Simple Criterion for Asymptotic Stability of Interval Neutral Delay-Differential Systems

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Abstract—In this paper, the asymptotic stability of interval neutral delay-differential systems is investigated. A delay-independent criterion for the stability of the system is derived in terms of the spectral radius. Numerical computations are performed to illustrate the result. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Over the decades, the stability analysis of various neutral delay-differential systems has received considerable attention [1-3]. The theory of neutral delay-differential systems is of both theoretical and practical interest. For a large class of electrical networks containing lossless transmission lines, the describing equations can be reduced to neutral delay-differential equations; such networks arise in high speed computers where nearly lossless transmission lines are used to interconnect switching circuits. Also, the neutral systems often appear in the study of automatic control, population dynamics, and vibrating masses attached to an elastic bar, etc. In the literature, various analysis techniques utilizing Lyapunov technique, characteristic equation method, or state solution approach have been used to derive stability criteria for asymptotic stability of the systems. The developed stability criteria are often classified into two categories according to their dependence on the size of delays. Several delay-independent sufficient conditions for the asymptotic stability of the systems are presented by some researchers [4-8]. Also, a few delay-dependent sufficient conditions have been exploited in [9-11].

In this paper, we are interested in the following linear systems of neutral type described by

\[ \dot{x}(t) = A_I x(t) + B_I x(t - h) + C_I \dot{x}(t - h), \]
\[ x(t) = \phi(t), \quad t \in [-h, 0], \]

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where \( x(t) \in \mathbb{R}^n \) is the state vector, \( A_t, B_t, \) and \( C_t \in \mathbb{R}^{n \times n} \) are matrices whose elements vary in prescribed ranges, e.g., \( A_t, B_t, \) and \( C_t \) are such that

\[
\begin{align*}
A_t &= \{(a_{ij}^t) : a_{ij}^t \leq a_{ij}^t \leq a_{ij}^t, \; i, j = 1, 2, \ldots, n\}, \\
B_t &= \{(b_{ij}^t) : b_{ij}^t \leq b_{ij}^t \leq b_{ij}^t, \; i, j = 1, 2, \ldots, n\}, \\
C_t &= \{(c_{ij}^t) : c_{ij}^t \leq c_{ij}^t \leq c_{ij}^t, \; i, j = 1, 2, \ldots, n\},
\end{align*}
\]  

(2)

\( h \) is the positive time-delay, and \( \psi(.) \) is the given continuously differentiable function on \([-h, 0]\).

It is well known that interval matrices, which are caused by unavoidable system parametric variations, changes in operating conditions, aging, etc., are real matrices in which all elements are known only to belong to a specified closed interval. In the past, a number of reports have been proposed for the stability analysis of interval systems [12-14]. However, up to now, the stability of interval neutral differential systems has not yet received great interest from the researchers. The goal of this paper is to deal with the stability analysis of the system.

In this paper, we present a delay-independent criterion for asymptotic stability of the system given in (1). The derived sufficient condition is expressed in terms of the spectral radius of the matrix which is the combination of the modulus matrices.

2. PRELIMINARIES

Before we develop our main result, we state some notations and a lemma. Let \( \rho[R] \) denote the largest modulus of the eigenvalues of the matrix \( R \), which is known as the spectral radius of \( R \). \( |R| \) denotes a matrix formed by taking the absolute value of every element of \( R \), and it is called the modulus matrix of \( R \). \( I \) denotes the identity matrix of appropriate order. The relation \( R \leq T \) represents that all the elements of matrices, \( R \) and \( T \), satisfy \( r_{ij} \leq t_{ij} \) for all \( i \) and \( j \).

\( \|R\| \) denotes the matrix norm of \( R \).

Also, the following lemma is used for the main result.

**Lemma 2.1.** Consider any \( n \times n \) matrices \( R, T, \) and \( V \).

**Part I.** (See [15].) If \( |R| \leq V \), then

(a) \( |RT| \leq |R||T| \leq V|T| \),

(b) \( |R + T| \leq |R| + |T| \leq V + |T| \),

(c) \( \rho[R] \leq \rho[|R|] \leq \rho[V] \),

(d) \( \rho[RT] \leq \rho[|R||T|] \leq \rho[V|T|] \), and

(e) \( \rho[R + T] \leq \rho[|R + T|] \leq \rho[|R| + |T|] \leq \rho[V + |T|] \).

**Part II.** (See [16].) If \( \rho[R] < 1 \), then \( \det(I + R) \neq 0 \).

**Part III.** (See [17].) If \( \|R\| < 1 \), then \( (I - R)^{-1} \) exists, \( (I - R)^{-1} = I + R + R^2 + \cdots \), and

\[ \|(I - R)^{-1}\| \leq 1/(1 - \|R\|) \.
\]

3. MAIN RESULT

In this section, we derive a sufficient condition for the asymptotic stability of the system given in (1).

Denote

\[
A_1 = (a_{ij}^1), \quad A_2 = (a_{ij}^2), \quad B_1 = (b_{ij}^1), \quad B_2 = (b_{ij}^2), \quad C_1 = (c_{ij}^1), \quad C_2 = (c_{ij}^2)
\]

(3)

and let

\[
A = (a_{ij}) = \frac{1}{2} (a_{ij}^1 + a_{ij}^2) = \frac{1}{2} (A_1 + A_2),
\]

\[
B = (b_{ij}) = \frac{1}{2} (b_{ij}^1 + b_{ij}^2) = \frac{1}{2} (B_1 + B_2),
\]

\[
C = (c_{ij}) = \frac{1}{2} (c_{ij}^1 + c_{ij}^2) = \frac{1}{2} (C_1 + C_2),
\]

(4)
where \(A, B,\) and \(C\) are the average matrices between \(A_1\) and \(A_2, B_1\) and \(B_2,\) and \(C_1\) and \(C_2,\) respectively. In this paper, it is assumed that \(A\) is a Hurwitz. Furthermore, we have

\[
\Delta A = (a_{ij} - a_{ij}) = A_1 - A, \quad \Delta B = (b_{ij} - b_{ij}) = B_1 - B, \quad \Delta C = (c_{ij} - c_{ij}) = C_1 - C, \tag{5}
\]

where \(\Delta A, \Delta B,\) and \(\Delta C\) are the bias matrices between \(A_1\) and \(A, B_1\) and \(B,\) and \(C_1\) and \(C,\) respectively. Also,

\[
A_M = (a_{ij}^2 - a_{ij}) = A_2 - A, \quad B_M = (b_{ij}^2 - b_{ij}) = B_2 - B, \quad C_M = (c_{ij}^2 - c_{ij}) = C_2 - C, \tag{6}
\]

where \(A_M, B_M,\) and \(C_M\) are the maximal bias matrices between \(A_2\) and \(A, B_2\) and \(B,\) and \(C_2\) and \(C,\) respectively. Note that

\[
|\Delta A| \leq A_M, \quad |\Delta B| \leq B_M, \quad |\Delta C| \leq C_M. \tag{7}
\]

Let \(F(s) = (sI - A)^{-1},\) and \(F_M\) be the matrix formed by taking the maximum magnitude of each element of \(F(s)\) for \(Rs \geq 0.\) Then, we have the following theorem.

**Theorem 3.1.** The interval neutral delay-differential system given in (1) is asymptotically stable, if the following inequalities are satisfied:

\[
\|C\| + \|C_M\| < 1 \tag{8}
\]

and

\[
\rho \left[ F_M \cdot \left\{ A_M + |B| + B_M + \frac{1}{1 - (\|C\| + \|C_M\|)} \cdot (|CA| + |C|A_M \right.ight.
\]

\[
\left. \left. + |CB| + |C|B_M + C_M (|A| + A_M + |B| + B_M) \right) \right] < 1. \tag{9}
\]

**Proof.** The characteristic equation of the system given in (1) is

\[
\lambda(s) = \det [sI - A_I - (B_I + C_I s) \exp(-hs)] = 0. \tag{10}
\]

Since \(\det[RT] = \det[R] \det[T]\) for any two \(n \times n\) matrices \(R\) and \(T,\) so we have

\[
\lambda(s) = \det[I - C_I \exp(-hs)] \det [sI - (I - C_I \exp(-hs))^{-1} \cdot (A_I + B_I \exp(-hs))]. \tag{11}
\]

In view of \(\|C\| + \|C_M\| < 1,\) the relation

\[
\|C_I\| = \|C + \Delta C\| \leq \|C\| + \|\Delta C\| \leq \|C\| + \|C_M\| < 1
\]

follows. Hence, the matrix \((I - C_I \exp(-hs))^{-1}\) exists and \(\det(I - C_I \exp(-hs)) \neq 0\) because \(\|C_I \exp(-hs)\| \leq \|C_I\| < 1\) for \(Rs \geq 0.\)

Therefore, if we can show that

\[
\det [sI - (I - C_I \exp(-hs))^{-1} \cdot (A_I + B_I \exp(-hs))] \neq 0, \quad \text{for } Rs \geq 0, \tag{12}
\]

then

\[
\lambda(s) \neq 0, \quad \text{for } Rs \geq 0. \tag{13}
\]

Here, equations (12) and (13) guarantee the asymptotic stability of system (1) [5, Theorem 1].
For simplicity, let us define $\xi = \exp(-hs)$ and $T = \xi C_I$. Then, using the inequality $(I - T)^{-1} = I + (I - T)^{-1}T$, the left-hand side of (12) becomes

$$\det[sI - (I - T)^{-1}(A_I + \xi B_I)] = \det[sI - (I + (I - T)^{-1}T)(A + \Delta A + \xi(B + \Delta B))]
$$

$$= \det[(sI - A) - \Delta A - \xi(B + \Delta B)
- (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)]
= \det[sI - A] \det[I - (sI - A)^{-1}\{\Delta A + \xi B + \xi \Delta B
+ (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)]
= \det[sI - A] \det[I - F(s)\{\Delta A + \xi B + \xi \Delta B
+ (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)]$$

Therefore, equation (12) can be rewritten as

$$\det[sI - A] \det[I - F(s)\{\Delta A + \xi B + \xi \Delta B + (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)] \neq 0. \quad (15)$$

Since $A$ is a Hurwitz matrix, $\det[sI - A] \neq 0$ for $\Re s \geq 0$. So, equation (15) is further simplified as

$$\det[I - F(s)\{\Delta A + \xi B + \xi \Delta B + (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)] \neq 0,
\text{ for } \Re s \geq 0. \quad (16)$$

Thus, if we can show that $\rho[F(s)\{\Delta A + \xi B + \xi \Delta B + (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)] < 1$ for $\Re s \geq 0$, equation (16) is satisfied by Part II of Lemma 2.1.

Here, note that using Part I of Lemma 2.1, for $\Re s \geq 0$, we have

$$\|T\| = \|\xi C_I\| \leq \|C_I\| < \|C\| + \|CM\|,
\|TA\| = \|\xi(C + \Delta C)A\| \leq \|\xi CA\| + \|\xi \Delta CA\| \leq |CA| + CM|A|. \quad (17)$$

Similarly,

$$|\Delta A| \leq |C|A_M + CMAM, \quad |\Delta B| < |CB| + CM|B|, \quad |\Delta T\Delta B| \leq |C|B_M + CMB_M. \quad (18)$$

Now, using Parts I and III of Lemma 2.1 and (17),(18), we obtain

$$\rho[F(s)\{\Delta A + \xi B + \xi \Delta B + (I - T)^{-1}(A + \Delta A + \xi B + \xi \Delta B)]
\leq \rho[F(s)\{(|\Delta A| + |\xi B| + |\xi \Delta B| + (I - T)^{-1}(TA + T\Delta A + \xi TB + \xi T\Delta B)]
\leq \rho[F(s)\{(|\Delta A| + |B| + |\Delta B| + (I - T)^{-1}[(TA + T\Delta A + \xi TB + \xi T\Delta B)]
\leq \rho[F_M\{A_M + |B| + B_M + \|T\|^{-1}\cdot(|\Delta A| + C_M|A|) + |C|A_M + CMAM
+ |CB| + CM|B| + C_MB_M\}]\]
\leq \rho[F_M\{A_M + |B| + B_M + \|C\| + CM|B| + C_MB_M\}] < 1. \quad (19)$$

Thus, this completes the proof.

REMARK 3.1. Since the matrix $A$ is a Hurwitz, the matrix $F_M$ always exists and it can obtained for some $s$ on imaginary axis by the maximum modulus principle.
COROLLARY 3.1. If $C_I = 0$, system (1) becomes a interval retarded delay differential system, i.e.,

$$\dot{x}(t) = A_1 x(t) + B_1 x(t - h).$$

Then, by Theorem 3.1, sufficient condition for asymptotic stability of system (20) is simplified as

$$\rho \left[ F_M(A_M + |B| + B_M) \right] < 1.$$  \hfill (21)

REMARK 3.2. The sufficient conditions for the stability of the interval retarded delay-differential system (20) by Tissir and Hmamed [14] are as follows:

$$\mu(A + zB) + \|A_M\| + \|B_M\| < 0,$$

$$\mu(A) + \mu(zB) + \|A_M\| + \|B_M\| < 0,$$

$$\mu(A) + \|B\| + \|A_M\| + \|B_M\| < 0,$$

where $z = e^{jw}$ with $w \in [0, 2\pi]$ and $j^2 = -1$.

To apply the above sufficient conditions, it needs the requirement that $\mu(A + zB) < 0$ or $\mu(A) < 0$ while Corollary 3.1 allows a more relaxed requirement that $A$ is a Hurwitz matrix. Furthermore, the stability criterion (21) of Corollary 3.1 is expressed in terms of the spectral radius of the matrices which is the combination of the modulus of the matrices. Therefore, there is a better possibility that the proposed criterion is less conservative than those in (22), which uses the matrix norms and matrix measure.

To demonstrate the applications of the result, we give the following examples.

EXAMPLE 3.1. Consider the interval neutral differential system described by (1) where

$$A_1 = \begin{bmatrix} -7 & 0.5 \\ 0 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & 1.5 \\ 0 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.4 & 0.2 \\ 0 & -0.3 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.2 & \nu \\ 0.2 & 0.3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

where $\nu$ is a parameter scalar for which we shall find the upper bound that guarantees the stability of the systems.

From (4) and (6), the average matrices are

$$A = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -0.3 & 0.5(\nu + 0.2) \\ 0.1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.2 \end{bmatrix},$$

and the matrices $A_M$, $B_M$, and $C_M$ are

$$A_M = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad B_M = \begin{bmatrix} 0.1 & 0.5\nu - 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \quad C_M = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The rational function matrices $F(s)$ and $F_M$ are computed as

$$F(s) = \begin{bmatrix} \frac{1}{(s + 6)} & \frac{1}{(s + 4)(s + 6)} \\ 0 & \frac{1}{s + 4} \end{bmatrix}, \quad F_M = \begin{bmatrix} \frac{1}{6} & \frac{1}{24} \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Then, by simple computation of inequality (8) and (9), the bound of $\nu$ for guaranteeing the asymptotic stability of the system is

$$0.2 < \nu < 1.4.$$
It is interesting to note that when $C_I = 0$, the bound of $\nu$ in [14] for the stability is $0.2 < \nu < 3.96$. However, by applying our criterion (21), the stability bound is $0.2 < \nu < 60.6$, which gives a less conservative result than in [14].

**EXAMPLE 3.2.** Consider the interval retarded delay-differential system described by (20) where

$$A_1 = \begin{bmatrix} -4 & -3 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & -0.1 \\ -0.1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}.$$  

From (4) and (6), we obtain

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_M = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}.$$  

Since $\mu(A + zB) = \mu(A) = 0.0811 > 0$, the criteria (22) in [14] are not applicable. Hence, one cannot state the stability from the result of Tissir and Hmamed [14]. However, the matrix $A$ is a Hurwitz, and therefore, Corollary 1 can be applied. Then, the rational function matrices $F(s)$ and $F_M$ are as follows:

$$F(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s & -2 \\ 1 & s + 3 \end{bmatrix}, \quad F_M = \begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 1 \end{bmatrix}.$$  

Then, by checking criterion (21) of Corollary 1, we obtain

$$\rho [F_M \cdot (A_M + |B| + B_M)] = 0.9833 < 1.$$  

This gives the asymptotic stability of the system.

**REFERENCES**


