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ON THE RELATION BETWEEN SPHERICAL AND PRIMITIVE HOMOLOGY CLASSES IN TOPOLOGICAL GROUPS†

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SUPPOSE that G is a connected topological group with $H_*(G; \mathbf{Z})$ of finite type. A theorem of Cartan and Serre [5] then implies that the Hurewicz map induces a monomorphism

$$\pi_*(G)/\text{torsion} \to PH_*(G; \mathbf{Z})/\text{torsion}$$

onto a subgroup of maximal rank. (Here $PH_*(G; \mathbf{Z})$ denotes the module of primitive elements in the coalgebra $H_*(G; \mathbf{Z})$.) This leads naturally to the following:

Problem. Let G be a connected topological group with $H_*(G; \mathbf{Z})$ of finite type. Let $x \in PH_*(G; \mathbf{Z})/\text{torsion}$ be an element of degree t. What is the smallest integer $N(t)$ such that $N(t) \cdot x$ is spherical?

As a step towards answering this question we shall establish:

THEOREM. *Let G be an s-connected topological group*, $s > 0$, with $H_*(G; \mathbf{Z})$ *of finite type. Let t be a positive integer and $x \in H_t(G; \mathbf{Z})/\text{torsion}$ a primitive element. Then $N_s(t) \cdot x$ is a spherical class where*

$$N_s(t) = \prod_{\substack{2(p-1)<t-s \\ p \text{ a prime}}} p^{\left[\frac{t-s-1}{2(p-1)-1}\right]}$$

and $[a]$ denotes the integral part of $[a]$.

If G is a simply connected topological group with $H_*(G; \mathbf{Z})$ finitely generated as an abelian group then G is also 2-connected [1]. Moreover, $PH_*(G; \mathbf{Z})/\text{torsion}$ is zero in even dimensions and so the spherical elements in $H_*(G; \mathbf{Z})/\text{torsion}$ are all odd dimensional. Thus by a slight reindexing of the above result we obtain:

THEOREM. *Let G be a simply connected topological group with $H_*(G; \mathbf{Z})$ finitely generated as an abelian group. Let r be a positive integer. If $x \in H_{2r-1}(G; \mathbf{Z})$ is a primitive element then $N(r) \cdot x$ is a spherical class where*

$$N(r) = \prod_{\substack{p<r \\ p \text{ a prime}}} p^{\left[\frac{2(r-2)}{2(p-1)-1}\right]}.$$

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The first few values of $N(r)$ are:

r	2	3	4	5
$N(r)$	1	4	48	576

The number $N(r)$ compares nicely with the number

$$m(r) = \prod_{\substack{p<r \\ p \text{ a prime}}} p^{[r/p-1]}$$

studied by Adams in [0].

It also compares favorably with the number $(r-1)!$ which we know from Bott's work to be the number required when $G = SU(n)$. More precisely we have

$$(r-1)! = \prod_{\substack{p<r \\ p \text{ a prime}}} p^{\frac{r-1-\sigma_p(r-1)}{p-1}}$$

where $\sigma_p(r-1)$ denotes the sum of the coefficients in the p-adic expansion of $r-1$. Note that when $r = p^k + 1$ that $(r-1)!$ and $N(r)$ contain almost the same power of p in their factorizations.

Our study of spherical and primitive classes in topological groups is closely related to the method of Adams in [0] and entails studying the connective coverings of the group G; the main technical tool being [7], [11]. This treatment owes much to the work of W. Singer on divisibilities of Chern classes [10].

The restriction that G be a topological group may be weakened but would involve us with several delicate questions concerning Postnikov systems and connective coverings which would be best postponed until another occasion.

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§1. FORMULATION OF THE PROBLEM

In this section we show how to convert the homotopy problem that we are interested in to a cohomology problem.

Convention. The word space will always mean a connected pointed topological space with compactly generated topology, of the homotopy type of a *cw*-complex. All base points will be assumed non-degenerate.

The entire discussion will take place in the obvious category whose objects are spaces.

Given a space B we may construct a tower of fibrations over B

$$
\begin{array}{c}
\vdots \\
\downarrow \\
B(n, \ldots, \infty) \\
\downarrow {\scriptstyle \pi(n-1,n)} \\
B(n-1, \ldots, \infty) \\
\downarrow \\
\vdots \\
\vdots \\
\downarrow \\
B(2, \ldots, \infty) \\
\downarrow {\scriptstyle \pi(1,2)} \\
B = B(1, \ldots, \infty)
\end{array}
$$

with the following properties: (see [0], [8])

(1) $\pi_i(B(n, \ldots, \infty)) = 0$ for $i < n$

(2) $\pi(n-1, n)_* : \pi_j(B(n, \ldots, \infty)) \to \pi_j(B(n-1, \ldots, \infty))$ is an isomorphism for $j \geq n$

(3) Each $\pi(n-1, n): B(n, \ldots, \infty) \to B(n-1, \ldots, \infty)$ is a principal $\mathbf{K}(\pi_{n-1}(B), n-2)$ bundle.

This tower is referred to as the connective tower of B (or sometimes the upside down Postnikov tower of B). It is unique in a suitable homotopy category of towers.

If B is simply connected then applying the Ω-functor to the connective tower of B yields the connective tower of ΩB, i.e., $\Omega(B(n, \ldots, \infty)) = (\Omega B)(n-1, \ldots, \infty)$ and similarly for the maps.

Thus if G is a group all the spaces in the connective tower of G may be assumed to be groups [4]. In addition the classifying diagram for the fibration $\pi(n-1, n): G(n, \ldots, \infty) \to G(n-1, \ldots, \infty)$, i.e..

$$
\mathscr{T}(n) \qquad
\begin{array}{ccc}
G(n, \ldots, \infty) & \longrightarrow & \mathbf{L}(\pi_{n-1}(G), n-1) \\
{\scriptstyle \pi(n-1,n)} \downarrow & & \downarrow \\
G(n-1, \ldots, \infty) & \xrightarrow{\varphi(n)} & \mathbf{K}(\pi_{n-1}(G), n-1)
\end{array}
$$

is a Hopf fibre square in the sense of [7].

Suppose now that G is a connected topological group. Then we have a commutative diagram

$$
\begin{array}{ccc}
\pi_n(G(n, \ldots, \infty) & \xrightarrow[\cong]{h} & H_n(G(n, \ldots, \infty); \mathbf{Z}) \\
{\scriptstyle \pi(n, \ldots, 0)_*} \downarrow \cong & & \downarrow {\scriptstyle \pi(n, \ldots, 0)_*} \\
\pi_n(G) & \xrightarrow{h} & H\,(G; \mathbf{Z})
\end{array}
$$

where h is the Hurewicz map.

Notation. If $\mathbf{A}$ is an augmented algebra over the ring $\mathbf{K}$ let $Q\mathbf{A} = \mathbf{K} \otimes_{\mathbf{A}} I\mathbf{A}$, where $I\mathbf{A}$ is the augmentation ideal of A. The elements of the $\mathbf{K}$ module $Q\mathbf{A}$ are called the indecomposable elements of $\mathbf{A}$.

PROPOSITION 1.1. *Suppose that G is a connected topological group with $H_*(G; \mathbf{Z})$ of finite type. Then there are (unnatural!) isomorphisms f, g making the diagram*

$$\begin{array}{ccc} \pi_n(G)/torsion & \xrightarrow{h} & PH_n(G; \mathbf{Z})/torsion \\ \cong \updownarrow f & & \cong \updownarrow g \\ QH^n(G; \mathbf{Z})/torsion & \xrightarrow{Q\pi(n, \ldots, 0)_*} & QH^n(G(n, \ldots, \infty)\mathbf{Z})/torsion \end{array}$$

commutative.

Proof. From our discussion above we have the natural commutative diagram

$$\begin{array}{ccc} \pi_n(G)/\text{torsion} & \xrightarrow{h} & PH_n(G; \mathbf{Z})/\text{torsion} \\ \downarrow \cong & & \updownarrow \cong \\ PH_n(G(n, \ldots, \infty); \mathbf{Z})/\text{torsion} & \xrightarrow{P\pi(n, \ldots, 0)_*} & PH_n(G; \mathbf{Z})/\text{torsion}. \end{array}$$

Applying the functor $\mathrm{Hom}_{\mathbf{Z}}(\ , \mathbf{Z})$ to the bottom row, and using the fact that $PH_n(G(n, \ldots, \infty); \mathbf{Z})$/torsion and $PH_n(G; \mathbf{Z})$/torsion are free abelian groups we obtain the desired conclusion. $\square$

Thus we have converted the problem of how $\pi_n(G)$/torsion is imbedded in $PH_n(G, \mathbf{Z})$/torsion by the Hurewicz map, to the study of how $QH^n(G, \mathbf{Z})$/torsion is imbedded by $Q\pi(n, \ldots, 0)_*$ into $QH^n(G(n, \ldots, \infty); \mathbf{Z})$. The remainder of this paper is devoted to a study of this cohomology problem by the methods of [11].

§2. WHAT PRIMES CAN DIVIDE?

Definition. Let X be a topological space and $x \in H_*(X; \mathbf{Z})$/torsion a spherical homology class. We say that x is spherically divisible iff there exists a spherical homology class $y \in H_*(X; \mathbf{Z})$/torsion such that $x = my$ for some $m \in \mathbf{Z}$, $m \neq 0, \pm 1$. If x is not spherically divisible then we say that x is spherically indivisible.

Our objective in this section will be to demonstrate:

THEOREM 2.1. *Let G be an s-connected topological group, $s > 0$, with $H_*(G; \mathbf{Z})$ of finite type. If t is a positive integer and p is a prime that divides a spherically indivisible spherical homology class in $H_t(G; \mathbf{Z})$/torsion then $2(p - 1) < t - s$.*

(It is of course implicit in the above theorem that $H_t(G; \mathbf{Z})$/torsion contains a non-zero spherical homology class; for otherwise the statement is trivial.)

The proof of Theorem 2.1 will be based on the results of [7], [11]. We give below a short summary of the results that we need. For an introduction to Eilenberg–Moore theory the reader should consult [3], [6], [7], or [12].

Recollections. We now assume that all spaces have homology of finite type. A Hopf space is a homotopy associative H-space [7]. A Hopf fibre square $\mathscr{F}$ is a diagram of spaces

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{f} & B_0 \end{array}$$

where

(1) $\pi_0 \colon E_0 \to B_0$ is a fibration,
(2) $f \colon B \to B_0$ is a continuous map.
(3) $\pi \colon E \to B$ is the fibration induced by the map $f \colon B \to B_0$,
(4) all the spaces are Hopf spaces and all the maps are homotopy multiplicative,
(5) B_0 is simply connected.

Associated with such a fibre square and a prime p we have an Eilenberg–Moore spectral sequence [3], [12], $\{\mathbf{E}_r, \mathbf{d}_r\}$ with the following properties:

(1) $\mathbf{E}_r \Rightarrow H^*(E; \mathbf{Z}_p)$ in the naive sense,
(2) $\mathbf{E}_2 = \operatorname{Tor}_{H^*(B_0;\mathbf{Z}_p)}(H^*(B; \mathbf{Z}_p), H^*(E_0; \mathbf{Z}_p))$
(3) Each $\mathbf{E}_r$ is a Hopf algebra, $\mathbf{d}_r$ is a derivation of Hopf algebras and the convergence in (1) is as Hopf algebras.

In [7] this spectral sequence was studied in some detail. Among the results established was the following ([7; Corollary 4.6]; compare [11; Proposition 5.5]):

PROPOSITION 2.2. *Let p be a prime and $\mathscr{F}$*

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{f} & B_0 \end{array}$$

be a Hopf fibre square with

(1) *$\pi_0 \colon E_0 \to B_0$ the path space fibration over B_0,*
(2) *$H^*(B_0; \mathbf{Z}_p)$ an abelian Hopf algebra which as an algebra is isomorphic to a free commutative algebra,*
(3) *$H^*(\Omega B_0; \mathbf{Z}_p)$ is a primitive Hopf algebra.*

Set $R^* = H^*(B; \mathbf{Z}_p)//f^*$. Then the sequence

$$0 \to QR^* \to QH^*(E; \mathbf{Z}_p)$$

is exact. □

Before turning to the proof of Theorem 2.1 we record an elementary lemma.

LEMMA 2.3. *Let $0 \to \mathbf{Z} \to \mathbf{Z} \xrightarrow{\rho} \mathbf{Z}_p \to 0$ be the usual exact sequence determined by multiplication by the prime p. Let $f \colon X \to Y$ be a map of spaces and $y \in H^*(Y; \mathbf{Z})$. Then $f^*(y)$ is divisible by p iff $\rho f_*(y) = 0 \in H^*(X; \mathbf{Z}_p)$.*

Proof. A routine consequence of the exact cohomology triangle determined by the exact coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0$. □

Proof of Theorem 2.1. Combining Lemma 2.3 and Proposition 1.1 we see that it suffices to show the following:

$(*)$: If $x \in H^t(G; \mathbf{Z}_p)$ is an indecomposable element that is the reduction of an indecomposable element of $H^t(G; \mathbf{Z})$/torsion and there exists an integer n such that $t > n > 0$ and

$$\pi(n, \ldots, 0)^*(x) \neq 0 \in QH^*(G(n, \ldots, \infty); \mathbf{Z}_p)$$

while

$$\pi(n+1, n)^*\pi(n, \ldots, 0)^*(x) = 0 \in QH^*(G(n+1, \ldots, \infty); \mathbf{Z}_p)$$

then $2(p-1) < t - s$.

So suppose that the conditions of $(*)$ obtain. Consider the Hopf fibre square $\mathscr{T}(n+1)$

$$\begin{array}{ccc} G(n+1, \ldots, \infty) & \longrightarrow & \mathbf{L}(\pi_n(G), n) \\ {\scriptstyle \pi(n+1,n)}\downarrow & & \downarrow \\ G(n, \ldots, \infty) & \xrightarrow{\varphi(n)} & \mathbf{K}(\pi_n(G), n). \end{array}$$

Observe that by the results of Cartan [2] and Serre [9] we may apply Proposition 2.2 to $\mathscr{T}(n+1)$. So doing we deduce that

$$\pi(n+1, n)^*\pi(n, \ldots, 0)^*(x) = 0 \in QH^*(G(n+1, \ldots, \infty); \mathbf{Z}_p)$$

iff

$$\pi(n, \ldots, 0)^*(x) = 0 \in QH^*(G(n, \ldots, \infty); \mathbf{Z}_p)//\varphi(n)^*$$

iff

$$\pi(n, \ldots, 0)^*(x) = \varphi(n)^*(y)$$

for some $y \in QH^*(\pi_n(G), n; \mathbf{Z}_p)$. From Cartan [2] and Serre [9] it follows that

$$y = \sum P_p^{I_j}\iota$$

where $P_p^{I_j}$ are admissible monomials of positive degree (recall $t > n$).

†A straightforward calculation shows that $\pi(n, \ldots, 0)^*(x)$ is the reduction of an integral class that represents a non-zero element of $[H^*(G(n, \ldots, \infty); \mathbf{Z})/\text{torsion prime to } p] \otimes \mathbf{Z}_p$. Hence at least one $P_p^{I_j}$ does not begin in a β [1]. Therefore since G is s-connected we have

$$t = \deg x = \deg y \geq 2(p-1) + \deg \iota \geq 2(p-1) + s + 1$$

and hence $t - s \geq 2(p-1) + 1$, i.e., $t - s > 2(p-1)$ as claimed. □

COROLLARY 2.4. *Let G be a simply connected topological group with $H_*(G; \mathbf{Z})$ finitely generated as an abelian group. If r is a positive integer and p is a prime that divides a spherically indivisible spherical homology class in $H_{2r-1}(G; \mathbf{Z})$/torsion then $p < r$.*

Proof. From [1] it follows that $\pi_2(G) = 0$. The result now follows easily from Theorem 2.1. □

† For $\langle \pi(n, \ldots, 0)^*(x), \sigma \rangle \neq 0 \in \mathbf{Z}$ for a suitable spherical class $\sigma \in H_t(G(n, \ldots, \infty; \mathbf{Z})$.

Remark. Corollary 2.4 was clearly known to Serre. While not explicit in [8] it appears as a step in the proof of Proposition IV.6 of [8] and we include it simply for the sake of completeness. Our proof is in a sense the Eckmann–Hilton dual of Serre's.

§3. WHAT MULTIPLES ARE SPHERICAL?

Our objective in this section will be to establish:

THEOREM 3.1. *Let G be an s-connected topological group, $s > 0$, with $H_*(G; \mathbf{Z})$ of finite type. Let t be a positive integer and $x \in H_t(G; \mathbf{Z})/torsion$ a primitive element. Then $N_s(t)x$ is a spherical class where*

$$N_s(t) = \prod_{\substack{2(p-1)<t-s \\ p \text{ a prime}}} p^{\left[\frac{t-s-1}{2(p-1)-1}\right]}$$

and $[a]$ denotes the integral part of a.

As in the proof of Theorem 2.1 this will be a consequence of properties of the connective tower of G. This requires that we review some additional results of [7].

Recollections. Consider a Hopf fibre square $\mathscr{F}$

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{f} & B_0 \end{array}$$

where $B_0 = \mathbf{K}(\pi, n)$, $n > 1$, π is a finitely generated abelian group, and $\pi_0: E_0 \to B_0$ is the path space fibration. Let $\{\mathbf{E}_r, \mathbf{d}_r\}$ denote the Eilenberg–Moore spectral sequence of $\mathscr{F}$ with $\mathbf{Z}_p$-coefficients, p a prime.

Since $H^*(B_0; \mathbf{Z}_p)\backslash\backslash f^*$ is a sub-Hopf algebra of $H^*(B_0; \mathbf{Z}_p)$ it follows from [2] and [9] and the Borel structure theorem [5; Theorem 7.11] that $H^*(B_0; \mathbf{Z}_p)\backslash\backslash f^*$ is a free commutative algebra. Since $H^*(B_0; \mathbf{Z}_p)$ is primitive so is $H^*(B_0; \mathbf{Z}_p)\backslash\backslash f^*$. Thus

$$H^*(B_0; \mathbf{Z}_p)\backslash\backslash f^* = \mathrm{S}[\{x_i\}]$$

where

$$x_i = \sum a_{ij} \beta^{\varepsilon_{i,j}} P_p^{I_{i,j}} \iota$$

and $P_p^{I_{i,j}}$ is an admissible monomial satisfying various conditions while $\varepsilon_{i,j} = 0, 1$, $a_{i,j} \in \mathbf{Z}_p$. (See [2] for the precise conditions.)

We turn now to the proof of Theorem 3.1. We shall require the following technical lemma. Our notation is that of [7].

LEMMA 3.2. *Suppose that p is a prime and $\mathscr{F}$*

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{f} & B_0 \end{array}$$

is a Hopf fibre square where $B_0 = \mathbf{K}(\pi, n)$, $n > 1$, π *a finitely generated abelian group, and* $\pi_0 : E_0 \to B_0$ *is the path space fibration. Let* $x \in H^*(B; \mathbf{Z})$ *be a primitive indecomposable having reduction* $\rho(x) \in H^*(B; \mathbf{Z}_p)$ *also indecomposable. Suppose that* $\rho(x) = f^*(y)$, *then*

$$\rho\left(\frac{1}{p}\,\pi^*(x)\right) = s^{-1}\beta y \quad \textit{mod decomposables}$$

in $H^*(E; \mathbf{Z}_p)$. (Note that $\pi^*(x)$ is divisible by p by Lemma 2.3.)

Proof. Let

$$g : B \to \mathbf{K}(\mathbf{Z}, \deg x)$$

be a map such that $g^*\iota_{\deg x} = x \in H^*(B; \mathbf{Z})$ and let

$$h : B_0 \to \mathbf{K}(\mathbf{Z}_p, \deg y)$$

be a map such that $h^*\iota_{\deg y} = y \in H^*(B_0; \mathbf{Z}_p)$. Form the Hopf fibre square

$$\begin{array}{ccc} \mathbf{K}(\mathbf{Z}, \deg x) & \longrightarrow & \mathbf{L}(\mathbf{Z}_p, \deg y) \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow \\ \mathbf{K}(\mathbf{Z}, \deg x) & \xrightarrow{\theta} & \mathbf{K}(\mathbf{Z}_p, \deg y) \end{array}$$

where $\theta^*\iota_{\deg y} = \rho\iota_{\deg x} \in H^*(\mathbf{Z}, \deg x; \mathbf{Z}_p)$. (Recall that $\deg x = \deg y$.) It is immediate that

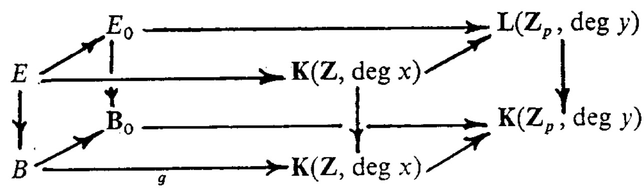
$$\rho\left(\frac{1}{p}\,\hat{\pi}^*(\iota_{\deg x})\right) = \rho\iota_{\deg x}$$

and that

$$\rho\iota_{\deg x} = s^{-1}\beta\iota_{\deg y} \in H^*(\mathbf{Z}, \deg x, \mathbf{Z}_p).$$

(Note that $\mathbf{K}(\mathbf{Z}, \deg x)$ is both the total and base space of the fibration $\hat{\pi}: \mathbf{K}(\mathbf{Z}, \deg x) \to \mathbf{K}(\mathbf{Z}, \deg x)$. This should cause no confusion in the above formulas.)

Next note that we have a morphism of Hopf fibre squares



and the result now follows by naturality of the Eilenberg–Moore spectral sequence. □

PROPOSITION 3.3. *Suppose that p is a prime and that*

$$\begin{array}{ccc} E & \longrightarrow & E_0 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_0} \\ B & \xrightarrow{f} & B_0 \end{array}$$

is a Hopf fibre square where $B_0 = \mathbf{K}(\pi, n)$, $n > 1$ *and* π *a finitely generated abelian group.* Let $x \in H^*(B; \mathbf{Z})$ *be a primitive indecomposable with* $p \cdot x \neq 0$ *and with reduction* $\rho(x) \in H^*(B; \mathbf{Z}_p)$ *also indecomposable. Suppose that* $\rho(x) = f^*(y)$ *and that* $\deg x > n$. *Then* $\pi^*(x) \in H^*(E; \mathbf{Z})$ *is divisible by p but not by* p^2.

Proof. From Lemma 2.3 we learn that $\pi^*(x)$ is divisible by p. From Lemma 3.2 we see that if $s^{-1}\beta y \neq 0 \in QH^*(E; \mathbf{Z}_p)$ then $\pi^*(x)$ is not divisible by p^2. Consider first the case when p is odd. Since y is indecomposable and $\deg x = \deg y < n$

$$y = \Sigma a_{s'} P_p^{I's'}\iota + \beta\Sigma a_{s''} P_p^{I''s''}\iota$$

where $P_p^{I's'}$, $P_p^{I''s''}$ are admissible monomials not beginning in β, with excess $<n$, and $a_{s'}$, $a_{s''} \neq 0 \in \mathbf{Z}_p$. From [7; Theorem 5.5] it follows that $s^{-1}\beta y = 0 \in QH^*(E; \mathbf{Z}_p)$ iff there exist admissible monomials $P_p^{J's'}$ with

(1) $\deg P_p^{J's'} = 2T + 1 - n$

(2) $\beta P_p^{I's'} = \beta P_p^T P_p^{J's'}$

(3) $\Sigma a_{s'} P_p^{J's'}\iota \in H^*(B_0; \mathbf{Z}_p)f^*\backslash\backslash$ is an indecomposable element.

(Note that $\Sigma a_{s'}\beta P_p^{I's'}\iota$ is a non-zero indecomposable element of $H^*(B_0; \mathbf{Z}_p)$ by the results of [2].)

Since $\beta P_p^{I's'}$ is again an admissible monomial and the admissible monomials are a basis for $\mathscr{A}^*(p)$ it follows from (2) that $P_p^{I's'} = P_p^T P^{J's'}$ and hence

$$\begin{aligned} f^*\Sigma a_{s'} P^{I's'}\iota &= f^*\Sigma a_{s'} P_p^T P_p^{J's'}\iota \\ &= P_p^T f^*\Sigma a_{s'} P_p^{J'}\iota = 0 \end{aligned}$$

by (3).

Hence

$$\rho(x) = f^*\beta\Sigma a_{s''} P_p^{I''s''}\iota = \beta f^*\Sigma a_{s''} P_p^{I''s''}\iota.$$

But this contradicts the fact that $p\cdot x \neq 0$ [1]. Hence if p is odd $\pi^*(x)$ is divisible by p but not by p^2.

Consider now the case $p = 2$. It then follows from [7] (compare [11; Proposition 2.1 and Proposition 2.2]) that $s^{-1}\beta y \neq 0 \in QH^*(E; \mathbf{Z}_p)$ iff $\beta y = 0$ in $H^*(B_0; \mathbf{Z}_p)$. Since $p = 2$, $\beta = Sq^1$. Let

$$y = \Sigma Sq^{I_s}\iota.$$

Note that not all of the Sq^{I_s} can begin on the left in an odd Sq^i. For if this were the case the Adem relation $Sq^1Sq^{2j} = Sq^{2j+1}$ would show

$$\rho(x) = f^*(y) = Sq^1 f^*\Sigma Sq^{I''s''}\iota$$

contrary to the hypothesis that $2\cdot x \neq 0$. Therefore it follows that

$$\beta y = Sq^1 y = \sum_s Sq^1 Sq^{I_s}\iota \neq 0$$

and the result follows for $p = 2$. □

Remark. The reader should compare Proposition 3.3 with the results obtained by W. Singer in Section 9 of [10].

LEMMA 3.4. *Let X be a simply connected space and*

$$\mathscr{F}(n) \qquad \begin{array}{ccc} X(n+1, \ldots, \infty) & \longrightarrow & \mathbf{L}(\pi_n(X), n) \\ \downarrow & & \downarrow \\ X(n, \ldots, \infty) & \xrightarrow{\varphi(n)} & \mathbf{K}(\pi_n(X), n) \end{array}$$

the nth-*connective fibre square over* X. *Then*

$$\varphi(n)^*: H^n(\pi_n(X), n; \mathbf{Z}_p) \to H^n(X(n, \ldots, \infty); \mathbf{Z}_p)$$

is an isomorphism and

$$\varphi(n)^*: H^{n+1}(\pi_n(X), n; \mathbf{Z}_p) \to H^{n+1}(X(n, \ldots, \infty); \mathbf{Z}_p)$$

is a monomorphism.

Proof. An elementary consequence of the generalized Whitehead Theorem [8]. □

THEOREM 3.5 *Let* G *be an* s*-connected topological group,* $s > 0$, *with* $H_*(G; \mathbf{Z})$ *of finite type. Let* t *be a positive integer and* p *a prime with* $2(p-1) < t - s$. *Then the highest power of* p *that divides a spherically indivisible spherical homology class in* $H_t(G; \mathbf{Z})$ *is no greater than* $p^{\left[\frac{t-s-1}{2(p-1)-1}\right]}$, *where* $[a]$ *denotes the integral part of* a.

Proof. Suppose that $x \in H^t(G; \mathbf{Z})$ is an indecomposable element. Since G is simply connected we may assume that $t > 1$. From Proposition 3.3 we see that in passing from $G(n, \ldots, \infty)$ to $G(n+1, \ldots, \infty)$ that $\pi(n, \ldots, 0)^*(x)$ becomes divisible by at most one more power of p. Moreover, from Lemma 3.2 we see that if $\pi(n, \ldots, 0)^*(x)$ becomes divisible by p then

$$\rho\left(\frac{1}{p}\pi(n+1, n)^*\pi(n, \ldots, 0)^*(x)\right) = s^{-1}\beta\alpha\iota \notin \operatorname{Im} \pi(n+1, n)^*.$$

Since the element of lowest possible degree in $H^*(\pi_n(G), n; \mathbf{Z}_p)\backslash\backslash f^*$ is $P_p{}^1\iota$ (by Lemma 3.4) which has dimension $2(p-1) + n$ it follows that the element of lowest possible dimension of $H^*(G(n+1, \ldots, \infty); \mathbf{Z}_p)$ not in $\operatorname{Im} \pi(n+1, n)^*$ is $s^{-1}P_p{}^1\iota$ which has dimension $n - 1 + 2(p-1)$. Since $\operatorname{Im} \pi(n+1, n)^*$ is an $\mathscr{A}^*(p)$-submodule of $H^*(G(n+1, \ldots. \infty); \mathbf{Z}_p)$ the lowest possible dimension of an element $y \in H^*(G(n+1, \ldots, \infty); \mathbf{Z}_p)$ with

$$s^{-1}\beta\alpha\iota = \zeta y$$

$\zeta \in \mathscr{A}^*(p)$, $\deg \zeta > 0$, is $n - 1 + 2(p-1)$. In passing to still higher connective fiberings $G(n+k+1, \ldots, \infty)$ one obtains in view of Lemma 3.4 no new cohomology classes of dimension $< n - 1 + 2(p-1)$. Therefore by Proposition 2.3

$$\pi(n + 2(p-1), \ldots, n)^*(s^{-1}\beta\alpha\iota) \neq 0.$$

Repeating the above argument for each $n > s$ yields the result. □

Proof of Theorem 3.1. It follows from Theorem 3.5 that a spherically indivisible spherical class in $H_t(G; \mathbf{Z})$/torsion is divisible by at most $N_s(t) = \prod_{\substack{2(p-1)<t-s \\ p \text{ a prime}}} p^{\left[\frac{t-s-1}{2(p-1)-1}\right]}$.
Since the spherical elements are a subgroup of $PH_*(G; \mathbf{Z})$/torsion of maximal rank it follows that some multiple of any primitive class must be spherical. Combining these two facts the result follows. □

COROLLARY 3.6 *Let* G *be a simply connected topological group with* $H_*(G; \mathbf{Z})$ *finitely* *generated as an abelian group. Let r be a positive integer. If* $x \in H_{2r-1}(G; \mathbf{Z})$ *is a primitive element then* $N(r).x$ *is a spherical class where*

$$N(r) = \prod_{\substack{p<r \\ p \text{ a prime}}} p^{[2(\frac{r-2}{2(p-1)-1})]}.$$

Proof. From [1] it follows that G is 2-connected and the result now follows from Theorem 3.1 by a slight reindexing. □

§4. CONCLUDING REMARKS

The problem that we have been studying has a natural companion problem which may be described as follows:

Problem. Let G be an s-connected topological group with $H_*(G; \mathbf{Z})$ of finite type. What is the largest integer $n_s(t)$ such that any spherical class $H_t(G; \mathbf{Z})$/torsion is divisible by $n_s(t)$?

The case when $H_*(G; \mathbf{Z})$ is finitely generated as an abelian group is of particular interest (e.g. G a Lie group). In this case if G has no 2-torsion and $t \neq 2^j - 1$, then work of E. Thomas [14] implies that any spherical class of dimension t is divisible by 2. Additional results of Thomas [13] seem to indicate further that if $t \neq 1, 3, 7, 15$ then any spherical class of dimension t is divisible by 2.

Computations of sundry examples lends credence to the following:

CONJECTURE. *Let G be a connected topological group with* $H_*(G; \mathbf{Z})$ *torsion free of finite rank. If* $\sigma \in H_{2r-1}(G; \mathbf{Z})$ *is a spherical class and p is a prime satisfying*

(1) $p < r$
(2) $r \neq kp^i, 0 < k < p, i \geq 0$ *then p divides* σ.

We remark that the restriction that $H^*(G; \mathbf{Z})$ be torsion free is important as the following example shows.

Recall the $H^*(F_4; \mathbf{Z}_5) = E[x_3, x_{11}, x_{15}, x_{23}]$ and that $H^*(F_4; \mathbf{Z}_5)$ has no 5-torsion. Note that $15 = 2(8) - 1$, $8 \neq k5^i$, $0 < k < 5$, and that degree $P_5{}^1 = 8$. Simple calculations show

$$H^i(F_4(n, \ldots, \infty); \mathbf{Z}_5) = 0 \qquad 3 < i < 10.$$

It therefore follows easily that there is a spherical class $\sigma_{15} \in H_{15}(F_4; \mathbf{Z})$/torsion that is not divisible by 5.

I am indebted to Allan Clark for bringing the above example to my attention.

In the special case that G is a Lie group it would be interesting to have formulas for $n(r)$ and $N(r)$ in terms of the Lie algebra and Weyl group of G.

REFERENCES

0. J. F. ADAMS: On Chern characters and the structure of the unitary group, *Proc. Camb. phil. Soc. math. phys. Sci.* **57** (1961), 189–199.
1. W. BROWDER: Torsion in *H*-spaces, *Ann. Math.* (2) **74** (1961), 24–51.
2. H. CARTAN: Algèbres d'Eilenberg-MacLane et homotopie, *Seminar Cartan* 1954/1955, Ecole Normale Supérieure.
3. S. EILENBERG and J. C. MOORE: Homology and fibrations I, *Comment. math. helvet.* **40** (1966), 199–236.
4. J. MILNOR: Construction of universal bundles I, *Ann. Math.* (2) **63** (1956), 272–284.
5. J. MILNOR and J. C. MOORE: The structure of Hopf algebras, *Ann. Math.* (2) **81** (1965), 211–264.
6. J. C. MOORE: Algèbre homologique et homologie des espaces classifiants, *Seminar Carten et Moore* 1959/1960, Exposé 7, Ecole Normale Supérieure.
7. J. C. MOORE and L. SMITH: Hopf algebras and multiplicative fibrations I, II, *Am. J. Math.* (to appear).
8. J. P. SERRE: Groupes d'homotopie et classes des groupes abelienes, *Ann. Math.* (2) **58** (1953), 258–294.
9. J. P. SERRE: Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, *Comment. math. helvet.* **27** (1953), 198–234.
10. W. SINGER: Connective fiberings over *BU* and *U*, *Topology* **7** (1968), 189–225.
11. L. SMITH: The cohomology of stable two Stage Postnikov systems, *Ill. J. Math.* **11** (1967), 310–329.
12. L. SMITH: Homological algebra and the Eilenberg-Moore spectral sequence, *Trans. Am. math. Soc.* **129** (1967), 58–93.
13. E. THOMAS: On the mod 2 cohomology of certain *H*-spaces, *Comment. math. helvet.* **37** (1962), 132–140.
14. E. THOMAS: Steenrod squares and *H*-spaces II, *Ann. Math.* (2) **81** (1965), 473–495.

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