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Covers for regular semigroups and an application to complexity

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Abstract

A major result of D.B. McAlister for inverse semigroups is generalised in the paper to classes of regular semigroups, including the class of all regular semigroups. It is shown that any regular semigroup is a homomorphic image of a regular semigroup whose least full self-conjugate subsemigroup is unitary; the homomorphism is injective on the subsemigroup. As an application, the group complexity of any finite E-solid regular semigroup is shown to be the same as, or one more than that of its least full self-conjugate subsemigroup (the subsemigroup is completely regular and is the type II subsemigroup). In an addition to the paper, by P.R. Jones, it is shown that any finite locally orthodox semigroup has group complexity 0 or 1.

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0. Introduction

McAlister proved that any inverse semigroup is an idempotent separating homomorphic image of an E-unitary inverse semigroup. This was later extended by Takizawa [17] and Szendrei [16] to the class of orthodox semigroups. The covering theorem is of major significance to the structure theory of inverse semigroups in that it strongly relates arbitrary inverse semigroups to inverse semigroups with relatively simple structure (see [10, 11 or 14]). Notice that an inverse or orthodox semigroup is E-unitary if and only if the idempotent class of its least group congruence is its subsemigroup of idempotents.

The aim here is to generalise McAlister's result to the class of all regular semigroups. Define the *self-conjugate core* of a regular semigroup to be its least full self-conjugate subsemigroup. It will be seen that the idempotent class of the least group congruence on a regular semigroup contains the self-conjugate core and that

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equality occurs if and only if the self-conjugate core is unitary. The generalization of McAlister's theorem that is proved in this paper is that any regular semigroup is a homomorphic image of a regular semigroup with a unitary self-conjugate core, where the homomorphism separates the self-conjugate core. The cover for a finite regular semigroup can be chosen to be finite. Furthermore, this result holds within any variety of regular unary semigroups that includes all groups and within any e-variety that has e-free objects and includes all groups (these have been specified in [21]).

An *E-solid regular semigroup* is a regular semigroup whose idempotent generated subsemigroup is completely regular; by [19] an equivalent condition is that the self-conjugate core is completely regular. The cover for a finite E-solid regular semigroup can be chosen to be also finite and E-solid. In Section 4, as a consequence of this refinement, it is shown that the group complexity (as defined in [1]) of a finite E-solid regular semigroup is the same as, or is one more than that of its self-conjugate core. By [1, Ch. 9], the group complexity of a completely regular semigroup is decidable.

An observation by P.R. Jones is included in an additional section where it is shown that any finite locally orthodox semigroup has group complexity 0 or 1.

1. Preliminaries

Let S be a regular semigroup. Denote by $V(x)$ the set of inverses in S of $x \in S$. Let $E(S)$ be the set of idempotents of S; the *core* $C(S) = \langle E(S) \rangle$ of S is its idempotent generated subsemigroup. Define

$$C_c(S) = \langle xC(S)x';\ x' \in V(x),\ x \in S \rangle, \qquad C_\infty(S) = C_{cc\cdots}(S).$$

Then $C_\infty(S)$ is the *self-conjugate core* of S; of course, $C_\infty(S)$ is the least self-conjugate full subsemigroup of S. By [19, Lemma 2.3], $C(S)$, $C_c(S)$ and $C_\infty(S)$ are all full regular subsemigroups of S that have the property that they include all inverses of their elements.

Lemma 1.1 (LaTorre [9]). *Let $\phi: S \to T$ be a surjective homomorphism of regular semigroups. Then $C_\infty(T) = (C_\infty(S))\phi$.*

For any congruence ρ on a regular semigroup S define the trace and kernel of ρ respectively by

$$\operatorname{tr}\rho = \rho|_{E(S)}, \qquad \ker\rho = \bigcup\{e\rho;\ e \in E(S)\}.$$

A congruence is uniquely determined by its trace and kernel (see [5] or [18]).

A subsemigroup A of a regular semigroup S is *unitary* if and only if for each $a \in A$, $s \in S$, if $as \in A$ or $sa \in A$ then $s \in A$.

Lemma 1.2. *Let σ be the least group congruence on a regular semigroup S.*
(i) [5, 9] $\operatorname{tr}\sigma = E(S)\times E(S)$, $\ker\sigma \supseteq C_\infty(S)$,
(ii) *$C_\infty(S)$ is unitary in S if and only if* $\ker\sigma = C_\infty(S)$.

Proof. (ii) By [9], $\sigma = \{(a,b)\in S\times S;\ ua = bv \text{ for some } u, v\in C_\infty(S)\}$. Suppose $C_\infty(S)$ is unitary in S and $a\sigma e\in E(S)$. Then $ua = ev$ for some u, $v\in C_\infty(S)$. Since $ev\in C_\infty(S)$ then $a\in C_\infty(S)$ thus $C_\infty(S) \supseteq \ker\sigma$, so by (i) $C_\infty(S) = \ker\sigma$. The converse is easy to see.

Notice that by Lemma 1.1, for any congruence ρ on a regular semigroup S, $C_\infty(S/\rho) = \{s\rho;\ s\in C_\infty(S)\}$. Generalise the definition of kernels to define

$$\ker_\infty \rho = \bigcup\{s\rho;\ s\in C_\infty(S)\}.$$

Define ρ to be *C_∞-pure* if $C_\infty(S) = \ker_\infty\rho$. Also define ρ to be *C_∞-separating* if no two elements of $C_\infty(S)$ are ρ-related. If $C_\infty(S)$ is unitary in S then define S to be *C_∞-unitary*. If S is orthodox then $C_\infty(S) = E(S)$ and these notions are the familiar "idempotent pure", "idempotent separating" and "E-unitary" notions, respectively. In this paper the aim is to construct a self-conjugate core unitary cover T for S; define a regular semigroup T to be a *C_∞-unitary cover* for S if and only if T is C_∞-unitary and there is a C_∞-separating congruence ρ on T such that $T/\rho \cong S$.

For any congruence ρ on a regular semigroup S let ρ_c denote the least congruence on S such that $\rho_c|_{C_\infty(S)} = \rho|_{C_\infty(S)}$. By Lemma 1.1 then ρ/ρ_c is a C_∞-separating congruence on S/ρ_c.

2. A covering theorem

Let $\boldsymbol{U}$ be the variety of all *unary semigroups*; that is, of type $\langle 2, 1\rangle$ algebras where the binary operation is associative. For any subvariety $\boldsymbol{V}$ of $\boldsymbol{U}$ let $F\boldsymbol{V}(X)$ denote the free object in $\boldsymbol{V}$ on a non-empty set X. This means that for any map $\phi: X \to S\in \boldsymbol{V}$ there is a unique unary semigroup homomorphism $\Phi: F\boldsymbol{V}(X) \to S$ that extends ϕ.

For a non-empty set X let F denote the free semigroup on the set $X\cup\{(,)^{-1}\}$. By [6], $F\boldsymbol{U}(X)$ is the least subsemigroup of F that contains X and is such that for each $w\in F\boldsymbol{U}(X)$ then $(w)^{-1}\in F\boldsymbol{U}(X)$. We will write $w^{-1} = (w)^{-1}$ and $X^{-1} = \{x^{-1};\ x\in X\}$. Clearly X^{-1} is a bijective copy of X and $X\cap X^{-1} = \emptyset$.

Let $\boldsymbol{RU}$ be the subvariety of $\boldsymbol{U}$ consisting of all regular unary semigroups; its defining identities are $xx^{-1}x = x$, $x^{-1}xx^{-1} = x^{-1}$. For any regular semigroup S, any choice of inverse $v^{-1}\in V(v)$ for each $v\in S$, and any map $\phi: X \to S$, there is a unique semigroup homomorphism $\Phi: F\boldsymbol{RU}(X) \to S$ extending ϕ such that $(w^{-1})\Phi = (w\Phi)^{-1}$ for all $w\in F\boldsymbol{RU}(X)$.

Lemma 2.1. *Let $\boldsymbol{V}$ be a subvariety of $\boldsymbol{RU}$. As a semigroup $F\boldsymbol{V}(X)$ is generated by $X\cup X^{-1}\cup E(F\boldsymbol{V}(X))$.*

Proof. Let $E = E(FV(X))$. The result is a consequence of reductions of the following types, where u, $v \in FV(X)$;

$$(u^{-1})^{-1} = (u^{-1})^{-1}u^{-1} \cdot u \cdot u^{-1}(u^{-1})^{-1} \in EuE,$$

$$(uv)^{-1} = (uv)^{-1}uv \cdot v^{-1} \cdot v(uv)^{-1}u \cdot u^{-1}uv(uv)^{-1} \in Ev^{-1}Eu^{-1}E.$$

The variety $\boldsymbol{G}$ of all groups is a subvariety of $\boldsymbol{RU}$. As a subvariety of $\boldsymbol{U}$ it is defined by the identities $xx^{-1}x = x$ and $xx^{-1} = y^{-1}y$; any unary semigroup S that satisfies these laws has an identity element $vv^{-1} = v^{-1}v$ for any $v \in S$ and is therefore a group. The free group $FG(X)$ is generated as a semigroup by $X \cup X^{-1}$ so for any $w \in FG(X)$ there exists $y_1, \dots, y_n \in X \cup X^{-1}$, $n \geq 1$, such that $w = y_1 \dots y_n$. If $w = 1$ is the identity element then $y_i = y_{i+1}^{-1}$ for various i, $1 \leq i < n$, and $y_1 \dots y_n$ is the result of a sequence of conjugations and products constructed from the various pairs $y_i y_{i+1}$.

Theorem 2.2. *Let V be a subvariety of $\boldsymbol{RU}$ such that $V \supseteq \boldsymbol{G}$. Then $FV(X)$ is C_∞-unitary.*

Proof. Let σ be the least group congruence on $FV(X)$, so $FV(X)/\sigma \cong FG(X)$. By Lemma 1.2, $\ker\sigma \supseteq C_\infty(FV(X)) \supseteq E(FV(X))$. Suppose $u \in \ker\sigma$; we may assume by Lemma 2.1 that u is expressed as a product of elements from $X \cup X^{-1} \cup E(FV(X))$. Construct v from u by deleting all entries from $E(FV(X))$ in the expression for u; the deleted entries are from $\ker\sigma$ so now $v \in \ker\sigma$ and v is a product of elements from $X \cup X^{-1}$. By the comment preceding this theorem it follows that $v \in C_\infty(FV(X))$, and therefore $u \in C_\infty(FV(X))$. Thus $C_\infty(FV(X)) = \ker\sigma$ and the result follows from Lemma 1.2.

Theorem 2.3. *Let V be a subvariety of $\boldsymbol{RU}$ such that $V \supseteq \boldsymbol{G}$. Then any $S \in V$ has a C_∞-unitary cover in V.*

Proof. There is a non-empty set X and a congruence ρ such that $S \cong FV(X)/\rho$. Let θ be the least group congruence on $FV(X)$ such that $\theta \supseteq \rho$; so θ/ρ is the least group congruence on $FV(X)/\rho$. Recall that ρ_c is least congruence on the semigroup $FV(X)$ such that $\rho_c|_{C_\infty(FV(X))} = \rho|_{C_\infty(FV(X))}$, while ρ/ρ_c is a C_∞-separating congruence on $FV(X)/\rho_c$. Since $\theta \supseteq \rho$ then $\theta_c \supseteq \rho_c$. By Lemma 1.2, θ is a group congruence on $FV(X)$ if and only if $\theta|_{C_\infty(FV(X))}$ is the universal congruence on $C_\infty(FV(X))$, so θ_c is the least group congruence on $FV(X)$. Hence $\ker\theta_c = C_\infty(FV(X))$ by Theorem 2.2. Of course θ_c/ρ_c is the least group congruence on $FV(X)/\rho_c$ and by Lemma 1.1, $\ker(\theta_c/\rho_c) = C_\infty(FV(X))/\rho_c = C_\infty(FV(X)/\rho_c)$. We have shown that $FV(X)/\rho_c$ is a C_∞-unitary cover of S.

With $\boldsymbol{RU}$ in place of V in the above proof we get

Corollary 2.4. *Any regular semigroup has a C_∞-unitary cover.*

Remark 2.5. (a) An *e-variety* $\mathscr{V}$ is a class of regular semigroups that is closed under the taking of homomorphic images, direct products and regular subsemigroups (see [7]). An e-free object in $\mathscr{V}$ is defined as follows. A mapping $\theta: X \cup X^{-1} \to S$, $S \in \mathscr{V}$, is *paired* if $x^{-1}\theta \in V(x\theta)$ and $x\theta = y\theta$ implies $x^{-1}\theta = y^{-1}\theta$ for all $x, y \in X$. An object $F\mathscr{V}(X) \in \mathscr{V}$ is e-free on X if there is a paired map $\iota: X \cup X^{-1} \to F\mathscr{V}(X)$ such that for any $S \in \mathscr{V}$ and paired map $\theta: X \cup X^{-1} \to S$ then there is a unique homomorphism $\phi: F\mathscr{V}(X) \to S$ such that $\iota\phi = \theta$.

In [21] it is shown that e-free objects $F\mathscr{V}(X)$ exist for $|X| > 1$ if and only if $\mathscr{V}$ is an e-subvariety of the e-variety of locally inverse semigroups or of the e-variety of E-solid regular semigroups. If $F\mathscr{V}(X)$ exists we may easily modify the proof of Lemma 2.1 to show that $F\mathscr{V}(X)$ is generated by $X \cup X^{-1} \cup E(F\mathscr{V}(X))$. The proofs of Theorems 2.2 and 2.3 can then be used, with $F\mathscr{V}(X)$ in place of $FV(X)$, to show that, if $\mathscr{V} \supseteq \mathbf{G}$ then $F\mathscr{V}(X)$ is C_∞-unitary and that any $S \in \mathscr{V}$ has a C_∞-unitary cover in $\mathscr{V}$.

(b) The covering theorems of [10, 16, 17] for inverse and orthodox semigroups respectively are corollaries of Theorem 2.3 since any inverse or orthodox semigroup lies in the regular unary semigroup variety, and in the e-variety, of inverse or orthodox semigroups respectively.

Theorem 2.6. *Let V be a variety of completely regular semigroups. Then $FV(X)$ is C_∞-unitary.*

Proof. It is known (for example see [19]) that the least inverse semigroup congruence on a completely regular semigroup has the self-conjugate core for its kernel. So with γ as the least inverse semigroup of congruence on $FV(X)$ then $FV(X)/\gamma$ is a relatively free semilattice of groups and $\ker\gamma = C_\infty(FV(X))$. But by [14, XII.9.9], relatively free semilattices of groups are E-unitary. Hence if σ is the least group congruence on $FV(X)$ we have $\ker\sigma = C_\infty(FV(X))$ and $FV(X)$ is C_∞-unitary.

3. An alternative derivation

The referee of this paper has pointed out that S.W. Margolis has been aware of the covering theorem for some time; he has known of the result as being a consequence of a construction of McAlister in [12]. Margolis has privately indicated to the referee (and to the author) how the connections can be made. The following description has been suggested by the referee.

Throughout this section S denotes a regular semigroup. The *natural partial order* $\leq$ on S is defined by

$$a \leq b \Leftrightarrow a \in bS \quad \text{and } a = aa'b \text{ for some } a' \in V(a).$$

Following [12], we define a map $\phi: S \to T$ of S into a regular semigroup T to be a *prehomomorphism* if $(ab)\phi \leq a\phi b\phi$ for all $a, b \in S$.

Lemma 3.1 [12; Lemma 1.3]. *Let ϕ be a prehomomorphism of S into a regular semigroup T and let $a, b \in S$. Then $a'\phi \in V(a\phi)$ for each $a' \in V(a)$ and ϕ maps $E(S)$ into $E(T)$.*

Define

$$\pi = \left\{ \begin{matrix} (a, b) \in S \times S; \text{ there exists } e, f \in E(S) \text{ and } u, v \in C_\infty(S) \text{ such} \\ \text{that } a \mathscr{R} e \mathscr{L} u \mathscr{R} b,\ a \mathscr{L} f \mathscr{R} v \mathscr{L} b \text{ and } b = uav \end{matrix} \right\},$$

and let I be the symmetric inverse semigroup on S/π.

Lemma 3.2 [12; Constructions 3.1 and 1.13]. *There is a prehomomorphism $\psi : S \to I$ given by*

$$(x\pi)(a\psi) = (xa)\pi \quad \text{for all } a \in S,\ x \in Saa' \text{ and any } a' \in V(a).$$

Let P be a set such that $P \supseteq S/\pi$ and $|P \backslash (S/\pi)| = |S/\pi|$. Since any $\alpha \in I$ is a one to one partial transformation of S/π, and hence of P, then there exist permutations of P that extend α. Define a binary relation $\theta \subseteq I \times \mathscr{S}_P$, where $\mathscr{S}_P$ is the symmetric group on P, such that $\alpha\theta = \{\beta \in \mathscr{S}_P; \beta \text{ extends } \alpha\}$. Let $R = \{(a, \beta); a \in S, \beta \in a\psi\theta\}$ be the graph of the relation $\psi\theta$.

Lemma 3.3. *R is a regular subsemigroup of $S \times \mathscr{S}_P$ whose projection onto S is surjective.*

Proof. For $a, b \in S$, $(ab)\psi \le a\psi b\psi$ in I so $a\psi b\psi$ is a partial transformation of S/π that extends $(ab)\psi$. Hence $(a\psi\theta)(b\psi\theta) \subseteq (a\psi b\psi)\theta \subseteq (ab)\psi\theta$. It follows that R is a subsemigroup of $S \times \mathscr{S}_P$. That the projection is surjective is a consequence of the definition of R. Notice that in the terminology of [2] we have shown that $\psi\theta$ is a relational morphism. It follows from Lemma 3.1 that (a', β^{-1}) is an inverse of $(a, \beta) \in R$ where $a' \in V(a)$.

Lemma 3.4. *R is a C_∞-unitary cover of S.*

Proof. A consequence of Lemma 3.1 is that ψ maps $C_\infty(S)$ into $C_\infty(T)$. Since I is an inverse semigroup then $(C_\infty(S))\psi \subseteq E(I)$. Let 1 be the identity element of $\mathscr{S}_P$; since precisely the idempotents of I are identity maps on their domains and therefore extend to 1 then $1\theta^{-1} = E(I)$. Now suppose $1 \in a\psi\theta$, so $a\psi \in E(I)$; then $a\psi$ is the identity map on its domain whence $x\pi = (xa)\pi$ for any $x \in Saa'$ and $a' \in V(a)$. So $(a', a'a) \in \pi$ and by the definition of π, $a' = ua'av$ for some $u, v \in C_\infty(S)$. Thus $a' \in C_\infty(S)$ and by [19; Lemma 2.3] then $a \in C_\infty(S)$. We have shown that $1 \in a\psi\theta$ if and only if $a \in C_\infty(S)$.

Let α be the restriction to R of the projection of $S \times \mathscr{S}_P$ onto $\mathscr{S}_P$. By the last paragraph $\ker \alpha = \{(a, 1); a \in C_\infty(S)\}$; this is isomorphic to $C_\infty(S)$ under the restriction of the surjective projection of R onto S. But $C_\infty(R) = \{(a, 1); a \in C_\infty(S)\}$, so by Lemma 1.2(ii), R is a C_∞-unitary cover for S.

Corollary 3.4 has now been reproved. Furthermore, R is a subsemigroup of the direct product of S by a group, so (as pointed out in Remark 2.5) R is in any e-variety containing S and the e-variety of all groups. If S is finite then so is I and $\mathscr{S}_P$.

Corollary 3.5. *Any finite regular semigroup has a finite C_∞-unitary cover.*

It should be noted that this result is also a consequence of a recently discovered covering theorem for finite semigroups [20].

4. Complexity of E-solid regular semigroups

This section begins with some observations about congruences.

An element x of a semigroup S is *aperiodic* if and only if there is a natural number n such that $x^{n+1} = x^n$. Let $A(S)$ denote the set of aperiodic elements of S. A congruence ρ on S is *aperiodic pure* if and only if ρ saturates $A(S)$; that is, $A(S) = \bigcup\{a\rho;\, a\in A(S)\}$. A congruence θ on S is called *aperiodic* if and only if for each aperiodic element $t\theta\in S/\theta$ then $\{x;\, x\in t\theta\} \subseteq A(S)$. By [4, XII, Proposition 4.6] for a finite semigroup S, θ is aperiodic if and only if for each subgroup G of S, θ separates the elements of G.

Theorem 4.1. *A congruence θ on a finite semigroup S is aperiodic if and only if it is aperiodic pure.*

Proof. If θ is aperiodic then, since $a\theta$ is aperiodic for any $a\in A(S)$, θ saturates $A(S)$. Conversely if θ is aperiodic pure and $x(\theta\cap\mathscr{H})e = e^2$ then the idempotent $e\in A(S)$, so $x\in A(S)$ and x is in a subgroup of S so $x = e$; thus θ separates the members of subgroups of S.

For a semigroup S define γ_S to be the greatest congruence on S that saturates $A(S)$. So γ_S is the greatest aperiodic pure congruence on S and (as in [3])

$$\gamma_S = \{(a, b)\in S\times S;\, xay\in A(S) \Leftrightarrow xby\in A(S) \text{ for all } x, y\in S^1\},$$

Theorem 4.2. *Let S be a C_∞-unitary regular semigroup. Then $A(S)\subseteq C_\infty(S)$, γ_S is C_∞-pure and $\gamma_{C_\infty(S)} = \gamma_S|_{C_\infty(S)}$.*

Proof. If $a\in A(S)$ then $a^{n+1} = a^n$ for some $n\geq 1$ and therefore $a^{2n} = a^n\in E(S)\subseteq C_\infty(S)$; now a^n, $a^{n+1}\in C_\infty(S)$ which is unitary so $a\in C_\infty(S)$. Hence $A(S)\subseteq C_\infty(S)$.

Observe for $a\in C_\infty(S)$ and $p, q\in S^1$ such that $paq\in A(S)$ then $(paq)^n = (paq)^{n+1}$ for some $n\geq 1$ so $(aqp)^{n+1} = aq(paq)^n p = aq(paq)^{n+1}p = (aqp)^{n+2}$ whence $aqp\in A(S)$; since $C_\infty(S)$ is unitary then $qp\in C_\infty(S)$. It follows that if $(a, b)\in\gamma_S$ then $bqp\in C_\infty(S)$ and by the unitary property $b\in C_\infty(S)$; that is γ_S is C_∞-pure. Similarly, if $aqp\in A(S)$ then $paq\in A(S)$.

Suppose $(a, b) \in \gamma_{C_\infty(S)}$ and $paq \in A(S)$ for some $p, q \in S^1$. By the above $aqp \in A(S)$ and $qp \in C_\infty(S)$, so $bqp \in A(S)$ and therefore $pbq \in A(S)$. Likewise $pbq \in A(S)$ gives $paq \in A(S)$ so $\gamma_{C_\infty(S)} \subseteq \gamma_S|_{C_\infty(S)}$. The reverse inclusion is immediate.

Suppose S is a regular semigroup. Denote by L the greatest congruence under the $\mathscr{L}$-relation on $C_\infty(S)$. So (see [3])

$$L = \{(a, b) \in C_\infty(S) \times C_\infty(S);\ paq \mathrel{\mathscr{L}} pbq \text{ for all } p, q \in C_\infty(S)^1\}.$$

Let L^+ be the greatest congruence on S that lies under the relation $\mathscr{L}|_{C_\infty(S)} \cup \mathscr{L}|_{S \setminus C_\infty(S)}$. Then

$$L^+ = \{(a, b) \in S \times S;\ paq \mathrel{\mathscr{L}} pbq \text{ and } paq \in C_\infty(S) \Leftrightarrow pbq \in C_\infty(S) \text{ for all } p, q \in S^1\}.$$

Notice that L^+ and L are respectively under the $\mathscr{L}$-relations on S and $C_\infty(S)$. In the terminology of [4, VII.4], L^+ and L are U_2-free congruences.

Theorem 4.3. *Let S be a C_∞-unitary regular semigroup. Then $L = L^+|_{C_\infty(S)}$.*

Proof. Suppose $p, q \in S^1$. If $p, q \in S$ let $p' \in V(p)$ and $q' \in V(q)$, and put $1' = 1$. Then $p'p$, $qq' \in C_\infty(S)^1$. Suppose $(a, b) \in L$, so $p'paqq' \mathrel{\mathscr{L}} p'pbqq'$ in $C_\infty(S)$. Hence in S, $p'paq \mathrel{\mathscr{L}} p'pbq$ and therefore $paq \mathrel{\mathscr{L}} p'paq \mathrel{\mathscr{L}} p'pbq \mathrel{\mathscr{L}} pbq$. Now assume $paq \in C_\infty(S)$; we have $a, b \in C_\infty(S)$ since $(a, b) \in L$. It follows that $p'(paq)p \in C_\infty(S)$, so $qp \in C_\infty(S)$ since $C_\infty(S)$ is unitary. Therefore $p'pbqp \in C_\infty(S)$ so $pbqpp' = p(p'pbqp)p' \in C_\infty(S)$. Again by the unitary property, $pbq \in C_\infty(S)$. We have shown that $L \subseteq L^+|_{C_\infty(S)}$; the reverse inclusion is immediate.

The pseudo-variety of all finite semigroups is denoted **Sgp**, while $\mathbb{N}^\circ$ denotes the set of natural numbers with 0 adjoined. By [8], the *complexity function* $c: \mathbf{Sgp} \to \mathbb{N}^\circ$ is the largest function (in a pointwise sense) that satisfies

(1) $S_C = 0$ if S is an aperiodic semigroup,
(2) $S_C \leq 1$ if S is a group,
(3) $S_C \leq T_C$ if S divides T,
(4) $(S \times T)_C \leq \max\{S_C, T_C\}$,
(5) $(S * T)_C \leq S_C + T_C$.

Here $S * T$ denotes the wreath product. The *local complexity function* is the largest function $\ell: \mathbf{Sgp} \to \mathbb{N}^\circ$ satisfying properties (1)–(5) and

(6) $S\ell = \max\{(eSe)\ell;\ e = e^2 \in S\}$.

It is shown in [1] that for any finite completely regular semigroup S the value of S_C is decidable and $S_C = S_\ell$.

As mentioned in the introduction, a regular semigroup S is E-solid if and only if the self conjugate core $C_\infty(S)$ is a completely regular semigroup. If S is a finite then S has a finite C_∞-unitary cover T by Corollary 3.5; this cover is E-solid since $C_\infty(T) \cong C_\infty(S)$ is completely regular.

Theorem 4.4. *Let S be a finite C_∞-unitary E-solid regular semigroup. Then* $(C_\infty(S))c \leq Sc \leq (C_\infty(S))c + 1$.

Proof. For any C_∞-unitary regular semigroup T put $T^\gamma = T/\gamma_T$ and $T^{L^+} = T/L^+$. By Theorem 4.2 and definitions, γ and L^+ are C_∞-pure so T^γ and T^{L^+} are both also C_∞-unitary. Write $C_\infty(T)^\gamma = C_\infty(T)/\gamma_{C_\infty(T)}$ and $C_\infty(T)^L = C_\infty(T)/L$; these are respectively the self-conjugate cores of T^γ and T^{L^+}, by Lemma 1.1 and Theorems 4.2 and 4.3. By these comments there is a sequence of C_∞-unitary regular semigroups S, $S^\gamma, S^{\gamma L^+}, S^{\gamma L^+\gamma}, \ldots$ and a corresponding sequence of self-conjugate cores $C_\infty(S), C_\infty(S)^\gamma$, $C_\infty(S)^{\gamma L}$, $C_\infty(S)^{\gamma L\gamma}, \ldots$. By [1; Lemma 8.3.18(b)] there is a first term of the form $C_\infty(S)^{\gamma L\gamma \cdots \gamma L\gamma}$ in the second sequence that is the trivial semigroup. Since $C_\infty(S)$ is completely regular then by [1; Definition 9.2.4(e) and Theorem 9.2.5], $(C_\infty(S))c$ is the number of superscripts L in the aforementioned term.

A regular semigroup with trivial self-conjugate core is a group. So $G = S^{\gamma L^+\gamma \cdots \gamma L^+\gamma}$ is the maximal group homomorphic image of S, where there are $(C_\infty(S))c$ superscripts L^+. In our sequence from S to $\{1\}$ there are $(C_\infty(S))c$ or $(C_\infty(S))c + 1$ superscripts L^+ according as $G = \{1\}$ or not; this is the number of congruences under $\mathscr{L}$ (or U_2-free congruences [4]) in our sequence so by [4; Corollary XII.5.2], $Sc \leq (C_\infty(S))c + 1$. By property (3) of the complexity function, $(C_\infty(S))c \leq Sc$.

Corollary 4.5. *Let S be a finite E-solid regular semigroup. Then* $(C_\infty(S))c \leq Sc \leq (C_\infty(S))c + 1$. *Also* $S\ell \leq Sc \leq S\ell + 1$.

Proof. Let T be a finite C_∞-unitary cover of S; T exists by Corollary 3.5. Then $C_\infty(S) \cong C_\infty(T)$ so $(C_\infty(S))c \leq Sc \leq Tc \leq (C_\infty(T))c + 1 = (C_\infty(S))c + 1$. By [1; Definition 9.2.4(i) and Theorem 9.2.5], along with [8, Theorem 4.2], c and ℓ take identical values on completely regular semigroups so $(C_\infty(S))c = (C_\infty(S))\ell$. From definitions $(C_\infty(S))\ell \leq S\ell \leq Sc$.

Remark 4.6. For any finite semigroup S the notion of its *type II subsemigroup* S_{II} was introduced in [15] to aid in the study of lower bounds for complexity. It is shown in [2; Facts 2.3, 2.4] that for any finite regular semigroup, $S_{II} = C_\infty(S)$.

5. An observation by P.R. Jones

A regular semigroup S is *locally orthodox* if for each $e \in E(S)$, the local subsemigroup eSe is orthodox. In [13], McAlister characterizes any locally orthodox semigroup S as being a homomorphic image of a regular Rees matrix semigroup $\mathscr{M}(S)$ over an orthodox semigroup $\mathrm{O}(S)$ where the homomorphism acts injectively on the local subsemigroups. Such semigroups need not be E-solid.

Suppose that S is a finite locally orthodox semigroup. It can be readily checked from McAlister's construction [13, Theorem 3.2] that $\mathscr{M}(S)$ and $\mathrm{O}(S)$ are also finite.

By [4, Proposition XI.3.1], the matrix semigroup $\mathscr{M}(S)$ over O(S) divides a wreath product of O(S) by a finite aperiodic semigroup. It follows from properties (1), (3) and (5) of the complexity function that $Sc = (\mathrm{O}(S))c$. An orthodox semigroup is an E-solid regular semigroup whose self conjugate core is a band. Since a band is aperiodic, it has a complexity of 0, so by Corollary 4.5 any orthodox semigroup has complexity 0 or 1. It follows that $Sc = 0$ or 1.

Theorem 5.1. *For any finite locally orthodox semigroup S, $Sc = 0$ or* 1.

References

[1] M.A. Arbib, The Algebraic Theory of Machines, Languages and Semigroups (Academic Press, New York, 1968).

[2] J. Birget, S. Margolis and J. Rhodes, Semigroups whose idempotents form a subsemigroup, Bull. Austral. Math. Soc. 41 (1990) 161–184.

[3] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. II, Math. Surveys 7 (Amer. Math. Soc., Providence, RI, 1967).

[4] S. Eilenburg, Automata, Languages and Machines, Vol. B (Academic Press, New York, 1976).

[5] R. Feigenbaum, Regular semigroup congruences, Semigroup Forum 17 (1979) 373–377.

[6] J.A. Gerhard, Free completely regular semigroups I. Representation, J. Algebra 82 (1983) 135–142.

[7] T.E. Hall, Identities for existence varieties of regular semigroups, Bull. Austral. Math. Soc. 40 (1989) 59–77.

[8] K. Henckell, S.W. Margolis, J.E. Pin and J. Rhodes, Ash's type II theorem, profinite topology and Malcev products, Internat. J. Algebra Comput. 1 (1991) 411–436.

[9] D.R. LaTorre, Group congruences on regular semigroups, Semigroup Forum 24 (1982) 327–340.

[10] D.B. McAlister, Groups, semilattices and inverse semigroups. Trans. Amer. Math. Soc. 192 (1974) 227–244.

[11] D.B. McAlister, Groups, semilattices and inverse semigroups II, Trans. Amer. Math. Soc. 196 (1974) 351–370.

[12] D.B. McAlister, Regular semigroups, fundamental semigroups and groups, J. Austral. Math. Soc. 29 (1980) 475–503.

[13] D.B. McAlister, Rees matrix covers for regular semigroups, J. Algebra 84 (1984) 264–279.

[14] M. Petrich, Inverse Semigroups (Wiley, New York, 1984).

[15] J. Rhodes and B. Tilson, Improved lower bounds for complexity of finite semigroups, J. Pure Appl. Algebra 2 (1972) 13–71.

[16] M.B. Szendrei, On a pull back diagram for orthodox semigroups, Semigroup Forum 20 (1980) 1–10: Corrigendum 25 (1982) 311–324.

[17] K. Takizawa, Orthodox semigroups and E-unitary regular semigroups, Bull. Tokyo Gakugai Uni. Ser. IV 31 (1979) 41–43.

[18] P.G. Trotter, Congruences on regular and completely regular semigroups, J. Austral. Math. Soc. 32 (1982) 388–398.

[19] P.G. Trotter, Congruence extensions in regular semigroups, J. Algebra 137 (1991) 166–179.

[20] P.G. Trotter and Zhonghao Jiang, Covers for regular semigroups, Southeast Asian Bull. Maths., 18 (1994) 157–161.

[21] Y.T. Yeh, The existence of e-free objects in e-varieties of regular semigroups, Internat. J. Algebra Comput. 2 (1992) 471–484.