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Existence and multiplicity of positive solutions for singular fractional boundary value problems[☆]

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ABSTRACT

In this paper, we discuss the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$D^{\alpha}_{0+}u(t) + f\left(t, u(t), D^{\nu}_{0+}u(t), D^{\mu}_{0+}u(t)\right) = 0,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0,$$

where $3 < \alpha \le 4, 0 < \nu \le 1, 1 < \mu \le 2, D^{\alpha}_{0+}$ is the standard Riemann–Liouville fractional derivative, f is a Carathédory function and $f(t, x, y, z)$ is singular at the value 0 of its arguments x, y, z. By means of a fixed point theorem, the existence and multiplicity of positive solutions are obtained.

1. Introduction

This paper investigates the singular fractional boundary value problem

$$D^{\alpha}_{0+}u(t) + f\left(t, u(t), D^{\nu}_{0+}u(t), D^{\mu}_{0+}u(t)\right) = 0, \tag{1.1}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $3 < \alpha \le 4, 0 < \nu \le 1, 1 < \mu \le 2$ are real numbers. Here f satisfies the local Carathéodory condition on $[0, 1] \times \mathscr{D}, \mathscr{D} \subset \mathbb{R}^3$ $(f \in \mathrm{Car}([0, 1] \times \mathscr{D}))$, $f(t, x, y, z)$ may be singular at the value 0 of all its space variables x, y, z, and D^{α}_{0+} is the standard Riemann–Liouville fractional derivative of order α.

We say that f satisfies the local Carathéodory condition on $[0, 1] \times \mathscr{D}, \mathscr{D} \subset \mathbb{R}^3$, if

(i) $f(\cdot; x, y, z) : [0, 1] \to \mathbb{R}$ is measurable for all $(x, y, z) \in \mathscr{D}$,
(ii) $f(t; \cdot, \cdot, \cdot) : \mathscr{D} \to \mathbb{R}$ is continuous for a.e. $t \in [0, 1]$,
(iii) for each compact set $\mathscr{K} \subset \mathscr{D}$ there is a function $\varphi_{\mathscr{K}} \in L^1[0, 1]$ such that

$$|f(t, x, y, z)| \le \varphi_{\mathscr{K}}(t), \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in \mathscr{K}.$$

A function $u \in C^2[0, 1]$ is called a positive solution of problem (1.1), (1.2), if $u > 0$ on $(0, 1]$, $D^{\alpha}_{0+}u \in L^1[0, 1]$, u satisfies boundary condition (1.2) and equality (1.1) holds for a.e. $t \in [0, 1]$.

Recently, fractional differential equations have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of science and engineering. For examples and details, see [1–22] and the references therein.

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Papers, such as [10,23–28], discuss fractional boundary value problems with nonlinearities having singularities in space variables.

Paper [29] investigates positive solutions of the singular Dirichlet problem

$$D_{0+}^{\alpha}u(t) + f\,(t, u(t), D^{\mu}u(t)) = 0,$$
$$u(0) = u(1) = 0,$$

where $1 < \alpha < 2$, $0 < \mu \le \alpha - 1$, and f is a Carathéodory function on $[0, 1] \times (0, \infty) \times \mathbb{R}$. The existence of positive solutions is obtained by the combination of regularization and sequential techniques with the Guo–Krasnosel'skii fixed point theorem on cones.

In [1], the authors discuss the (n, p) boundary value problem

$$y^{(n)} + Q\left(t, y, y', \ldots, y^{(q)}\right) = 0, \quad 0 < t < 1,$$
$$y^{(i)}(0) = 0, \quad 0 \le i \le n-2,$$
$$y^{(p)}(1) = 0,$$

where $0 \le q \le p \le n-1$, but fixed. The function $Q(t, y, y', \ldots, y^{(q)})$ is singular at $y_i = 0$, $0 \le i \le q$. The existence results are ascertained by using topological transversality results.

The singular problem

$$D_{0+}^{\alpha}u(t) + q(t)f\left(u(t), u'(t), \ldots, u^{(n-2)}(t)\right) = 0, \quad n-1 < \alpha \le n,\ n \ge 2,$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u^{(n-2)}(1) = 0,$$

was discussed in [27], where $f \in C\left((0, \infty)^{n-1}\right)$ and $q \in L^r[0, 1]$ $(r > 0)$ are positive. The existence results of positive solutions are proved by the combination of regularization and sequential techniques with a fixed point theorem for mixed monotone operators on normal cones.

More recently, Staněk [25] investigates the problem

$$D_{0+}^{\alpha}u(t) + f(t, u(t), u'(t), D^{\mu}u(t)) = 0,$$
$$u(0) = 0, \qquad u'(0) = u'(1) = 0.$$

Here $2 < \alpha < 3$, $0 < \mu < 1$ and f satisfies the local Carathéodory condition on $[0, 1] \times \mathscr{D}$, $\mathscr{D} \subset \mathbb{R}^3$. The existence of positive solutions is obtained by means of Guo–Krasnosel'skii fixed point theorem on cones.

Inspired by above works, we consider the existence and the multiplicity of positive solutions of problem (1.1), (1.2), where f satisfies the local Carathéodory condition and $f(t, x, y, z)$ can be singular at the value 0 of all its space variables x, y, z.

Throughout the paper, $\|x\|_1 = \int_0^1 |x(t)|\mathrm{d}t$ is the norm in $L^1[0, 1]$ and $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$ is the norm in the space $C[0, 1]$, while $\|x\|_* = \max\{\|x\|, \|x'\|, \|x''\|\}$ is the norm in $C^2[0, 1]$. AC[0, 1] and $\mathrm{AC}^k[0, 1]$ are sets of absolutely continuous functions and functions having absolutely continuous k-th derivatives on [0, 1], respectively.

We work with the following conditions on f in (1.1),

(H_1) $f \in \mathrm{Car}([0, 1] \times \mathscr{D})$, $\mathscr{D} = (0, \infty)^3$, and there exists a positive constant m such that, for a.e. $t \in [0, 1]$ and all $(x, y, z) \in \mathscr{D}$,

$$f(t, x, y, z) \ge m.$$

(H_2) f satisfies the estimate, for a.e. $t \in [0, 1]$ and all $(x, y, z) \in \mathscr{D}$,

$$f(t, x, y, z) \le p(x, y, z) + \gamma(t)h(x, y, z),$$

where $\gamma \in L^1[0, 1]$, $p \in C(\mathscr{D})$ and $h \in C([0, \infty)^3)$ are positive, p and h are nonincreasing and nondecreasing in all their arguments, respectively,

$$\int_0^1 p\left(Mt^{\alpha-1}, \frac{(2-\nu)M}{6}t^{3-\nu}, \frac{(2-\mu)M}{6}t^{3-\mu}\right)\mathrm{d}t < \infty,$$
$$M = \frac{m}{(\alpha-2)\Gamma(\alpha+1)}, \qquad \lim_{x\to\infty}\frac{h(x, x, x)}{x} = 0.$$

(H_3) f satisfies the estimate, for a.e. $t \in [0, 1]$ and all $(x, y, z) \in \mathscr{D}$,

$$f(t, x, y, z) \le p(x, y, z) + \gamma(t)h(x, y, z),$$

where $\gamma \in L^1[0, 1]$, $p \in C(\mathscr{D})$ and $h \in C([0, \infty)^3)$ are positive, p and h are nonincreasing and nondecreasing in all their arguments, respectively,

$$\int_0^1 p\left(Mt^{\alpha-1}, \frac{(2-\nu)M}{6}t^{3-\nu}, \frac{(2-\mu)M}{6}t^{3-\mu}\right)\mathrm{d}t < \infty,$$
$$M = \frac{m}{(\alpha-2)\Gamma(\alpha+1)}, \qquad \lim_{x\to 0}\frac{h(x, x, x)}{x} = 0.$$

Since (1.1) is a singular equation, we use regularization and sequential techniques to prove the existence and the multiplicity of positive solutions of problem (1.1), (1.2). To this end, we define χ_n and $f_n (n \in \mathbb{N})$ by the following formulas

$$\chi_n(t) = \begin{cases} t, & \text{if } t \geq \dfrac{1}{n}; \\ \dfrac{1}{n}, & \text{if } t < \dfrac{1}{n}, \end{cases}$$

for a.e. $t \in [0, 1]$ and all $(x, y, z) \in \mathbb{R}^3$,

$$f_n(t, x, y, z) = f(t, \chi_n(x), \chi_n(y), \chi_n(z)).$$

Then condition (H_1) provides that $f_n \in \mathrm{Car}([0, 1] \times \mathbb{R}^3)$ and

$$f_n(t, x, y, z) \geq m, \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in \mathbb{R}^3. \tag{1.3}$$

Condition (H_2) gives

$$\left.\begin{aligned} & f_n(t, x, y, z) \leq p\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) + \gamma(t)h\left(x + \frac{1}{n}, y + \frac{1}{n}, z + \frac{1}{n}\right), \\ & \qquad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in [0, \infty)^3, \end{aligned}\right\} \tag{1.4}$$

$$\left. f_n(t, x, y, z) \leq p(x, y, z) + \gamma(t)h\left(x + \frac{1}{n}, y + \frac{1}{n}, z + \frac{1}{n}\right), \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in \mathscr{D}. \right\} \tag{1.5}$$

Condition (H_3) gives

$$\left.\begin{aligned} & f_n(t, x, y, z) \leq p\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) + \gamma(t)h\left(x + \frac{1}{n}, y + \frac{1}{n}, z + \frac{1}{n}\right), \\ & \qquad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in [0, \infty)^3, \end{aligned}\right\} \tag{1.6}$$

$$\left. f_n(t, x, y, z) \leq p(x, y, z) + \gamma(t)h\left(x + \frac{1}{n}, y + \frac{1}{n}, z + \frac{1}{n}\right), \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y, z) \in \mathscr{D}. \right\} \tag{1.7}$$

We investigate the regular fractional differential equation

$$D_{0+}^{\alpha} u(t) + f_n(t, u(t), D_{0+}^{\nu} u(t), D_{0+}^{\mu} u(t)) = 0. \tag{1.8}$$

This paper is organized as follows. Section 2 contains some results of fractional calculus theory and auxiliary technical lemmas, which are used in the next two sections. Section 3 deals with the auxiliary regular problem (1.8), (1.2). We reduce the solvability of this problem to the existence of a fixed point of an operator Q_n. By the fixed point theorem of cone expansion and compression, the existence of one or at least two fixed points of Q_n is obtained. In Section 4, we prove the existence and multiplicity of positive solutions of problem (1.1), (1.2) by applying the results of Sections 2 and 3. Two examples are also presented to demonstrate the application of our results.

2. Background materials and preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature, such as [9,10,14,25,29].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : [0, 1] \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) \mathrm{d}s,$$

provided the right hand side is pointwise defined on [0, 1].

Definition 2.2. The Riemann–Liouville fractional derivative of order $\beta > 0$ of a function $\upsilon \in C(0, 1]$ is defined by

$$D_{0+}^{\beta} \upsilon(t) = \frac{1}{\Gamma(n - \beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t - s)^{n - \beta - 1} \upsilon(s) \mathrm{d}s,$$

provided that the right hand side is pointwise defined on (0, 1], where $n = [\beta] + 1$ and $[\beta]$ means the integral part of the number β. Γ is the Euler gamma function.

Lemma 2.1 ([25]). *We have*

$$I_{0+}^{\alpha}: L^1[0,1] \to \begin{cases} L^1[0,1], & \text{if } \alpha \in (0,1); \\ \mathrm{AC}^{[\alpha]-1}[0,1], & \text{if } \alpha \geq 1, \end{cases}$$

where $[\alpha]$ *means the integral part of* α *and* $\mathrm{AC}^0[0,1] = \mathrm{AC}[0,1]$.

Lemma 2.2 ([10, Lemma 2.3]). *If* $x \in L^1[0,1]$ *and* $\alpha + \beta \geq 1$, *then the equation* $\left(I_{0+}^{\alpha} I_{0+}^{\beta} x\right)(t) = \left(I_{0+}^{\alpha+\beta} x\right)(t)$ *holds at any* $t \in [0,1]$, *that is,*

$$\int_0^t (t-s)^{\alpha-1} \left(\int_0^s (s-\xi)^{\beta-1} x(\xi) \mathrm{d}\xi \right) \mathrm{d}s = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} x(s) \mathrm{d}s. \tag{2.1}$$

Lemma 2.3 ([23]). *Suppose that* $\alpha > 0, \alpha \notin \mathbb{N}$. *If* $x \in C(0,1]$ *and* $D_{0+}^{\alpha} x \in L^1[0,1]$. *Then*

$$x(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) + \sum_{k=1}^{n} c_k t^{\alpha-k}, \quad \text{for } t \in (0,1],$$

where $n = [\alpha] + 1$ *and* $c_k \in \mathbb{R}, k = 1, 2, \ldots, n$.

Lemma 2.4. *Suppose that* $\mu \in (1,2)$ *and* $x \in C^2[0,1], x(0) = x'(0) = 0$. *Then* $D_{0+}^{\mu} x \in C[0,1]$ *and*

$$D_{0+}^{\mu} x(t) = \frac{1}{\Gamma(2-\mu)} \int_0^t (t-s)^{1-\mu} x''(s) \mathrm{d}s. \tag{2.2}$$

Proof. According to the integration by parts, we have

$$\int_0^t (t-s)^{1-\mu} x(s) \mathrm{d}s = \frac{1}{2-\mu} \int_0^t (t-s)^{2-\mu} x'(s) \mathrm{d}s,$$
$$\int_0^t (t-s)^{2-\mu} x'(s) \mathrm{d}s = \frac{1}{3-\mu} \int_0^t (t-s)^{3-\mu} x''(s) \mathrm{d}s.$$

Hence

$$\begin{aligned} D_{0+}^{\mu} x(t) &= \frac{1}{\Gamma(2-\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \int_0^t (t-s)^{1-\mu} x(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(3-\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \int_0^t (t-s)^{2-\mu} x'(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(4-\mu)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \int_0^t (t-s)^{3-\mu} x''(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(2-\mu)} \int_0^t (t-s)^{1-\mu} x''(s) \mathrm{d}s. \end{aligned}$$

The fact that $\int_0^t (t-s)^{1-\mu} x''(s) \mathrm{d}s$ is continuous on $[0,1]$ gives the continuity of $D_{0+}^{\mu} x$. □

Remark 2.1. If $\mu = 2$ and $x \in C^2[0,1], x(0) = x'(0) = 0$, it is easy to obtain that $x''(t) \in C[0,1]$.

Lemma 2.5. *Suppose that* $\nu \in (0,1]$ *and* $x \in C^2[0,1], x(0) = x'(0) = 0$. *Then* $D_{0+}^{\nu} x \in C[0,1]$ *and*

$$D_{0+}^{\nu} x(t) = \frac{1}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} x''(s) \mathrm{d}s. \tag{2.3}$$

Proof. The proof is similar to Lemma 2.4, so we omit it here. □

Lemma 2.6. *Given* $g \in L^1[0,1]$, *then for* $t \in [0,1]$,

$$u(t) = \int_0^1 G(t,s) g(s) \mathrm{d}s, \tag{2.4}$$

is the unique solution in $C^2[0,1]$ *of the equation*

$$D_{0+}^{\alpha}u(t)+g(t)=0, \tag{2.5}$$

satisfying the boundary condition (1.2)*, where* $\alpha\in(3,4]$ *and*

$$G(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0\le s\le t\le 1;\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, & \text{if } 0\le t\le s\le 1.\end{cases} \tag{2.6}$$

Proof. By Lemma 2.3, we can see that

$$u(t)=\begin{cases}-I_{0+}^{\alpha}\rho(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4}, & \text{for } 3<\alpha<4;\\ -I_{0+}^{\alpha}\rho(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4}+c_5t^{\alpha-5}, & \text{for } \alpha=4,\end{cases}$$

are all solutions of (2.5) in $C(0,1]$, where $c_j\in\mathbb{R}$. Since Lemma 2.1 guarantees that $I_{0+}^{\alpha}\rho\in AC^2[0,1]$ for $3<\alpha<4$ and $I_{0+}^{\alpha}\rho\in AC^3[0,1]$ for $\alpha=4$, therefore

$$u(t)=\begin{cases}-I_{0+}^{\alpha}\rho(t)+c_1t^{\alpha-1}, & \text{for } 3<\alpha<4;\\ -I_{0+}^{\alpha}\rho(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}, & \text{for } \alpha=4\end{cases}$$

are all solutions of (2.5) in $C^2[0,1]$, where $c_1,c_2\in\mathbb{R}$. Considering that solutions should satisfy $u(0)=0$ and $u'(0)=u''(0)=u''(1)=0$, we get $c_1=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-3}\rho(s)ds$ and $c_2=0$. As a result,

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}\int_0^1(1-s)^{\alpha-3}\rho(s)ds-\int_0^t(t-s)^{\alpha-1}\rho(s)ds\right)\\&=\int_0^1G(t,s)\rho(s)ds,\end{aligned}$$

is the unique solution of problem (2.5), (1.2) in $C^2[0,1]$. □

Lemma 2.7. *Let G be as defined in* (2.6)*. Then*

(1) $G(t,s)\in C([0,1]\times[0,1])$ *and* $G(t,s)>0$ *on* $(0,1)\times(0,1)$,
(2) $G(t,s)\le\frac{1}{\Gamma(\alpha)}$ *for* $(t,s)\in[0,1]\times[0,1]$,
(3) $\int_0^1G(t,s)ds\ge\frac{t^{\alpha-1}}{(\alpha-2)\Gamma(\alpha+1)}$ *for* $t\in[0,1]$,
(4) $\frac{\partial}{\partial t}G(t,s)\in C([0,1]\times[0,1])$ *and* $\frac{\partial}{\partial t}G(t,s)>0$ *on* $(0,1)\times(0,1)$,
(5) $\frac{\partial}{\partial t}G(t,s)\le\frac{1}{\Gamma(\alpha-1)}$ *for* $(t,s)\in[0,1]\times[0,1]$,
(6) $\int_0^1\frac{\partial}{\partial t}G(t,s)ds\ge\frac{t^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}$ *for* $t\in[0,1]$,
(7) $\frac{\partial^2}{\partial t^2}G(t,s)\in C([0,1]\times[0,1])$ *and* $\frac{\partial^2}{\partial t^2}G(t,s)>0$ *on* $(0,1)\times(0,1)$,
(8) $\frac{\partial^2}{\partial t^2}G(t,s)\le\frac{1}{\Gamma(\alpha-2)}$ *for* $(t,s)\in[0,1]\times[0,1]$,
(9) $\int_0^1\frac{\partial^2}{\partial t^2}G(t,s)ds\ge\frac{t(1-t)}{\Gamma(\alpha-1)}$ *for* $t\in[0,1]$.

Proof. (1) It follows from the definition of G that G is continuous on $[0,1]\times[0,1]$. If $0<s<t<1$, then

$$t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}=(t-s)^{\alpha-3}\left(t^{\alpha-1}\left(\frac{1-s}{t-s}\right)^{\alpha-3}-(t-s)^2\right).$$

Since $(1-s)/(t-s)$ is increasing in s on $(0,t)$, we have $(1-s)/(t-s)>1/t$. Hence,

$$t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}\ge(t-s)^{\alpha-3}\left(t^2-(t-s)^2\right)\ge 0,$$

where $0<s<t<1$. In addition, it is clear that $G(t,t)=\frac{t^{\alpha-1}(1-t)^{\alpha-3}}{\Gamma(\alpha)}>0$ for $t\in(0,1)$, and $t^{\alpha-1}(1-s)^{\alpha-3}>0$ for $0<t<s<1$. Therefore we get that $G>0$ on $(0,1)\times(0,1)$.

(3) For $t \in [0, 1]$, we have

$$\begin{aligned}\int_0^1 G(t,s)\mathrm{d}s &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}\mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}\int_0^1 (1-s)^{\alpha-3}\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}\mathrm{d}s\right) \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\alpha-2}}{\alpha-1}-\frac{t^{\alpha}}{\alpha}\right) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{\alpha-t(\alpha-2)}{\alpha(\alpha-2)} \\ &\geq \frac{t^{\alpha-1}}{(\alpha-2)\Gamma(\alpha+1)}.\end{aligned}$$

(4) Since

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \dfrac{t^{\alpha-2}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & \text{if } 0 \leq s \leq t \leq 1; \\ \dfrac{t^{\alpha-2}(1-s)^{\alpha-3}}{\Gamma(\alpha-1)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

It is clear that $\frac{\partial}{\partial t}G(t,s)$ is continuous on $[0, 1] \times [0, 1]$. The proof of $\frac{\partial}{\partial t}G(t,s) > 0$ on $[0, 1] \times [0, 1]$ is supported by the following facts that:

if $0 < t \leq s < 1$, we have

$$\frac{\partial}{\partial t}G(t,s) = \frac{t^{\alpha-2}(1-s)^{\alpha-3}}{\Gamma(\alpha-1)} > 0;$$

if $0 < s < t < 1$, we have

$$\begin{aligned}t^{\alpha-2}(1-s)^{\alpha-3}-(t-s)^{\alpha-2} &= (t-s)^{\alpha-3}\left(t^{\alpha-2}\left(\frac{1-s}{t-s}\right)^{\alpha-3}-(t-s)\right) \\ &> (t-s)^{\alpha-3}s > 0.\end{aligned}$$

(6) For $t \in [0, 1]$, we have

$$\begin{aligned}\int_0^1 \frac{\partial}{\partial t}G(t,s)\mathrm{d}s &= \int_0^t \frac{t^{\alpha-2}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s + \int_t^1 \frac{t^{\alpha-2}(1-s)^{\alpha-3}}{\Gamma(\alpha-1)}\mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha-1)}\left(t^{\alpha-2}\int_0^1 (1-s)^{\alpha-3}\mathrm{d}s - \int_0^t (t-s)^{\alpha-2}\mathrm{d}s\right) \\ &= \frac{1}{\Gamma(\alpha-1)}\left(\frac{t^{\alpha-2}}{\alpha-2}-\frac{t^{\alpha}}{\alpha-1}\right) = \frac{t^{\alpha-2}\,(\alpha-1-(\alpha-2)t)}{(\alpha-2)\Gamma(\alpha)} \\ &\geq \frac{t^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}.\end{aligned}$$

(7) Since

$$\frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} \dfrac{t^{\alpha-3}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}}{\Gamma(\alpha-2)}, & \text{if } 0 \leq s \leq t \leq 1; \\ \dfrac{t^{\alpha-3}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

We can see that $\frac{\partial^2}{\partial t^2}G(t,s)$ is continuous on $[0, 1] \times [0, 1]$. The proof of $\frac{\partial^2}{\partial t^2}G(t,s) > 0$ on $[0, 1] \times [0, 1]$ is supported by the following facts that:

if $0 < t \leq s < 1$, we have

$$\frac{\partial^2}{\partial t^2}G(t,s) = \frac{t^{\alpha-3}(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} > 0;$$

if $0 < s < t < 1$, we have

$$t^{\alpha-3}(1-s)^{\alpha-3}-(t-s)^{\alpha-3} = (t-s)^{\alpha-3}\left(t^{\alpha-3}\left(\frac{1-s}{t-s}\right)^{\alpha-3}-1\right)$$
$$> (t-s)^{\alpha-3}(1-1) = 0.$$

(9) We have for $t \in [0, 1]$ that

$$\begin{aligned}\int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)\mathrm{d}s &= \int_0^t \frac{t^{\alpha-3}(1-s)^{\alpha-3}-(t-s)^{\alpha-2}}{\Gamma(\alpha-2)}\mathrm{d}s + \int_t^1 \frac{t^{\alpha-3}(1-s)^{\alpha-3}}{\Gamma(\alpha)}\mathrm{d}s\\ &= \frac{1}{\Gamma(\alpha-2)}\left(t^{\alpha-3}\int_0^1 (1-s)^{\alpha-3}\mathrm{d}s - \int_0^t (t-s)^{\alpha-3}\mathrm{d}s\right)\\ &= \frac{1}{\Gamma(\alpha-2)}\left(\frac{t^{\alpha-3}}{\alpha-2}-\frac{t^{\alpha-2}}{\alpha-2}\right) = \frac{t^{\alpha-3}(1-t)}{\Gamma(\alpha-1)}\\ &\geq \frac{(1-t)}{\Gamma(\alpha-1)}.\end{aligned}$$

It is obvious that (2), (5) and (8) hold. □

3. Auxiliary regular problem (1.8), (1.2)

Let $X = C^2[0, 1]$ and define a cone P in X as

$$P = \{x \in X : x(0) = x'(0) = 0, x(t) \geq 0, x'(t) \geq 0, x''(t) \geq 0, \text{ for } t \in [0, 1]\}.$$

By Lemmas 2.4 and 2.5 and (2.2), (2.3), we can obtain that

$$\left.\begin{array}{lll} D_{0+}^{\nu}x \in C[0,1], & D_{0+}^{\mu}x \in C[0,1] & \text{and}\\ D_{0+}^{\nu}x(t) \geq 0, & D_{0+}^{\mu}x(t) \geq 0, & \text{for } x \in P \text{ and } t \in [0,1].\end{array}\right\} \tag{3.1}$$

Define an operator Q_n on P by the formula

$$(Q_nx)(t) = \int_0^1 G(t,s)f_n(s, x(s), D_{0+}^{\nu}x(s), D_{0+}^{\mu}x(s))\mathrm{d}s. \tag{3.2}$$

Lemma 3.1. *$Q_n : P \to P$ is a completely continuous operator.*

Proof. Given $x \in P$ and let $\rho(t) = f_n(t, x(t), D_{0+}^{\nu}x(t), D_{0+}^{\mu}x(t))$. Then, by (1.3) and (3.1), we have that $\rho \in L^1[0, 1]$ and $\rho(t) \geq m$ for a.e. $t \in [0, 1]$. It follows from Lemma 3.1 that G, $\frac{\partial}{\partial t}G$, $\frac{\partial^2}{\partial t^2}G$ are nonnegative and continuous on $[0, 1] \times [0, 1]$ and $G(0, s) = 0$ for $s \in [0, 1]$. Therefore, we get $Q_nx \in C^2[0, 1]$, $(Q_nx)(0) = (Q_nx)'(0) = (Q_nx)''(0) = 0$ and $Q_nx \geq 0$, $(Q_nx)' \geq 0$, $(Q_nx)'' \geq 0$ on $[0, 1]$. As a result, $Q_n : P \to P$.

In order to prove Q_n is a continuous operator, let $\{x_k\} \subset P$ be a convergent sequence. Suppose that $\lim_{k\to\infty} x_k = x$. Then $\lim_{k\to\infty} x_k^{(j)} = x^{(j)}(t)$ uniformly on $[0, 1]$, where $j = 0, 1, 2$. Since, for $\mu \in (1, 2)$, $\nu \in (0, 1]$,

$$|D_{0+}^{\mu}x_k(t) - D_{0+}^{\mu}x(t)| \leq \frac{\|x_k''(t)-x''(t)\|}{\Gamma(2-\mu)}\int_0^1 (t-s)^{1-\mu}\mathrm{d}s \leq \frac{\|x_k''(t)-x''(t)\|}{\Gamma(3-\mu)},$$
$$|D_{0+}^{\nu}x_k(t) - D_{0+}^{\nu}x(t)| \leq \frac{\|x_k''(t)-x''(t)\|}{\Gamma(2-\nu)}\int_0^1 (t-s)^{1-\nu}\mathrm{d}s \leq \frac{\|x_k''(t)-x''(t)\|}{\Gamma(3-\nu)},$$

we have $\lim_{k\to\infty} D_{0+}^{\mu}x_k(t) = D_{0+}^{\mu}x(t)$ and $\lim_{k\to\infty} D_{0+}^{\nu}x_k(t) = D_{0+}^{\nu}x(t)$ uniformly on $[0, 1]$. In addition, it follows from (2.2) and (2.3) that, for $\mu \in (1, 2)$, $\nu \in (0, 1]$,

$$\|D_{0+}^{\mu}x_k\| \leq \frac{\|x_k''\|}{\Gamma(2-\mu)}\int_0^t (t-s)^{1-\mu}\mathrm{d}s \leq \frac{\|x_k''\|}{\Gamma(3-\mu)}, \tag{3.3}$$

$$\|D_{0+}^{\nu}x_k\| \leq \frac{\|x_k''\|}{\Gamma(2-\nu)}\int_0^t (t-s)^{1-\nu}\mathrm{d}s \leq \frac{\|x_k''\|}{\Gamma(3-\nu)}. \tag{3.4}$$

Let

$$\rho_k(t) = f_n(t, x_k(t), D_{0+}^{\nu}x_k(t), D_{0+}^{\mu}x_k(t)). \tag{3.5}$$

Then we can obtain from the Lebesgue dominated convergence theorem that $\lim_{k\to\infty}\rho_k(t)=\rho(t)$ for a.e. $t\in[0,1]$. Since $f_n\in \mathrm{Car}([0,1]\times\mathbb{R}^3)$ and $\{x_k\}$ is bounded in $C^2[0,1]$, inequalities (3.3) and (3.4) imply that $\{D^{\mu}_{0+}x_k\}$ and $\{D^{\nu}_{0+}x_k\}$ are bounded in $C[0,1]$. It is obvious that $\{D^{\mu}_{0+}x_k\}$ is also bounded in $C[0,1]$ for $\mu=2$. As a result, there exists $\varphi\in L^1[0,1]$, such that

$$m\le\rho_k(t)\le\varphi(t),\quad \text{for a.e. } t\in[0,1] \text{ and all } k\in\mathbb{N}. \tag{3.6}$$

It follows from the Lebesgue dominated convergence theorem and from the following relations (see Lemma 2.7)

$$|(Q_nx_k)(t)-(Q_nx)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^1|\rho_k(s)-\rho(s)|\mathrm{d}s,$$
$$|(Q_nx_k)'(t)-(Q_nx)'(t)|\le\frac{1}{\Gamma(\alpha-1)}\int_0^1|\rho_k(s)-\rho(s)|\mathrm{d}s,$$
$$|(Q_nx_k)''(t)-(Q_nx)''(t)|\le\frac{1}{\Gamma(\alpha-2)}\int_0^1|\rho_k(s)-\rho(s)|\mathrm{d}s,$$

that $\lim_{k\to\infty}(Q_nx_k)^{(j)}(t)=(Q_nx)^{(j)}(t)$ uniformly on $[0,1]$, where $j=0,1,2$. Consequently, Q_n is a continuous operator.

Now, what we should do is to prove that, for any bounded sequence $\{x_k\}\subset P$, the sequence $\{Q_nx_k\}$ is relatively compact in X. In order to apply the Arzelà–Ascoli theorem, we need to prove that $\{Q_nx_k\}$ is bounded in X and $\{(Q_nx_k)''\}$ is equicontinuous on $[0,1]$. Let $\{x_k\}\in P$ be bounded and suppose that S is a positive number and $\|x_k\|<S$, $\|x_k'\|<S$, $\|x_k''\|<S$, for $k\in\mathbb{N}$. Then, (3.3) implies that $\|D^{\mu}_{0+}x_k\|\le\frac{S}{\Gamma(3-\mu)}$ and (3.4) implies that $\|D^{\nu}_{0+}x_k\|\le\frac{S}{\Gamma(3-\nu)}$. Put ρ_k as in (3.5). Then (3.6) holds for some $\varphi\in L^1[0,1]$. Since

$$0\le(Q_nx_k)(t)=\int_0^1G(t,s)\rho_k(s)\mathrm{d}s\le\frac{1}{\Gamma(\alpha)}\int_0^1\varphi(s)\mathrm{d}s=\frac{\|\varphi\|_1}{\Gamma(\alpha)},$$
$$0\le(Q_nx_k)'(t)=\int_0^1\frac{\partial}{\partial t}G(t,s)\rho_k(s)\mathrm{d}s\le\frac{1}{\Gamma(\alpha-1)}\int_0^1\varphi(s)\mathrm{d}s=\frac{\|\varphi\|_1}{\Gamma(\alpha-1)},$$
$$0\le(Q_nx_k)''(t)=\int_0^1\left(\frac{\partial}{\partial t}\right)^2G(t,s)\rho_k(s)\mathrm{d}s\le\frac{1}{\Gamma(\alpha-2)}\int_0^1\varphi(s)\mathrm{d}s=\frac{\|\varphi\|_1}{\Gamma(\alpha-2)},$$

it shows that $\{Q_nx_k\}$ is bounded in X. In addition, for $0\le t_1\le t_2\le 1$, the following relation,

$$\begin{aligned}
|(Q_nx_k)''(t_2)-(Q_nx_k)''(t_1)| &\le \frac{t_2^{\alpha-3}-t_1^{\alpha-3}}{\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\rho_k\mathrm{d}s+\frac{1}{\Gamma(\alpha-2)}\left|\int_0^{t_2}(t_2-s)^{\alpha-3}\rho_k\mathrm{d}s\right.\\
&\quad\left.-\int_0^{t_1}(t_1-s)^{\alpha-3}\rho_k(s)\mathrm{d}s\right|\\
&\le\frac{\|\rho_k\|_1}{\Gamma(\alpha-2)}(t_2^{\alpha-3}-t_1^{\alpha-3})+\frac{1}{\Gamma(\alpha-2)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-3}\rho_k(s)\mathrm{d}s\right.\\
&\quad\left.+\int_0^{t_1}\left((t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}\right)\rho_k(s)\mathrm{d}s\right]\\
&\le\frac{\|\varphi\|_1}{\Gamma(\alpha-2)}(t_2^{\alpha-3}-t_1^{\alpha-3})+\frac{1}{\Gamma(\alpha-2)}\\
&\quad\times\left[(t_2-t_1)^{\alpha-3}\|\varphi\|_1+\int_0^{t_1}\left((t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}\right)\varphi(s)\mathrm{d}s\right],
\end{aligned}$$

holds. Now, we choose any $\epsilon>0$. Since $t^{\alpha-3}$ is uniformly continuous on $[0,1]$ and $|t-s|^{\alpha-2}$ on $[0,1]\times[0,1]$, there exists $\delta>0$ such that for each $0\le t_1<t_2\le1, t_2-t_1<\delta, 0\le s\le t$, we have $0<t_2^{\alpha-3}-t_1^{\alpha-3}<\epsilon$, $0<(t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}<\epsilon$. Hence, for $k\in\mathbb{N}$, $0\le t_1<t_2\le1$ and $t_2-t_1<\min\{\delta,\sqrt[\alpha-3]{\epsilon}\}$, the inequality,

$$|(Q_nx_k)''(t_2)-(Q_nx_k)''(t_1)|\le\frac{3\varepsilon}{\Gamma(\alpha-2)}\|\varphi\|_1,$$

holds. As a result, $\{(Q_nx_k)''\}$ is equicontinuous on $[0,1]$. □

Lemma 3.2 ([14]). *Let Y be a Banach space, and $P\subset Y$ be a cone in Y. Let Ω_1,Ω_2 be bounded open balls of Y centered at the origin with $\overline{\Omega}_1\subset\Omega_2$. Suppose that $\mathscr{A}:P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ is a completely continuous operator such that*

$$\|\mathscr{A}x\|\ge\|x\|,\quad \textit{for } t\in P\cap\partial\Omega_1,\qquad \|\mathscr{A}x\|\le\|x\|,\quad \textit{for } t\in P\cap\partial\Omega_2$$

holds. Then $\mathscr{A}$ has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Theorem 3.1. *Let* (H_1) *and* (H_2) *(or* (H_3)*) hold. Then problem (1.8), (1.2) has a solution* $u_n \in P$ *and*

$$u_n(t) \geq Mt^{\alpha-1}, \quad \text{for } t \in [0, 1]. \tag{3.7}$$

Proof. By Lemma 3.1, $Q_n : P \to P$ is a completely continuous operator. Suppose $x \in P$, then (1.3) and Lemma 2.7 yield that

$$(Q_n x)(t) \geq m \int_0^1 G(t, s)\mathrm{d}s \geq Mt^{\alpha-1}, \quad \text{for } t \in [0, 1], \tag{3.8}$$

where M is defined by (H_2) (or (H_3)). Therefore $\|Q_n x\| \geq M$, $\|Q_n x\|_* \geq M$ for $x \in P$. Suppose $\Omega_1 = \{x \in X : \|x\|_* < M\}$, then

$$\|Q_n x\|_* \geq \|x\|_*, \quad \text{for } x \in P \cap \partial\Omega_1. \tag{3.9}$$

Let $W_n = p(1/n, 1/n, 1/n)$. Lemma 2.7 and (1.4) imply that, for $x \in P$ and $t \in [0, 1]$,

$$\begin{aligned}
0 &\leq (Q_n x)''(t) \\
&\leq \frac{1}{\Gamma(\alpha-2)} \int_0^1 \left(W_n + \gamma(s) h \left(x(s) + \frac{1}{n}, D_{0+}^{\nu} x(s) + \frac{1}{n}, D_{0+}^{\mu} x(s) + \frac{1}{n} \right) \right) \mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(\|x\| + \frac{1}{n}, \|D_{0+}^{\nu} x\| + \frac{1}{n}, \|D_{0+}^{\mu} x\| + \frac{1}{n} \right) \|\gamma\|_1 \right), \\
0 &\leq (Q_n x)'(t) \\
&= \int_0^t (Q_n x)''(s)\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(\|x\| + \frac{1}{n}, \|D_{0+}^{\nu} x\| + \frac{1}{n}, \|D_{0+}^{\mu} x\| + \frac{1}{n} \right) \|\gamma\|_1 \right), \\
0 &\leq (Q_n x)(t) \\
&= \int_0^t (Q_n x)'(s)\mathrm{d}s \\
&\leq \frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(\|x\| + \frac{1}{n}, \|D_{0+}^{\nu} x\| + \frac{1}{n}, \|D_{0+}^{\mu} x\| + \frac{1}{n} \right) \|\gamma\|_1 \right).
\end{aligned}$$

In view of (3.3) and (3.4), hence for $x \in P$,

$$\|Q_n x\|_* \leq \frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(\|x\|_* + \frac{1}{n}, \frac{\|x\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|x\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right). \tag{3.10}$$

If (H_2) holds. By assumption that $\lim_{x\to\infty} h(x, x, x)/x = 0$, there exists $S > M > 0$ such that

$$\frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(S + \frac{1}{n}, \frac{S}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{S}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right) \leq S. \tag{3.11}$$

Suppose $\Omega_2 = \{x \in X : \|x\|_* < S\}$. Then it follows from (3.10) and (3.11) that

$$\|Q_n x\|_* \leq \|x\|_*, \quad \text{for } x \in P \cap \partial\Omega_2. \tag{3.12}$$

Applying Lemma 3.2, we conclude from (3.9) and (3.12) that the operator Q_n has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, hence u_n is a solution of problem (1.8), (1.2), and (3.8) guarantees that u_n fulfills (3.7).

If (H_3) holds. By assumption that $\lim_{x\to 0} h(x, x, x)/x = 0$, there exists $0 < L < M$ and a large enough number N_1 such that, for $n \geq N_1$,

$$\frac{1}{\Gamma(\alpha-2)} \left(W_n + h \left(L + \frac{1}{n}, \frac{L}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{L}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right) \leq L. \tag{3.13}$$

Suppose $\Omega_3 = \{x \in X : \|x\|_* < L\}$. Then it follows from (3.10) and (3.13) that

$$\|Q_n x\|_* \leq \|x\|_*, \quad \text{for } x \in P \cap \partial\Omega_3. \tag{3.14}$$

By applying Lemma 3.2, we conclude from (3.9) and (3.14) that the operator Q_n has a fixed point in $P \cap (\overline{\Omega}_1 \setminus \Omega_3)$, hence u_n is a solution of problem (1.8), (1.2) and (3.8) guarantees that u_n fulfills (3.7). □

In order to prove the main results, we also need the following lemmas.

Lemma 3.3. *If* (H_1) *and* (H_2) *(or* (H_3)*) hold. Let* u_n *be a solution of problem (1.8), (1.2). Then the sequence* $\{u_n\}$ *is relatively compact in* X.

Proof. Note that

$$u_n(t) = \int_0^1 G(t,s) f_n(s, x_n(s), D_{0+}^{\nu} x_n(s), D_{0+}^{\mu} x_n(s)) ds, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}, \tag{3.15}$$

and u_n fulfills (3.7). Moreover, Lemma 2.7 and (1.3) imply

$$u_n'(t) \geq m \int_0^1 \frac{\partial}{\partial t} G(t,s) ds \geq \frac{m t^{\alpha-2}}{(\alpha-2)\Gamma(\alpha)}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}, \tag{3.16}$$

$$u_n''(t) \geq m \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s) ds \geq \frac{m t(1-t)}{\Gamma(\alpha-1)}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}. \tag{3.17}$$

By (2.2) and (2.3),

$$\begin{aligned} D_{0+}^{\mu} u_n(t) &= \frac{1}{\Gamma(2-\mu)} \int_0^t (t-s)^{1-\mu} u_n''(s) ds \\ &\geq \frac{m}{\Gamma(2-\mu)\Gamma(\alpha-1)} \int_0^t (t-s)^{1-\mu} s(1-s) ds, \\ D_{0+}^{\nu} u_n(t) &= \frac{1}{\Gamma(2-\nu)} \int_0^t (t-s)^{1-\nu} u_n''(s) ds \\ &\geq \frac{m}{\Gamma(2-\nu)\Gamma(\alpha-1)} \int_0^t (t-s)^{1-\nu} s(1-s) ds. \end{aligned}$$

Since

$$\begin{aligned} \int_0^t (t-s)^{1-\mu} s(1-s) ds &= \frac{1}{2-\mu} \int_0^t (t-s)^{2-\mu} (1-2s) ds \\ &= \frac{1}{2-\mu} \left(\frac{t^{3-\mu}}{3-\mu} - \frac{2t^{4-\mu}}{(3-\mu)(4-\mu)} \right) \\ &= \frac{t^{3-\mu}}{2-\mu} \left(\frac{4-\mu-2t}{(3-\mu)(4-\mu)} \right) \\ &\geq \frac{t^{3-\mu}}{2-\mu} \left(\frac{2-\mu}{(3-\mu)(4-\mu)} \right) \\ &= \frac{t^{3-\mu}}{(3-\mu)(4-\mu)}, \end{aligned}$$

and

$$\int_0^t (t-s)^{1-\nu} s(1-s) ds \geq \frac{t^{3-\nu}}{(3-\nu)(4-\nu)}.$$

Hence,

$$D_{0+}^{\mu} u_n(t) \geq \frac{m(2-\mu)}{\Gamma(5-\mu)\Gamma(\alpha-1)} t^{3-\mu}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}, \tag{3.18}$$

$$D_{0+}^{\nu} u_n(t) \geq \frac{m(2-\nu)}{\Gamma(5-\nu)\Gamma(\alpha-1)} t^{3-\nu}, \quad \text{for } t \in [0,1],\ n \in \mathbb{N}. \tag{3.19}$$

As $m \cdot \min\{\frac{1}{(\alpha-2)\Gamma(\alpha+1)}, \frac{1}{(\alpha-2)\Gamma(\alpha)}, \frac{1}{(\alpha-1)}\} = M$ and $\Gamma(5-\mu) < \Gamma(4) = 6$, it follows from (3.7), (3.18) and (3.19) that, for $t \in [0,1]$, $n \in \mathbb{N}$,

$$u_n(t) \geq M t^{\alpha-1}, \qquad D_{0+}^{\nu} u_n(t) \geq \frac{(2-\nu)M}{6} t^{3-\nu}, \qquad D_{0+}^{\mu} u_n(t) \geq \frac{(2-\mu)M}{6} t^{3-\mu}. \tag{3.20}$$

Therefore, for $t \in [0,1]$, $n \in \mathbb{N}$,

$$p\left(u_n(t), D_{0+}^{\nu} u_n(t), D_{0+}^{\mu} u_n(t)\right) \leq p\left(M t^{\alpha-1}, \frac{(2-\nu)M}{6} t^{3-\nu}, \frac{(2-\mu)M}{6} t^{3-\mu}\right), \tag{3.21}$$

and it follows from Lemma 2.7, (1.5) (or (1.7)), (3.3), (3.4), (3.15) and (3.21) that

$$\begin{aligned}
0 \ &\le \ u_n''(t) = \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s) f_n(s, u_n(s), D_{0+}^{\nu} u_n(s), D_{0+}^{\mu} u_n(s)) \mathrm{d}s \\
&\le \ \frac{1}{\Gamma(\alpha-2)} \int_0^1 p\left(Ms^{\alpha-1}, \frac{(2-\nu)M}{6} s^{3-\nu}, \frac{(2-\mu)M}{6} s^{3-\mu} \right) \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha-2)} h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \int_0^1 \gamma(s) \mathrm{d}s \\
&= \ \frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right), \\
0 \ &\le \ u_n'(t) = \int_0^t u_n''(s) \mathrm{d}s \\
&\le \ \frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right), \\
0 \ &\le \ u_n(t) = \int_0^t u_n'(s) \mathrm{d}s \\
&\le \ \frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right),
\end{aligned}$$

where $t \in [0,1]$, $n \in \mathbb{N}$ and $\Lambda = \int_0^1 p\left(Ms^{\alpha-1}, \frac{(2-\nu)M}{6} s^{3-\nu}, \frac{(2-\mu)M}{6} s^{3-\mu} \right) \mathrm{d}s$.

If (H_2) holds. It follows from the assumption that $\Lambda < \infty$. Hence,

$$\|u_n\|_* \le \frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right),$$

where $n \in \mathbb{N}$. Since $\lim_{x\to\infty} h(x,x,x)/x = 0$, there exists $L > 0$ such that, for $\upsilon \ge L$,

$$\frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\upsilon + \frac{1}{n}, \frac{\upsilon}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\upsilon}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right) < \upsilon.$$

Consequently, $\|u_n\|_* < L$ for $n \in \mathbb{N}$, so that $\{u_n\}$ is bounded in X. We are now in a position to prove $\{u_n''\}$ is equicontinuous on $[0,1]$. Let $V_1 = h\left(L + \frac{1}{n}, \frac{L}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{L}{\Gamma(3-\mu)} + \frac{1}{n} \right)$ and

$$\Phi(t) = p\left(Mt^{\alpha-1}, \frac{(2-\nu)M}{6} t^{3-\nu}, \frac{(2-\mu)M}{6} t^{3-\mu} \right), \quad \text{for } t \in (0,1]. \tag{3.22}$$

Then $\Lambda = \int_0^1 \Phi(t) \mathrm{d}t$ and, for a.e. $t \in [0,1]$, all $n \in \mathbb{N}$,

$$f_n(t, u_n(t), D_{0+}^{\nu} u_n(t), D_{0+}^{\mu} u_n(t)) \le \Phi(t) + V_1 \gamma(t)$$

holds. Suppose that $0 \le t_1 < t_2 \le 1$, then

$$\begin{aligned}
|u_n''(t_2) - u_n''(t_1)| &= \left| \int_0^1 \left(\left(\frac{\partial}{\partial t} \right)^2 G(t_2, s) - \left(\frac{\partial}{\partial t} \right)^2 G(t_1, s) \right) f_n(s, u_n(s), D_{0+}^{\nu} u_n(s), D_{0+}^{\mu} u_n(s)) \mathrm{d}s \right| \\
&\le \frac{1}{\Gamma(\alpha-2)} \left[(t_2^{\alpha-3} - t_1^{\alpha-3}) \int_0^1 (\Phi(s) + V_1 \gamma(s)) \mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-3} (\Phi(s) + V_1 \gamma(s)) \mathrm{d}s \right. \\
&\quad \left. + \int_0^{t_1} \left((t_2 - s)^{\alpha-3} - (t_1 - s)^{\alpha-3} \right) (\Phi(s) + V_1 \gamma(s)) \mathrm{d}s \right] \\
&\le \frac{1}{\Gamma(\alpha-2)} \left[(t_2^{\alpha-3} - t_1^{\alpha-3}) (\Lambda + V_1 \|\gamma\|_1) + (t_2 - t_1)^{\alpha-3} (\Lambda + V_1 \|\gamma\|_1) \right. \\
&\quad \left. + \int_0^{t_1} \left((t_2 - s)^{\alpha-3} - (t_1 - s)^{\alpha-3} \right) (\Phi(s) + V_1 \gamma(s)) \mathrm{d}s \right].
\end{aligned}$$

The proof is similar with that of Lemma 3.1. We choose $\varepsilon > 0$. Then there exists $\delta_0 > 0$ such that $t_2^{\alpha-3} - t_1^{\alpha-3} < \varepsilon$, $(t_2 - s)^{\alpha-3} - (t_1 - s)^{\alpha-3} < \varepsilon$, for any $0 \le t_1 < t_2 \le 1$, $t_2 - t_1 < \delta_0$ and $0 \le s \le t_1$ suppose that $0 < \delta < \min\{\delta_0, \sqrt[\alpha-3]{\varepsilon}\}$. Then, for $t_1, t_2 \in [0,1]$, $0 < t_2 - t_1 < \delta$, $n \in \mathbb{N}$, we have $|u_n''(t_2) - u_n''(t_1)| \le \frac{3\varepsilon}{\Gamma(\alpha-2)} (\Lambda + V_1 \|\gamma\|_1)$. As a result, $\{u_n''\}$ is equicontinuous on $[0,1]$.

If (H_3) holds. It follows from the assumption that $\Lambda < \infty$. Hence,

$$\|u_n\|_* \le \frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\|u_n\|_* + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\|u_n\|_*}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right),$$

where $n \in \mathbb{N}$. By $\lim_{x\to 0} h(x,x,x)/x = 0$, there exists $L > 0$ and a large enough number N_1 such that, for $n \ge N_1$ and $\upsilon \le L$,

$$\frac{1}{\Gamma(\alpha-2)} \left(\Lambda + h\left(\upsilon + \frac{1}{n}, \frac{\upsilon}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{\upsilon}{\Gamma(3-\mu)} + \frac{1}{n} \right) \|\gamma\|_1 \right) < \upsilon.$$

Consequently, $\|u_n\|_* < L$ for $n \in \mathbb{N}$ and $n \ge N_1$, so that $\{u_n\}$ is bounded in X. Let $V_2 = h\left(L + \frac{1}{N_1}, \frac{L}{\Gamma(3-\nu)} + \frac{1}{N_1}, \frac{L}{\Gamma(3-\mu)} + \frac{1}{N_1}\right)$. Then $\Lambda = \int_0^1 \Phi(t)\mathrm{d}t$ and, for a.e. $t \in [0,1]$, all $n \in \mathbb{N}$,

$$f_n(t, u_n(t), D_{0+}^{\nu} u_n(t), D_{0+}^{\mu} u_n(t)) \le \Phi(t) + V_2 \gamma(t).$$

The proof is similar with above. As a result, $\{u_n''\}$ is also equicontinuous on $[0,1]$. □

4. Main results

Theorem 4.1. *Let* (H_1) *and* (H_2) *(or* (H_3)*) hold. Then the problem (1.1), (1.2) has a positive solution u and, for* $t \in [0,1]$,

$$u(t) \ge Mt^{\alpha-1}, \qquad D_{0+}^{\nu} u(t) \ge \frac{(2-\nu)M}{6} t^{3-\nu}, \qquad D_{0+}^{\mu} u(t) \ge \frac{(2-\mu)M}{6} t^{3-\mu}. \tag{4.1}$$

Proof. Theorem 3.1 shows us that problem (1.8), (1.2) has a solution $u_n \in P$. In addition, Lemma 3.3 provides us that $\{u_n\}$ is relatively compact in X and satisfies inequality (3.20) for $t \in [0,1]$, $n \in \mathbb{N}$. Now the Arzelà–Ascoli theorem can be applied. Without loss of generality, assume that $\{u_n\}$ is convergent in X and $\lim_{n\to\infty} u_n = u$. Then $u \in P$ fulfills the boundary conditions (1.2), and it follows from (2.2) and (2.3) that $\lim_{n\to\infty} D_{0+}^{\mu} u_n = D_{0+}^{\mu} u$ and $\lim_{n\to\infty} D_{0+}^{\nu} u_n = D_{0+}^{\nu} u$ in $C[0,1]$. Now, passing to the limit as $n \to \infty$ in (3.20). Hence, u satisfies (4.1). Furthermore, since, for a.e. $t \in [0,1]$,

$$\lim_{n\to\infty} f_n(t, u_n(t), D_{0+}^{\nu} u_n(t), D_{0+}^{\mu} u_n(t)) = f(t, u(t), D_{0+}^{\nu} u(t), D_{0+}^{\mu} u(t)).$$

Suppose $K = \sup\{\|u_n\|_* : n \in \mathbb{N}\}$. Then it follows from (3.3) and (3.4) that $\|D_{0+}^{\mu} u_n\| \le \frac{K}{\Gamma(3-\mu)}$ and $\|D_{0+}^{\nu} u_n\| \le \frac{K}{\Gamma(3-\nu)}$ for $n \in \mathbb{N}$. Hence, for a.e. $(t,s) \in [0,1] \times [0,1]$ and *all* $u_n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\le G(t,s) f_n(s, u_n(s), D_{0+}^{\nu} u_n(s), D_{0+}^{\mu} u_n(s)) \\ &\le \frac{1}{\Gamma(\alpha)} \left(\Phi(s) + h\left(K + \frac{1}{n}, \frac{K}{\Gamma(3-\nu)} + \frac{1}{n}, \frac{K}{\Gamma(3-\mu)} + \frac{1}{n} \right) \gamma(s) \right), \end{aligned}$$

where Φ is defined by (3.22). Putting $n \to \infty$, we can conclude

$$u(t) = \int_0^1 G(t,s) f(s, u(s), D_{0+}^{\nu} u(s), D_{0+}^{\mu} u(s)) \mathrm{d}s, \quad \text{for } t \in [0,1],$$

by the Lebesgue dominated convergence theorem. Consequently, u is a positive solution of problem (1.1), (1.2) and satisfies inequality (4.1). □

From Theorem 4.1, we can easily derive the following theorem:

Theorem 4.2. *Let* (H_1), (H_2) *and* (H_3) *hold. Then the problem (1.1), (1.2) has at least two positive solution* u_1, u_2 *and, for* $t \in [0,1]$, $i = 1, 2$,

$$N \le \|u_2\|_* \le M \le \|u_1\|_* \le S, \tag{4.2}$$

$$u_i(t) \ge Mt^{\alpha-1}, \qquad D_{0+}^{\nu} u_i(t) \ge \frac{(2-\nu)M}{6} t^{3-\nu}, \qquad D_{0+}^{\mu} u_i(t) \ge \frac{(2-\mu)M}{6} t^{3-\mu}. \tag{4.3}$$

Example 4.1. Suppose that $\rho \in L^1[0,1]$ and m is a positive constant. Let $a \in (0, \frac{1}{\alpha-1})$, $b \in (0, \frac{1}{3-\nu})$, $c \in (0, \frac{1}{3-\mu})$ and $a_1, b_1, c_1 \in (0,1)$. Then the function

$$f(t,x,y,z) = x^{-a} + y^{-b} + z^{-c} + m + |\rho(t)|(x^{a_1} + y^{b_1} + z^{c_1})$$

satisfies conditions (H_1) and (H_2) for

$$p(x, y, z) = x^{-a} + y^{-b} + z^{-c} + m,$$
$$\gamma(t) = |\rho(t)| + m,$$
$$h(x, y, z) = x^{a_1} + y^{b_1} + z^{c_1}.$$

Hence, Theorem 4.1 guarantees that for each $\alpha \in (3, 4]$, $\nu \in (0, 1]$ and $\mu \in (1, 2]$, the fractional differential equation,

$$D_{0+}^{\alpha}u(t) = (u(t))^{-a} + \left(D_{0+}^{\nu}u(t)\right)^{-b} + \left(D_{0+}^{\mu}u(t)\right)^{-c} + m + |\rho(t)| \left((u(t))^{a_1} + \left(D_{0+}^{\nu}u(t)\right)^{b_1} + \left(D_{0+}^{\mu}u(t)\right)^{c_1}\right),$$

has a positive solution u satisfying the boundary condition (1.2) and inequality (4.1) for $t \in [0, 1]$, where $M = \frac{m}{(\alpha-2)\Gamma(\alpha+1)}$.

Example 4.2. Suppose that $\rho \in L^1[0, 1]$ and m is a positive constant. Let $a \in (0, \frac{1}{\alpha-1})$, $b \in (0, \frac{1}{3-\nu})$, $c \in (0, \frac{1}{3-\mu})$ and $a_1, b_1, c_1 \in (1, 2)$. Then the function

$$f(t, x, y, z) = x^{-a} + y^{-b} + z^{-c} + m + |\rho(t)|(x^{a_1} + y^{b_1} + z^{c_1})$$

satisfies conditions (H_1) and (H_3) for

$$p(x, y, z) = x^{-a} + y^{-b} + z^{-c} + m,$$
$$\gamma(t) = |\rho(t)| + m,$$
$$h(x, y, z) = x^{a_1} + y^{b_1} + z^{c_1}.$$

Hence, Theorem 4.1 guarantees that for each $\alpha \in (3, 4]$, $\nu \in (0, 1]$ and $\mu \in (1, 2]$, the fractional differential equation,

$$D_{0+}^{\alpha}u(t) = (u(t))^{-a} + \left(D_{0+}^{\nu}u(t)\right)^{-b} + \left(D_{0+}^{\mu}u(t)\right)^{-c} + m + |\rho(t)| \left((u(t))^{a_1} + \left(D_{0+}^{\nu}u(t)\right)^{b_1} + \left(D_{0+}^{\mu}u(t)\right)^{c_1}\right),$$

has a positive solution u satisfying the boundary condition (1.2) and inequality (4.1) for $t \in [0, 1]$, where $M = \frac{m}{(\alpha-2)\Gamma(\alpha+1)}$.

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