4.2 holds.

Lemma 5.3. *Every non-σ-scattered space X contains a subset which is neither σ-scattered nor contains any compact perfect set.*

Proof. Of course we can assume that X contains a compact perfect set Y. By [16, Proposition 5.4.1], there is a continuous surjection $f: Y \to [0, 1]$. Let λ denote the Lebesgue measure on $[0, 1]$ and let ρ be a strong lifting for λ. By Lemma 5.2, there is a $\mathcal{T}_\rho$-continuous function $g: [0, 1] \to Y$ such that $f \circ g$ is the identity function of $[0, 1]$.

Now λ is a τ-additive measure on $([0, 1], \mathcal{T}_\rho)$ (see [1, Proposition 3] or use the arguments of the proof of [11, Theorem 2.2]) and so the measure $\mu = g(\lambda)$ is τ-additive on Y. It is clear that $\mu^*(g[0, 1])) = 1$ and therefore μ^* induces a probability τ-additive measure on the subspace $Z = g([0, 1])$. By the arguments of Lemma 3.2, Z is not σ-scattered.

We choose a Bernstein set B in $[0, 1]$, that is, a subset B of $[0, 1]$ such that neither B nor $[0, 1]\backslash B$ contains any compact perfect set (see [12, pp. 514–516]) and set $Z_1 = Z \cap f^{-1}(B)$ and $Z_2 = Z\backslash f^{-1}(B)$. Since $Z = Z_1 \cup Z_2$, some Z_i ($i = 1$ or 2) is not σ-scattered. Since $f|_{Z_i}$ is continuous and one-to-one, it follows from the choice of B that Z_i does not contain any compact perfect set. Therefore Z_i is the desired subset. □

Lemma 5.4. *Let X be a space such that for some $x \in X$, $X\backslash\{x\}$ is Prohorov and x has a countable neighborhood base. Then X is Prohorov.*

The proof follows from the arguments used in [13, Lemma 5.11] and is omitted.

Lemma 5.5. *Every first countable scattered space is Prohorov.*

Proof. Let F_ξ and G_ξ be defined as in Lemma 3.3 for some first countable scattered space X. Then $X = \bigcup_{\xi<\kappa} G_\xi$ for some ordinal κ. We prove by induction that $\bigcup_{\xi<\rho} G_\xi$ is Prohorov for every $\rho \leq \kappa$. So we assume that $\bigcup_{\xi<\zeta} G_\xi$ is Prohorov for every $\zeta < \rho$ and show that $\bigcup_{\xi<\rho} G_\xi$ is locally Prohorov (hence Prohorov by [13, Theorem 4.7]). If ρ is limit, this is trivial because $\bigcup_{\xi<\zeta} G_\xi$ is open for every $\zeta < \rho$. If $\rho = \zeta + 1$, then

$$\bigcup_{\xi<\rho} G_\xi = \bigcup_{\xi<\zeta} G_\xi \cup G_\zeta = \bigcup_{x\in G_\zeta} \left(\bigcup_{\xi<\zeta} G_\xi \cup \{x\}\right),$$

where each $\bigcup_{\xi<\zeta} G_\xi \cup \{x\}$ is Prohorov (by Lemma 5.4) and open in $\bigcup_{\xi<\rho} G_\xi$ because x is isolated in G_ζ. Therefore, $\bigcup_{\xi<\rho} G_\xi$ is locally Prohorov. □

Remark. By an example of Varadarajan [21, p. 225] the assumption of first countability in Lemmas 5.4 and 5.5 cannot be dropped.

We are now ready to prove the next two theorems. For metrizable spaces X some other equivalent conditions can be added to these theorems (see [18, Theorem 11] and [19, Theorem 2]).

Theorem 5.6. *For a first countable space X the following are equivalent*:

(i) *X is σ-scattered;*
(ii) *X is Prohorov-analytic and every Baire subspace of X has an isolated point;*
(iii) *X is Prohorov-analytic and every compact subset of X is countable;*
(iv) *X is hereditarily Prohorov-analytic.*

Proof. (i)⇒(iv). As in the proof of Lemma 5.1, there is $f : X \to \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{Gr}(f)$ is scattered. Now for every $Z \subset X$, $Z = \pi_1(\mathrm{Gr}(f|_Z))$ and $\mathrm{Gr}(f|_Z)$ is Prohorov by Lemma 5.5, so Z is Prohorov-analytic.

(iv)⇒(i). Suppose that X is not σ-scattered. Then, by Lemma 5.3, there is $Z \subset X$ such that Z is neither σ-scattered nor contains any compact perfect set. So, by Theorem 4.2, Z is not Prohorov-analytic.

(i)⇒(ii). Since (i)⇒(iv), X is Prohorov-analytic. The rest follows from Lemma 5.1.

(ii)⇒(iii) follows from the fact that every uncountable compact first countable space contains a compact perfect set (cf. [16, Theorem 3.5.1 and Proposition 5.4.1]).

(iii)⇒(i) follows from Theorem 4.2. □

Theorem 5.7. *For a first countable space X the following are equivalent:*

(i) *X is scattered;*
(ii) *X is Prohorov and every Baire subspace of X has an isolated point;*
(iii) *X is Prohorov and every compact subset of X is countable;*
(iv) *X is hereditarily Prohorov;*
(v) *X is σ-scattered and every closed subset of X is a Baire space;*
(vi) *X is σ-scattered and Prohorov.*

Proof. (i)⇒(iv) follows from Lemma 5.5 and (iv)⇒(i) from the fact that a non-scattered first countable space contains a copy of the rationals. The proof of the equivalence of (i)–(iii) is similar to the proof of the corresponding assertions of Theorem 5.6 (with Corollary 4.3(a) replacing Theorem 4.2). Finally, (i)⇒(vi)⇒(v)⇒(i) follow from Lemma 5.5, Theorem 4.4 and Lemma 5.1, respectively. □

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