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Ribbon tableaux, ribbon rigged configurations and Hall–Littlewood functions at roots of unity

Francois Descouens

Université de Marne-la-Vallée, France

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Abstract

Hall–Littlewood functions indexed by rectangular partitions, specialized at primitive roots of unity, can be expressed as plethysms. We propose a combinatorial proof of this formula using Schilling's bijection between ribbon tableaux and ribbon rigged configurations.

Keywords: Hall–Littlewood functions; Ribbon tableaux; Rigged configurations

1. Introduction

In [8,9], Lascoux, Leclerc and Thibon proved several formulas for Hall–Littlewood functions $Q'_\lambda(X;q)$ with the parameter q specialized at primitive roots of unity. These formulas imply a combinatorial interpretation of the plethysms $l_k^{(j)}[h_\lambda]$ and $l_k^{(j)}[e_\lambda]$, where h_λ and e_λ are respectively products of complete and elementary symmetric functions. The symmetric function $l_k^{(j)}$ is the Frobenius characteristic of the jth representation of the symmetric group $\mathfrak{S}_k$ induced by a cyclic subgroup of order k. More precisely, these representations are given, for all τ in $\mathfrak{S}_k$, by $\tau \mapsto \zeta^j$, where ζ is a kth primitive root of unity.

However, the combinatorial interpretation of the plethysms of Schur functions $l_k^{(j)}[s_\lambda]$ would be far more interesting. This question led the same authors to introduce a new basis $H_\lambda^{(k)}(X;q)$ of symmetric functions, depending on an integer $k \geqslant 1$ and a parameter q, which interpolate between Schur functions, for $k=1$, and Hall–Littlewood functions $Q'_\lambda(X;q)$, for $k \geqslant l(\lambda)$. These

E-mail address: francois.descouens@univ-mlv.fr.

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functions were conjectured to behave similarly under specialization at root of unity, and to provide a combinatorial expression of the expansion of the plethysm $l_k^{(j)}[s_\lambda]$ in the Schur basis for suitable values of the parameters. This conjecture has been proved only for the stable case $k = l(\lambda)$ and for $k = 2$. The stable case reduces to the result of Theorem 1 on Hall–Littlewood functions [8,9]. For the case $k = 2$, which gives the symmetric and antisymmetric squares $h_2[s_\lambda]$ and $e_2[s_\lambda]$, the proof given in [1] by Carré et Leclerc, relies upon the study of diagonal classes of domino tableaux, i.e. sets of domino tableaux having the same diagonals. They proved that the cospin polynomial of such a class has the form $(1+q)^a q^b$, and from this, they obtained the specialization $H^{(2)}_{\lambda\cup\lambda}(X; -1)$.

In [13], Schilling defines ribbon rigged configurations and gives a statistic preserving bijection between this kind of rigged configurations and ribbon tableaux corresponding to a product of row partitions. The aim of this note is to show that the result on Hall–Littlewood functions at roots of unity follows from an explicit formula for the cospin polynomials of certain diagonal classes of ribbon tableaux, which turn out to have a very simple characterization through Schilling's bijection.

2. Basic definitions

Consider $\lambda = (\lambda_1, \dots, \lambda_p)$ a partition (we use the French convention for Young diagrams). We denote by $l(\lambda)$ its length p, $|\lambda|$ its weight $\sum_i \lambda_i$, and λ' its conjugate partition. A k-ribbon is a connected skew diagram of weight k which does not contain a 2×2 square. The first (northwest) cell of a k-ribbon is called the *head* and the last one (southeast) the *tail*. By removing successively k-ribbons from λ such that, at each step, the remaining shape is still a partition, we obtain a partition $\lambda_{(k)}$ independent of the removing procedure. This partition is called the k-core of λ. The k-quotient is a sequence of k partitions derived from λ. The k-core and the k-quotient of a partition are defined in [11] (I.1 Example 8, p. 12). Denote by $\mathcal{P}$ the set of all partitions, $\mathcal{P}_{(k)}$ the set of all k-cores (i.e. all partitions from which we cannot remove any k-ribbon) and $\mathcal{P}^{(k)}$ the cartesian product $\mathcal{P} \times \cdots \times \mathcal{P}$ of length k. The quotient bijection Φ_k is defined by

$$\begin{aligned}\Phi_k : \mathcal{P} &\to \mathcal{P}_{(k)} \times \mathcal{P}^{(k)},\\ \lambda &\mapsto \big(\lambda_{(k)}, \lambda^{(k)}\big).\end{aligned} \tag{1}$$

Remark 1. For any partition λ, we denote by $l(\lambda)\lambda$ the partition $(l(\lambda)\lambda_1, \dots, l(\lambda)\lambda_{l(\lambda)})$. The $l(\lambda)$-core of $l(\lambda)\lambda$ is empty and the $l(\lambda)$-quotient of $l(\lambda)\lambda$ is the following product of single row partitions

$$\big(l(\lambda)\lambda\big)^{(l(\lambda))} = \big((\lambda_{l(\lambda)}), \dots, (\lambda_1)\big). \tag{2}$$

2.1. Preliminaries on k-ribbon tableaux

A k-ribbon tableau of shape λ and weight μ is a tiling of the skew diagram $\lambda/\lambda_{(k)}$ by labeled k-ribbons such that

(1) the head of a ribbon labeled i must not be on the right of a ribbon labeled $j > i$,
(2) the tail of a ribbon labeled i must not be on the top of a ribbon labeled $j \geqslant i$.

We denote by $\mathrm{Tab}^{(k)}_{\lambda,\mu}$ the set of all k-ribbon tableaux of shape λ and weight μ, and by $\mathrm{Tab}^{(k)}_{\lambda}$ the set of all k-ribbon tableaux of shape λ and evaluation any composition of $|\lambda|/k$.

The spin of a k-ribbon R is defined by

$$\mathrm{sp}(R) = \frac{\mathrm{h}(R) - 1}{2}, \tag{3}$$

where $\mathrm{h}(R)$ is the height of R. The spin of a k-ribbon tableau T of $\mathrm{Tab}^{(k)}_{\lambda,\mu}$ is the sum of the spins of all its k-ribbons. The cospin of T is the associated co-statistic on $\mathrm{Tab}^{(k)}_{\lambda,\mu}$, i.e.

$$\mathrm{cosp}(T) = \max\bigl(\mathrm{sp}(T'),\ T' \in \mathrm{Tab}^{(k)}_{\lambda,\mu}\bigr) - \mathrm{sp}(T). \tag{4}$$

We define the cospin polynomial $\tilde{G}^{(k)}_{\lambda,\mu}(q)$ as the generating polynomial of $\mathrm{Tab}^{(k)}_{\lambda,\mu}$ with respect to the cospin statistic

$$\tilde{G}^{(k)}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{Tab}^{(k)}_{\lambda,\mu}} q^{\mathrm{cosp}(T)}.$$

Example 1. The cospin polynomial for $\mathrm{Tab}^{(3)}_{(87651),(3321)}$ is

$$\tilde{G}^{(3)}_{(87651),(3321)}(q) = 3q^5 + 17q^4 + 33q^3 + 31q^2 + 18q + 5.$$

2.2. k-tuples of semi-standard Young tableaux

Let $T^\circ = (T^{\circ 1}, \ldots, T^{\circ k})$ be a k-tuple of semi-standard Young tableaux $T^{\circ i}$ of shape $\lambda^{\circ i}$ and evaluation $\mu^{\circ i}$. We call shape of T° the sequence of partitions $\lambda^\circ = (\lambda^{\circ 1}, \ldots, \lambda^{\circ k})$. The weight of T° is the composition $\mu = (\mu_1, \ldots, \mu_p)$, where $\mu_i = \sum_{j=1}^{k} \mu_i^{\circ j}$.

Definition 1. Let s be any cell in T°

- $\mathrm{pos}(s)$ is the integer such that the cell s is in the tableau $T^{\circ \mathrm{pos}(s)}$,
- $T^\circ(s)$ is the label of the cell s,
- $\mathrm{row}(s)$ (respectively $\mathrm{col}(s)$) represents the row (respectively the column) of s in $T^{\circ \mathrm{pos}(s)}$,
- $\mathrm{diag}(s)$ is the content of s, i.e. $\mathrm{diag}(s) = \mathrm{col}(s) - \mathrm{row}(s)$.

Denote by $\mathrm{Tab}_{\lambda^\circ,\mu}$ the set of all k-tuples of Young tableaux T° of shape $\lambda^\circ = (\lambda^{\circ 1}, \ldots, \lambda^{\circ k})$ and weight μ. In [15], Stanton and White extend the bijection Φ_k to a correspondence Ψ_k between the set of k-ribbon tableaux of shape λ and weight μ, and the set of k-tuples of semi-standard Young tableaux of shape $\lambda^{(k)}$ and weight μ.

Example 2. The bijection Ψ_3 sends the 3-ribbon tableau of Fig. 1 to the 3-tuple of tableaux

$$\left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}\,,\ \begin{array}{|c|c|c} \hline 2 & 3 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,,\ \begin{array}{|c|c|} \hline 1 & 4 \\ \hline \end{array} \right).$$

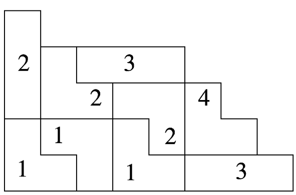


Fig. 1. A 3-ribbon tableau of shape (87651) and weight (3321).

Definition 2. (See [12].) Let $T^\circ = (T^{\circ 1}, \ldots, T^{\circ k})$ be a k-tuple of semi-standard Young tableaux and consider s, t two cells of T°. The couple (s, t) is an inversion in T° if the following conditions hold

(1) $\mathrm{diag}(s) = \mathrm{diag}(t)$ and $\mathrm{pos}(s) < \mathrm{pos}(t)$ or $\mathrm{diag}(s) = \mathrm{diag}(t) - 1$ and $\mathrm{pos}(s) > \mathrm{pos}(t)$,
(2) $\mathrm{row}(s) \leqslant \mathrm{row}(t)$,
(3) $T(t) < T(s) < T(t^\uparrow)$, where $t^\uparrow$ is the cell directly above t and $T(t^\uparrow) = \infty$ if $t^\uparrow \notin \lambda^\circ$.

The inversion statistic on T°, denoted by $\mathrm{Inv}(T^\circ)$, is the number of couples in T° which form an inversion in T°. This statistic permits to extend the correspondence Ψ_k to a bijection compatible with the inversion statistic and cospin [12], i.e.

$$\forall T \in \mathrm{Tab}^{(k)}_{\lambda,\mu}, \quad \mathrm{Inv}\big(\Psi_k(T)\big) = \mathrm{cosp}(T). \tag{5}$$

The inversion polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q)$ is the generating polynomial of $\mathrm{Tab}_{\lambda^\circ,\mu}$ with respect to the inversion statistic

$$\widetilde{I}_{\lambda^\circ,\mu}(q) = \sum_{T^\circ \in \mathrm{Tab}_{\lambda^\circ,\mu}} q^{\mathrm{Inv}(T^\circ)}. \tag{6}$$

Proposition 1. *(See [12].) The compatibility of the bijection Ψ_k with the inversion statistic and cospin implies the following property*

$$\widetilde{G}_{\lambda,\mu}(q) = \widetilde{I}_{\lambda^{(k)},\mu}(q). \tag{7}$$

In [4], Haglund, Haiman, Loehr, Remmel and Ulyanov use another notion of inversion statistic. Their statistic coincides with $\mathrm{Inv}(T^\circ)$ up to a constant and gives a combinatorial interpretation of the powers of q appearing in the decomposition of Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$ on monomials.

3. Specializations of Hall–Littlewood functions

3.1. Basic definitions

Let i, j be two nonnegative integers. The raising operator R_{ij} acts on partitions by

$$\forall \lambda \in \mathcal{P}, \quad R_{ij}(\lambda) = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_p). \tag{8}$$

This action can be extended on elementary functions as follows

$$R_{ij} \cdot h_\lambda = h_{R_{ij}(\lambda)}. \tag{9}$$

Hall–Littlewood functions $Q'_\lambda(X; q)$ are the symmetric functions defined by

$$Q'_\lambda(X; q) = \prod_{i<j} (1 - qR_{ij})^{-1} s_\lambda(X). \tag{10}$$

We shall need the $\widetilde{Q}'_\lambda(X; q)$ version of Hall–Littlewood functions defined by

$$\widetilde{Q}'_\lambda(X; q) = q^{\eta(\lambda)} Q'_\lambda\big(X; q^{-1}\big),$$

where $\eta(\lambda) = \sum_{i \geqslant 1} (i - 1)\lambda_i$.

Example 3. The expansion of the Hall–Littlewood function $\widetilde{Q}'_{222}(X;q)$ on Schur basis is

$$\widetilde{Q}'_{222}(X;q) = q^6 s_{222} + \left(q^5+q^4\right)s_{321} + q^3 s_{33} + q^3 s_{411} + \left(q^4+q^3+q^2\right)s_{42} + \left(q^2+q\right)s_{51} + s_6.$$

In [10], Lascoux, Leclerc and Thibon have shown that for any partition λ, the Hall–Littlewood function $\widetilde{Q}'_\lambda(X;q)$ can be expressed in terms of $l(\lambda)$-ribbon tableaux

$$\widetilde{Q}'_\lambda(X;q) = \sum_{\mu\dashv|\lambda|} \sum_{T\in \mathrm{Tab}^{(l(\lambda))}{}_{l(\lambda)\lambda,\mu}} q^{\operatorname{cosp}(T)} X^T = \sum_{\mu\dashv|\lambda|} \widetilde{G}^{(l(\lambda))}_{l(\lambda)\lambda,\mu}(q) m_\mu,$$

where X^T is $x_1^{\mu_1}\dots x_p^{\mu_p}$ with μ the weight of T.

Proposition 2. *Hall–Littlewood functions $\widetilde{Q}'_\lambda(X;q)$ can be expressed in terms of inversion polynomials using Proposition* 1*,*

$$\widetilde{Q}'_\lambda(X;q) = \sum_{\mu\dashv|\lambda|} \widetilde{I}_{(l(\lambda)\lambda)^{(l(\lambda))},\mu}(q)\, m_\mu.$$

Using Remark 1, this proposition means that the Hall–Littlewood function $\widetilde{Q}'_\lambda(X;q)$ can be expressed in terms of $l(\lambda)$-tuples of Young tableaux with shape the product of single row partitions $((\lambda_{l(\lambda)}),\dots,(\lambda_1))$.

Example 4. The expansion of the Hall–Littlewood function $\widetilde{Q}'_{211}(X;q)$ on monomials is

$$\widetilde{Q}'_{211}(X;q) = \left(q^3+3q^2+5q+3\right)m_{1111} + \left(q^3+q^2+2q\right)m_{22} + \left(q^3+2q^2+3q+1\right)m_{211} + \left(q^3+q^2+q\right)m_{31} + q^3 m_4.$$

3.2. Specialization at roots of unity

Denote by Λ_q the vector space of symmetric functions over the field $\mathbb{C}(q)$. Let k be a positive integer and λ be a partition. The plethysm of the power sum p_λ by the power sum p_k is defined by

$$p_k \circ p_\lambda = p_{k\lambda}. \tag{11}$$

Since power sums $(p_\lambda)_{\lambda\in\mathcal{P}}$ form a basis of the vector space Λ_q, the plethysm by a power sum p_k is defined on any symmetric function f.

Theorem 1. *(See [8].) Let n,k be two positive integers and ζ be a primitive kth root of unity. The specialization of the parameter q at ζ in $Q'_{n^k}(X;q)$ and $\widetilde{Q}'_{n^k}(X;q)$ yields the following identities*

$$Q'_{n^k}(X;\zeta) = (-1)^{(k-1)n} p_k \circ h_n(X) \quad \textit{and} \quad \widetilde{Q}'_{n^k}(X;\zeta) = p_k \circ h_n(X). \tag{12}$$

Example 5. The Hall–Littlewood function $\widetilde{Q}'_{222}(X;q)$ with q specialized at $j=e^{\frac{2i\pi}{3}}$ is

$$\begin{aligned}\widetilde{Q}'_{222}(X;j) &= s_{222}-s_{321}+s_{33}+s_{411}-s_{51}+s_6 &(13)\\ &= \frac{1}{2}(p_{33}+p_6) = p_3\circ\left(\frac{1}{2}\,p_{11}+\frac{1}{2}p_2\right) &(14)\\ &= p_3\circ h_2(X). &(15)\end{aligned}$$

In [8], Lascoux, Leclerc and Thibon have given an algebraic proof of this theorem.

4. Combinatorial proof of the specialization

We give a combinatorial proof of (12) for Hall–Littlewood functions specialized at primitive roots of unity. The sketch of the proof is to separate the set of k-tuples of single row Young tableaux into subsets called diagonal classes. The specialization at primitive roots of unity of the restriction of inversion polynomials on these classes are 0 or 1. In order to have an explicit expression for these restricted polynomials, we translate the problem into sets of ribbon rigged configurations according to [13]. These sets of configurations are interesting because the image of diagonal classes can be easily characterized and inversion polynomials have a nice expression in terms of fermionic formulas which behave well at primitive roots of unity. Due to Remark 1 and Proposition 2, we only consider in the following, k-tuples of Young tableaux with shape increasing sequences of row partitions $((\lambda_1^{\circ 1}),\dots,(\lambda_1^{\circ p}))$, i.e.,

$$\lambda_1^{\circ 1}\leqslant\cdots\leqslant\lambda_1^{\circ p}. \tag{16}$$

4.1. Diagonal classes of k-tuples of Young tableaux

Let $T^\circ=(T^{\circ 1},\dots,T^{\circ k})$ be a sequence of Young tableaux with shape $\lambda^\circ=(\lambda^{\circ 1},\dots,\lambda^{\circ k})$ an increasing sequence of single row partitions. Define the maximal content m of cells of λ° by

$$m=\max\bigl(\lambda_1^{\circ i}-1,\ i=1,\dots,k\bigr)=\lambda_1^{\circ k}-1. \tag{17}$$

For all $i\in\{0,\dots,m\}$, we call d_i the ith diagonal of T° defined by

$$d_i=\bigl\{T^\circ(s)\ \text{such that}\ \operatorname{diag}(s)=i\bigr\}.$$

We call diagonal vector of T° the vector $d_{T^\circ}=(d_0,\dots,d_m)$.

Example 6. The diagonal vector of the following 3-tuple of Young tableaux

$$\left(\boxed{1\,|\,3},\ \boxed{1\,|\,1},\ \boxed{2\,|\,2\,|\,3\,|\,4}\right)$$

is given by $d_0=\{1,1,2\}$, $d_1=\{1,2,3\}$, $d_2=\{3\}$ and $d_3=\{4\}$.

Two k-tuples of Young tableaux T° and T'° in $\mathrm{Tab}_{\lambda^\circ,\mu}$ are equivalent if and only if, for all i in $\{0,\dots,m\}$, the ith set in d_{T° and $d_{T'^\circ}$ are the same. A diagonal class in $\mathrm{Tab}_{\lambda^\circ,\mu}$ is a set $D_{\lambda^\circ,\mu}(d)$ of all equivalent k-tuples of tableaux with diagonal vector d. We denoted by $\Delta_{\lambda^\circ,\mu}$ the set of all diagonal vectors. Thus, we can write the following decomposition

$$\mathrm{Tab}_{\lambda^\circ,\mu}=\bigsqcup_{d\in\Delta_{\lambda^\circ,\mu}} D_{\lambda^\circ,\mu}(d). \tag{18}$$

The restriction of the inversion polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q)$ to a diagonal class $D_{\lambda^\circ,\mu}(d)$ is defined by

$$\widetilde{I}_{\lambda^\circ,\mu}(q;d) = \sum_{T^\circ \in D_{\lambda^\circ,\mu}(d)} q^{\mathrm{Inv}(T^\circ)}. \tag{19}$$

Hence, by (18) the inversion polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q)$ can be split into

$$\widetilde{I}_{\lambda^\circ,\mu}(q) = \sum_{d \in \Delta_{\lambda^\circ,\mu}} \widetilde{I}_{\lambda^\circ,\mu}(q;d). \tag{20}$$

Example 7. The diagonal vector of the 3-tuple of Young tableaux of Example 6 is

$$d = \big(\{1,1,2\},\{1,2,3\},\{3\},\{4\}\big).$$

The diagonal class $D_{((2),(2),(4)),(3221)}(d)$ has 12 elements and the restriction of the inversion polynomial is

$$\widetilde{I}_{((2),(2),(4)),(3221)}(q;d) = q^5 + 3q^4 + 4q^3 + 3q^2 + q = q(q+1)^2\big(q^2+q+1\big).$$

Since we are only interested in Hall–Littlewood functions indexed by partitions of the form kn^k, we will now restrict to the special case of k-tuples of partitions $\lambda^\circ = ((n),\dots,(n))$ for some $n \geqslant 1$. We describe in the following corollary the diagonal classes with only one element depending on the weight μ.

Proposition 3. *Let n be a positive integer, λ° be the k-tuple of partitions $((n),\dots,(n))$ and μ a partition of weight kn.*

- *If each part of μ is divisible by k*
  (1) *there is a unique diagonal class $D_{\lambda^\circ,\mu}(d)$ with only one element $T^\circ = (T^{\circ 1},\dots,T^{\circ k})$,*
  (2) *the ith cell of each single row tableau $T^{\circ j}$ is filled with the same value,*
  (3) $\mathrm{Inv}(T^\circ) = 0$ *and* $\widetilde{I}_{\lambda^\circ,\mu}(q;d) = 1$.
- *If one part of μ is not divisible by k, there is no diagonal class with only one element.*

Proof. If each part of μ is divisible by k, we can construct a diagonal vector d such that for all i in $\{0,\dots,n-1\}$, all letters of d_i are the same. And there is a unique way T° to fill λ° according to d. Thus, the diagonal class $D_{\lambda^\circ,\mu}(d)$ has only one element. Since each filling of $\lambda^{\circ i}$ must be increasing, this diagonal class is unique. This proves statements (1) and (2).

Since the Young tableaux of T° are the same, the conditions of Definition 2 never hold. Hence, we conclude that

$$\mathrm{Inv}\big(T^\circ\big) = 0.$$

On the other hand, if one part of μ is not divisible by k, this implies that for any filling T° of λ°, there exist two tableaux $T^{\circ a}$ and $T^{\circ b}$ in T° which have two different values in one position. By transposition of these two tableaux, we obtain another filling $T^{'\circ}$ which belongs to the same diagonal class than T°. Thus, for this kind of weight, all diagonal classes have more than 2 elements. $\square$

Proposition 4. *Let n be a positive integer, λ° be the k-tuple of partitions $((n), \dots, (n))$ and d be a diagonal vector such that $\# D_{\lambda^\circ,\mu}(d) \geqslant 2$. For any primitive kth root of unity ζ, the restriction of the inversion polynomial on $D_{\lambda^\circ,\mu}(d)$ satisfies the following specialization*

$$\widetilde{I}_{\lambda^\circ,\mu}(\zeta; d) = 0. \tag{21}$$

In order to prove this proposition we need an explicit formula for the polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q; d)$. In [13], using a bijection between ribbon tableaux and ribbon rigged configurations, Schilling gives a fermionic expression of the polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q)$ in terms of q-binomial coefficients. This formula decomposes well on the image of diagonal classes on ribbon rigged configurations.

One can define similarly diagonal classes on k-ribbon tableaux through the bijection Ψ_k^{-1}. The combinatorial interpretation of cospin polynomials on diagonal classes is still an open problem for length of ribbons $k > 2$. For $k = 2$ (case of domino tableaux), Carré and Leclerc have found in [1] a combinatorial construction of these classes using a notion of labyrinths and proved that cospin polynomials of diagonal classes are of the form $q^a(1+q)^b$ with a and b two positive integers. This paper give a solution for the stable case $k = l(\lambda)$ using ribbon rigged configurations. We will come back to the proof of Proposition 4 in Section 4.2.4 after finishing the general proof of the specialization of Hall–Littlewood functions.

Corollary 1. *Let λ° be the k-tuple of single row partitions $((n), \dots, (n))$, μ a partition of weight nk and ζ be a kth primitive root of unity. The specialization of the inversion polynomial at $q = \zeta$ yields*

- *if all parts of μ are divisible by k*
$$\widetilde{I}_{\lambda^\circ,\mu}(\zeta) = 1,$$
- *if there exists a part of μ which is not divisible by k*
$$\widetilde{I}_{\lambda^\circ,\mu}(\zeta) = 0.$$

Proof. By splitting the set $\Delta_{\lambda^\circ,\mu}$ with respect to the cardinality of diagonal classes, (20) can be decomposed into

$$\widetilde{I}_{\lambda^\circ,\mu}(q) = \sum_{d/\# D_{\lambda^\circ,\mu}(d)=1} \widetilde{I}_{\lambda^\circ,\mu}(q; d) + \sum_{d/\# D_{\lambda^\circ,\mu}(d)>1} \widetilde{I}_{\lambda^\circ,\mu}(q; d). \tag{22}$$

By specializing q at ζ, the previous expression becomes

$$\widetilde{I}_{\lambda^\circ,\mu}(\zeta) = \sum_{d/\# D_{\lambda^\circ,\mu}(d)=1} \widetilde{I}_{\lambda^\circ,\mu}(\zeta; d) + \sum_{d/\# D_{\lambda^\circ,\mu}(d)>1} \widetilde{I}_{\lambda^\circ,\mu}(\zeta; d). \tag{23}$$

Using the result of Proposition 3, we conclude that the first term gives 1 if all the parts of μ are divisible by k and 0 otherwise. By Proposition 4 the second term always gives 0, which proves the corollary. □

We are able to give a combinatorial proof of the specialization of Hall–Littlewood functions given in Theorem 1. Let n, k be two positive integers, denote by Λ_p^k the set of all partitions of weight p with all parts divisible by k. Using Corollary 1,

$$\widetilde{Q}'_{n^k}(X;\zeta) = \sum_{\mu \dashv nk} \widetilde{I}_{((n),\ldots,(n)),\mu}(\zeta) m_\mu \tag{24}$$

$$= \sum_{\mu \in \Lambda^k_{nk}} m_\mu = \sum_{\mu \in \Lambda^k_{nk}} p_k \circ m_{\mu/k}, \tag{25}$$

where μ/k denotes the partition $(\frac{\mu_1}{k}, \ldots, \frac{\mu_p}{k})$. The linearity of the plethysm by p_k implies

$$\widetilde{Q}'_{n^k}(X;\zeta) = p_k \circ \left(\sum_{\mu \in \Lambda^k_{nk}} m_{\mu/k} \right). \tag{26}$$

By the definition of complete functions, we conclude that

$$\widetilde{Q}'_{n^k}(X;\zeta) = p_k \circ h_n(X).$$

In the rectangular case, the constant $\eta(n^k)$ is equal to

$$\eta\big(n^k\big) = n \sum_{i=1}^{k} (i-1) = \frac{nk(k-1)}{2}.$$

Hence, $\zeta^{\eta(n^k)} = (-1)^{(k-1)n}$ and (12) is proved.

Remark 2. In the case of $\lambda = k^{nk}$ which corresponds to $\lambda^{(k)} = ((1^n), \ldots, (1^n))$, there exists a similar factorization formula to (12) for Hall–Littlewood functions $Q'_\lambda(X;q)$ and $\widetilde{Q}'_\lambda(X;q)$

$$Q'_{k^{nk}}(X;\zeta) = (-1)^{(k-1)n} p_k \circ e_n(X) \quad \text{and} \quad \widetilde{Q}'_{k^{nk}}(X;\zeta) = p_k \circ e_n(X). \tag{27}$$

The proof is the same as for the product of single row partitions but using ribbon rigged configurations defined for products of single column partitions [13].

4.2. Ribbon rigged configurations and diagonal classes

The aim of this section is to prove Proposition 4 using ribbon rigged configurations introduced by Schilling in [13].

Let n be a positive integer, we define the q-factorial $(q)_n$ by

$$(q)_n = \prod_{i=1}^{n} \big(1 + q + \cdots + q^{i-1}\big). \tag{28}$$

Let a and b be two positive integers. The q-binomial coefficient is defined by

$$\begin{bmatrix} a+b \\ a, b \end{bmatrix} = \frac{(q)_{a+b}}{(q)_a (q)_b}. \tag{29}$$

4.2.1. Definition of ribbon rigged configurations

Let λ, μ be two partitions and $\nu^\circ = (\nu^{\circ 1}, \ldots, \nu^{\circ p})$ be a sequence of partitions. The sequence ν° is a (λ, μ)-configuration if the following conditions hold

(1) $\emptyset \subset \nu^{\circ 1} \subset \cdots \subset \nu^{\circ p-1} \subset \nu^{\circ p} = \mu'$, the conjugate partition of μ,
(2) $\forall 1 \leqslant a < p$, $|\nu^{\circ a}| = \lambda_1 + \cdots + \lambda_a$.

We denote by $C_{\lambda,\mu}$ the set of all (λ,μ)-configurations. We associate to each element ν° in $C_{\lambda,\mu}$ a constant $\alpha(\nu^\circ)$ defined by

$$\alpha(\nu^\circ)=\sum_{\substack{1\leqslant a\leqslant p-1\\ 1\leqslant i\leqslant \mu_1}} \nu_{i+1}^{\circ a}\big(\nu_i^{\circ a+1}-\nu_i^{\circ a}\big). \tag{30}$$

For any (λ,μ)-configuration $\nu^\circ=(\nu^{\circ 1},\dots,\nu^{\circ p})$, the vacancy numbers $p_i^{(a)}$ and the constant $m_i^{(a)}$ are defined for all $1\leqslant a<p$ and $1\leqslant i\leqslant \mu_1$ by

$$p_i^{(a)}=\nu_i^{\circ a+1}-\nu_i^{\circ a} \quad \text{and} \quad m_i^{(a)}=\nu_i^{\circ a}-\nu_{i+1}^{\circ a}. \tag{31}$$

One can fill top cells of the ith column of $\nu^{\circ a}$ by a number j satisfying

$$\forall 1\leqslant a<p,\quad \forall 1\leqslant i\leqslant \mu_1,\quad 0\leqslant j\leqslant p_i^{(a)}. \tag{32}$$

Such a filling is called a rigging of $\nu^{\circ a}$ and numbers j are called quantum numbers. In the special case of $j=p_i^{(a)}$, j is called a singular quantum number.

For each partition $\nu^{\circ a}$ and height i, we can view a rigging as a partition $J_i^{\circ a}$ in a box of width $m_i^{(a)}$ and height $p_i^{(a)}$. We denote by $J^{\circ a}$ the $l(\nu^{\circ a})$-tuple of partitions $J^{\circ a}=(J_1^{\circ a},\dots,J_{l(\nu^{\circ a})}^{\circ a})$ and by J the $(p-1)$-tuple $J=(J^{\circ 1},\dots,J^{\circ p-1})$.

Any rigging of ν° by J is called a rigged configuration (ν°,J) of shape ν° and weight J. The set of all rigged configurations (ν°,J) with $\nu^\circ\in C_{\lambda,\mu}$ is denoted by $RC_{\lambda,\mu}$.

The graphical representation of a ribbon rigged configuration (ν°,J) is

(1) the filling of top cells of columns which are in the ith row of $\nu^{\circ a}$ with numbers of $J_i^{\circ a}$ (riggings which differ only by reordering of quantum numbers corresponding to columns of the same height in a partition are identified),
(2) the filling of the ith row of the frontier of $\nu^{\circ a}$ with the vacancy numbers $p_i^{(a)}$.

Example 8. The graphical representation of the ribbon rigged configuration in $RC_{(3111),(322)}$ given by $\nu^\circ=((21),(31),(32),(331))$ and $J=(((1),\emptyset),(\emptyset,(1)),(\emptyset,(1)))$ is

$$\begin{array}{llll} \boxed{0} & 0 & & \\ \boxed{} & \boxed{1} & 1 & \end{array} \qquad \begin{array}{lllll} \boxed{1} & 1 & & & \\ \boxed{} & \boxed{0} & \boxed{0} & 1 & \end{array} \qquad \begin{array}{lllll} \boxed{0} & \boxed{1} & 1 & & \\ \boxed{} & \boxed{} & \boxed{0} & 0 & \end{array} \qquad \begin{array}{lll} \boxed{} & & \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{array}.$$

Denote by $RC_{\lambda,\mu}(\nu^\circ)$ the set of all rigged configurations in $RC(\lambda,\mu)$ with shape ν° in $C_{\lambda,\mu}$. Hence

$$RC_{\lambda,\mu}=\coprod_{\nu\in C_{\lambda,\mu}} RC_{\lambda,\mu}\big(\nu^\circ\big). \tag{33}$$

There exists a cocharge statistic on ribbon rigged configurations defined by

$$\mathrm{cc}\big(\nu^\circ,J\big)=\alpha\big(\nu^\circ\big)+\sum_{\substack{1\leqslant a\leqslant p-1\\ 1\leqslant i\leqslant \mu_1}} \big|J_i^{\circ a}\big|. \tag{34}$$

We denote by $\widetilde{S}_{\lambda,\mu}(q)$ the generating polynomial of $RC_{\lambda,\mu}$ with respect to the cocharge, i.e.

$$\widetilde{S}_{\mu,\delta}(q)=\sum_{(\nu,J)\in RC(\lambda,\mu)} q^{\mathrm{cc}(\nu^\circ,J)}. \tag{35}$$

In [13], Schilling has given the following explicit expression for $\widetilde{S}_{\lambda,\mu}(q)$,

$$\widetilde{S}_{\lambda,\mu}(q)=\sum_{\nu^\circ\in C_{\lambda,\mu}} q^{\alpha(\nu^\circ)}\prod_{\substack{1\leqslant a\leqslant p-1\\1\leqslant i\leqslant\mu_1}}\begin{bmatrix}\nu_i^{\circ a+1}-\nu_{i+1}^{\circ a}\\ \nu_i^{\circ a}-\nu_{i+1}^{\circ a},\ \nu_i^{\circ a+1}-\nu_i^{\circ a}\end{bmatrix}. \tag{36}$$

This expression of $\widetilde{S}_{\lambda,\mu}(q)$ is a fermionic formula first appeared in [5,7]. Denote by $\widetilde{S}_{\lambda,\mu}(q;\nu^\circ)$ the cocharge polynomial $\widetilde{S}_{\lambda,\mu}(q)$ restricted to the subset $RC_{\lambda,\mu}(\nu^\circ)$, i.e.

$$\widetilde{S}_{\lambda,\mu}\big(q;\nu^\circ\big)=q^{\alpha(\nu^\circ)}\prod_{\substack{1\leqslant a\leqslant p-1\\1\leqslant i\leqslant\mu_1}}\begin{bmatrix}\nu_i^{\circ a+1}-\nu_{i+1}^{\circ a}\\ \nu_i^{\circ a}-\nu_{i+1}^{\circ a},\ \nu_i^{\circ a+1}-\nu_i^{\circ a}\end{bmatrix}. \tag{37}$$

4.2.2. Bijection with k-tuples of tableaux

Theorem 2. *(See [13].) Let $\lambda^\circ=(\lambda^{\circ 1},\dots,\lambda^{\circ k})$ be a k-tuple of single row partitions, μ be a partition of weight $|\lambda^\circ|$ and δ be the partition such that $\delta_i=|\lambda^{\circ i}|$. There exists a bijection Θ_k between* $\mathrm{Tab}_{\lambda^\circ,\mu}$ *and $RC(\mu,\delta)$ which is compatible with the cocharge and inversion statistic, i.e.*

$$\forall T^\circ\in\mathrm{Tab}_{\lambda^{(k)},\mu},\quad \mathrm{cc}\big(\Theta_k\big(T^\circ\big)\big)=\mathrm{Inv}\big(T^\circ\big). \tag{38}$$

We recall the steps of the algorithm permitting to compute $\Theta_k(T^\circ)$. This algorithm is implemented in the package `MuPAD-Combinat`.

Algorithm 1. (See [13].)

```
Input: T° = (T°1, ..., T°k) a k-tuple of Young tableaux of Tab_{λ°,μ}.

Initialization: ν° ← a sequence of p empty partitions.

For i from k down to 1 do
  For j from 1 to l(λ°i) do
    (1) For k from T_j^{°i} to p do
          Add a box in the jth row in the partition ν°k
        EndFor
    (2) Recompute all vacancy numbers,
    (3) Fill the new cells coming from step 1 with the vacancy
        number of their row,
    (4) Remove a maximal number in the (j − 1)th row of the
        partitions which have a new box from step 1 in
        the jth row.
  EndFor
EndFor
For a from 1 to p − 1 do
  For i from 1 to μ1 do
      Replace each number β in the row ν_i^{°a} by ν_i^{°a+1} − ν_i^{°a} − β
  EndFor
EndFor.
```

Proposition 5. *Let k be a positive integer and λ, μ be two partitions such that $k|\mu| = |\lambda|$. Let $\lambda^{(k)} = (\lambda^{\circ 1}, \dots, \lambda^{\circ k})$ be the k-quotient of λ and δ be the partition $(|\lambda^{\circ 1}|, \dots, |\lambda^{\circ k}|)$. Combining the bijection Θ_k and Ψ_k, we have*

$$\tilde{G}^{(k)}_{\lambda,\mu}(q) = \widetilde{I}_{\lambda^{(k)},\mu}(q) = \widetilde{S}_{\mu,\delta}(q) = \sum_{\nu^\circ \in C_{\mu,\delta}} q^{\alpha(\nu^\circ)} \prod_{\substack{1 \leqslant a \leqslant p-1 \\ 1 \leqslant i \leqslant \mu_1}} \begin{bmatrix} \nu_i^{\circ a+1} - \nu_{i+1}^{\circ a} \\ \nu_i^{\circ a} - \nu_{i+1}^{\circ a}, \nu_i^{\circ a+1} - \nu_i^{\circ a} \end{bmatrix}. \tag{39}$$

This proposition gives an explicit formula for the transition matrix between Hall–Littlewood functions and monomials in terms of q-binomial coefficients.

Corollary 2. *Let λ° be the $l(\lambda)$-quotient of the partition $l(\lambda)\lambda$ and δ be the partition defined by $\delta_i = |\lambda^{\circ i}|$. Hall–Littlewood function can be expressed as*

$$\widetilde{Q}'_\lambda(X; q) = \sum_{\mu \dashv |\lambda|} \tilde{S}_{\mu,\delta}(q) m_\mu. \tag{40}$$

4.2.3. A matricial recoding of the bijection

We now give a slight reformulation of the previous algorithm for finding the shape of the ribbon rigged configuration corresponding to a k-tuple of single row Young tableaux. Let p and q be two integers and $\mathcal{M}_{p,q}$ the set of all $(p \times q)$-matrices with integer coefficients. We define the operator A_E on $\mathcal{M}_{p,q}$ by

$$\begin{aligned} A_E &: \mathcal{M}_{p,q} \to \mathcal{M}_{p,q}, \\ M = (m_{i,j})_{i,j} &\mapsto N = \begin{cases} n_{i,1} = m_{i,1}, \\ n_{i,j} = \sum_{1 \leqslant k \leqslant j} m_{i,k}, & \text{for } 2 \leqslant j \leqslant q. \end{cases} \end{aligned} \tag{41}$$

Let T° be a k-tuple of Young tableaux of shape λ°, weight μ and diagonal vector d. Let $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$ be the ribbon rigged configuration corresponding to T° by Θ_k. We construct a $(m_2 \times l(\mu))$-matrix M^{T° using the following rule

$$M^{T^\circ}_{i,j} = \text{number of cells labeled } j \text{ in diagonal } d_{i+1}.$$

Proposition 6. *The jth column of $A_E(M^{T^\circ})$ is equal to the jth partition of ν_{T°.*

Proof. Let T° be a k-tuple of single row tableaux and $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$. In the Algorithm 1, we observe that boxes which appear in the ith line of a partition $\nu^{(j)}_{T^\circ}$ only come from elements of the ith diagonal of T° which are smaller than j. And by definition of the operator A_E, the entry (i, j) of the matrix $A_E(M^{T^\circ})$ corresponds to the number of cells less than j in the ith diagonal. □

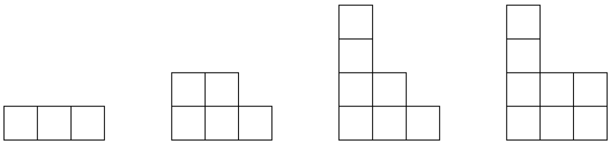
Example 9. Consider the following 3-tuple of single row tableaux T°

$$\left(\boxed{1\,|\,4}, \boxed{1\,|\,2}, \boxed{1\,|\,2\,|\,3\,|\,3} \right).$$

In this case, the matrices M^{T° and $A_E(M^{T^\circ})$ are

$$M^{T^\circ} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_E\big(M^{T^\circ}\big) = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The shape of the rigged configuration $\Theta_k(T^\circ)$ is



and can be read from $A_E(M^{T^\circ})$.

4.2.4. Diagonal classes on ribbon rigged configurations

We give an explicit formula for inversion polynomials restricted to diagonal classes. This formula permits to prove Proposition 4.

Proposition 7. *Let λ° be a sequence of single row partitions $(\lambda^{\circ 1}, \ldots, \lambda^{\circ k})$, μ be a partition of weight $\sum_i |\lambda^{\circ i}|$ and δ be the partition $(|\lambda^{\circ 1}|, \ldots, |\lambda^{\circ k}|)$. For each diagonal vector in $\Delta_{\lambda^\circ,\mu}$, there exists a unique (μ, δ)-configuration ν° in $C_{\mu,\delta}$ such that*

$$\Theta_k\big(D_{\lambda^\circ,\mu}(d)\big) = RC_{\mu,\delta}\big(\nu^\circ\big).$$

The explicit expression for the inversion polynomial restricted to the diagonal class $D_{\lambda^\circ,\mu}(d)$ is

$$\widetilde{I}_{\lambda^\circ,\mu}(q; d) = \widetilde{S}_{\mu,\delta}\big(q; \nu^\circ\big) = q^{\alpha(\nu^\circ)} \prod_{\substack{1\leqslant a\leqslant p-1\\ 1\leqslant i\leqslant \mu_1}} \begin{bmatrix} \nu_i^{\circ a+1} - \nu_{i+1}^{\circ a} \\ \nu_i^{\circ a} - \nu_{i+1}^{\circ a}, \nu_i^{\circ a+1} - \nu_i^{\circ a} \end{bmatrix}. \tag{42}$$

Proof. Let d be a diagonal vector and T°, T'° two elements in the diagonal class $D_{\lambda^\circ,\mu}(d)$. These two k-tuples of tableaux differ only by a permutation of cells which are in a same diagonal d_i. By construction, this property implies $M^{T^\circ} = M^{T'^\circ}$ and by Proposition 6, the ribbon rigged configuration $\Theta_k(T^\circ)$ has the same shape ν° than $\Theta_k(T'^\circ)$. Hence, since Θ_k is a bijection, $D_{\lambda^\circ,\mu}(d)$ is embedded into $RC_{\mu,\delta}(\nu^\circ)$.

Conversely, let T° and T'° be two k-tuples of tableaux in $\mathrm{Tab}_{\lambda^\circ,\mu}$ which are not in the same diagonal class. This implies that $M^{T^\circ} \neq M^{T'^\circ}$ and the shape of their corresponding ribbon rigged configurations are not the same. Finally, we conclude that

$$\Theta_k\big(D_{\lambda^\circ,\mu,}(d)\big) = RC_{\mu,\delta}(\nu).$$

The expression of inversion polynomials of diagonal classes in terms of q-supernomial coefficients follows immediately from the invariance of the statistics under Θ_k and (37). $\square$

Corollary 3. *Let n be a positive integer, λ° be the k-tuple of row partitions $((n), \ldots, (n))$ and μ a partition of weight nk satisfying the condition $\mu = (ks_1, \ldots, ks_p)$ for some positive integers $s_1, \ldots, s_p$ such that $s_1 + \cdots + s_p = n$. Let $D_{\lambda^\circ,\mu}(d)$ be a diagonal class with only one element T° and $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$ its corresponding ribbon rigged configuration. The ith partition of the shape ν_{T° is the rectangular shape $(k^{s_1+\cdots+s_i})$.*

Proof. By Proposition 3, since T° is alone in its diagonal class, T° is a k-tuple of tableaux of shape λ° with the same values at the same positions of each single row tableau. The corresponding matrix M^{T° is

$$M^{T^\circ} = \begin{pmatrix} k & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ k & 0 & \dots & \dots & \dots & 0 \\ 0 & k & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & k & 0 & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix},$$

where k occurs s_i times in the ith column. This implies that the shape is given by the matrix

$$A_E\left(M^{T^\circ}\right) = \begin{pmatrix} k & k & \dots & \dots & \dots & k \\ \vdots & \vdots & & & & \vdots \\ k & k & \dots & \dots & \dots & k \\ 0 & k & k & \dots & \dots & k \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & k & k & \dots & \dots & k \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix}.$$

Then, the ith partition in the shape ν_{T° is the rectangular partition $(k^{s_1+\dots+s_i})$. $\square$

Proof of Proposition 4. Let T° be a k-tuple of tableaux of shape $((n),\dots,(n))$ in a diagonal class $D_{\lambda^\circ,\mu}(d)$ with strictly more than one element. Write $\nu^\circ = (\nu^{\circ 1},\dots,\nu^{\circ p})$ the shape of the corresponding ribbon rigged configuration $\Theta_k(T^\circ)$. By Proposition 7, this shape is the same for all k-tuples of tableaux in $D_{\lambda^\circ,\mu}(d)$. Let h be the last position such that the hth diagonal d_h has at least two different elements. Then, the $(h+1)$th partition in ν_{T° is a rectangular partition of width k and height $s \leqslant r$. Since the last part of $\nu^{\circ h} = (\nu_1^{\circ h},\dots,\nu_l^{\circ h})$ is equal to a with $a < h$, the following coefficient appears in the inversion polynomial $\widetilde{I}_{\lambda^\circ,\mu}(q;d)$

$$\begin{bmatrix} \nu_l^{\circ h+1} - \nu_{l+1}^{\circ h} \\ \nu_l^{\circ h} - \nu_{l+1}^{\circ h}, \nu_l^{\circ h+1} - \nu_l^{\circ h} \end{bmatrix} = \begin{bmatrix} k \\ a-0, k-a \end{bmatrix}.$$

By definition, all kth primitive roots of unity ζ are roots of the q-binomial coefficient $\left[\begin{smallmatrix} k \\ a,k-a \end{smallmatrix}\right]$. Finally

$$\widetilde{I}_{\lambda^\circ,\mu}(\zeta;d) = 0. \qquad \square$$

5. Conclusion

The main tool of our combinatorial proof is the explicit expression of inversion polynomials on diagonal classes in terms of q-binomial coefficients. In this approach, we have used ribbon rigged configurations and fermionic formulas given in [13] which only exist in the case of products of row partitions or column partitions. The unrestricted rigged configurations [2] are another kind of configurations defined in the case of products of rectangles. Unfortunately, in the special case of product of rows, the number of shapes and the number of diagonal classes are not the same. Hence, the corresponding fermionic formula M cannot be used. We can mention [14] for a survey of the zoology of rigged configurations.

Acknowledgment

All the computations on ribbon rigged configurations, ribbon tableaux and Hall–Littlewood functions have been implemented by using MuPAD-Combinat (see [6] for details on this package and more especially [3] for implementation of symmetric functions).

References

[1] C. Carré, B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, J. Algebraic Combin. 4 (1995) 201–231.
[2] L. Deka, A. Schilling, New fermionic formula for unrestricted Kostka polynomials, J. Combin. Theory Ser. A 113 (2005) 1435–1461.
[3] F. Descouens, Making research on symmetric functions with MuPAD-Combinat, in: Lecture Notes in Comput. Sci., vol. 4151, 2006, pp. 407–418.
[4] J. Haglund, M. Haiman, N. Loehr, J.B. Remmel, A. Ulyanov, A combinatorial formula for the diagonal of the diagonal coinvariants, Duke Math. J. 126 (2) (2005) 195–232.
[5] G. Hatayama, A.N. Kirillov, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Character formulae of $\widehat{\mathrm{sl}}_n$-modules and inhomogeneous paths, Nuclear Phys. B 536 (3) (1999) 575–616.
[6] F. Hivert, N. Thiéry, MuPAD-Combinat, an open source package for research in algebraic combinatorics, Sém. Lothar. Combin. 51 (2004), http://mupad-combinat.sourceforge.net/.
[7] A.N. Kirillov, New combinatorial formula for modified Hall–Littlewood polynomials, in: q-Series From a Contemporary Perspective, South Hadley, MA, 1998, in: Contemp. Math., vol. 254, American Mathematical Society, Providence, RI, 2000, pp. 283–333.
[8] A. Lascoux, B. Leclerc, J.-Y. Thibon, Fonctions de Hall–Littlewood et polynomes de Kostka–Foulkes aux racines de l'unité, C. R. Acad. Sci. Paris 316 (1993) 1–6.
[9] A. Lascoux, B. Leclerc, J.-Y. Thibon, Green polynomials and Hall–Littlewood functions at roots of unity, European J. Combin. 15 (1994) 173–180.
[10] A. Lascoux, B. Leclerc, J.-Y. Thibon, Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras and unipotent varieties, J. Math. Phys. 38 (1997) 1041–1068.
[11] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, 1998.
[12] A. Schilling, M. Shimozono, D.E. White, Branching formula for q-Littlewood–Richardson coefficients, Adv. in Appl. Math. 30 (1–2) (2003) 258–272.
[13] A. Schilling, q-supernomial coefficients: From riggings to ribbons, in: MathPhys Odyssey 2001, Birkhäuser, Boston, Cambridge, MA, 2002, pp. 437–454.
[14] A. Schilling, $X = M$ theorem: Fermionic formulas and rigged configurations under review, in: MSJ Mem., vol. 17, 2007, pp. 75–104.
[15] D. Stanton, D. White, A Schensted algorithm for rim-hook tableaux, J. Combin. Theory Ser. A 40 (1985) 211–247.