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DISCRETE MATHEMATICS

Note

Calkin's binomial identity

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Abstract

We give a fairly direct proof of an identity involving powers of sums of binomial coefficients established recently by Neil J. Calkin, and present some associated formulae.

1. Introduction

In a recent communication, Neil J. Calkin defines

$$A_k = \sum_{j=0}^{k} \binom{n}{j}$$

and shows in a somewhat indirect manner that

$$A_0^3 + \cdots + A_n^3 = (n+2)2^{3n-1} - 3 \times 2^{n-2} n \binom{2n}{n}.$$

Let us write

$$S_p = A_0^p + \cdots + A_n^p.$$

I shall prove, as directly as I can, that

$$S_1 = (n+2)2^{n-1}, \qquad S_2 = (n+2)2^{2n-1} - \frac{1}{2} n \binom{2n}{n},$$

$$S_3 = (n+2)2^{3n-1} - 3 \times 2^{n-2} n \binom{2n}{n}.$$

[1] This research was carried out while the author was on leave from UNSW and visiting Universite Louis-Pasteur, Strasbourg, France.

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Furthermore, I shall show that, for $p > 0$,

$$S_{2p} = \binom{p}{1} 2^n S_{2p-1} - \binom{p}{2} 2^{2n} S_{2p-2} + - \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} S_p + (-1)^p P_p,$$

$$S_{2p+1} = \binom{p}{1} 2^n S_{2p} - \binom{p}{2} 2^{2n} S_{2p-1} + - \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} S_{p+1} + (-1)^p 2^{n-1} P_p,$$

where

$$P_p = A_0^p A_{n-1}^p + \cdots + A_{n-1}^p A_0^p.$$

2. Evaluation of S_1, S_2, S_3

First,

$$\begin{aligned} S_1 &= A_0 + \cdots + A_n = \tfrac{1}{2}[(A_0 + A_{n-1}) + \cdots + (A_{n-1} + A_0)] + A_n \\ &= \tfrac{1}{2}[2^n + \cdots + 2^n] + 2^n = \tfrac{1}{2}[n2^n] + 2^n \\ &= (n+2)2^{n-1}. \end{aligned}$$

Next we evaluate P_1:

$$\begin{aligned} P_1 &= A_0 A_{n-1} + \cdots + A_{n-1} A_0 \\ &= \binom{n}{0}\left[\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right] \\ &\quad + \left[\binom{n}{0} + \binom{n}{1}\right]\left[\binom{n}{2} + \cdots + \binom{n}{n}\right] + \cdots \\ &\quad + \left[\binom{n}{0} + \cdots + \binom{n}{n-1}\right]\binom{n}{n} \\ &= \sum_{j<k} \binom{n}{j}\binom{n}{k}(k-j). \end{aligned}$$

The term $(k-j)$ is the number of occurrences of the term $\binom{n}{j}\binom{n}{k}$ in the sum above Therefore

$$P_1 = \sum_{r=1}^{n} r \sum_{k-j=r} \binom{n}{j}\binom{n}{k} = \sum_{r=1}^{n} r \binom{2n}{n+r}.$$

The inner sum, $\sum_{k-j=r}\binom{n}{j}\binom{n}{k}$, is the coefficient of t^r in $(1+t^{-1})^n(1+t)^n = t^{-n}(1+t)^{2n}$, which is $\binom{2n}{n+r}$. The case $r=0$ is well-known. Hence

$$\begin{aligned} P_1 &= \sum_{r=1}^{n} n\left\{\binom{2n-1}{n-r} - \binom{2n-1}{n-r-1}\right\} = n\binom{2n-1}{n-1} \\ &= \frac{1}{2}n\binom{2n}{n}. \end{aligned}$$

Now,

$$\begin{aligned} S_2 &= A_0^2 + \cdots + A_n^2 \\ &= A_0(2^n - A_{n-1}) + \cdots + A_{n-1}(2^n - A_0) + A_n^2 \\ &= 2^n(A_0 + \cdots + A_{n-1}) - (A_0A_{n-1} + \cdots + A_{n-1}A_0) + A_n^2 \\ &= 2^n(A_0 + \cdots + A_n) - (A_0A_{n-1} + \cdots + A_{n-1}A_0) \\ &= 2^nS_1 - P_1 = 2^n\left[(n+2)2^{n-1}\right] - \frac{1}{2}n\binom{2n}{n} \\ &= (n+2)2^{2n-1} - \frac{1}{2}n\binom{2n}{n}, \end{aligned}$$

and

$$\begin{aligned} S_3 &= A_0^3 + \cdots + A_n^3 \\ &= \tfrac{1}{2}[(A_0^3 + A_{n-1}^3) + \cdots + (A_{n-1}^3 + A_0^3)] + A_n^3 \\ &= \tfrac{1}{2}[(A_0 + A_{n-1})(A_0^2 - A_0A_{n-1} + A_{n-1}^2) + \cdots \\ &\quad + (A_{n-1} + A_0)(A_{n-1}^2 - A_{n-1}A_0 + A_0^2)] + A_n^3 \\ &= \tfrac{1}{2}[2^n(A_0^2 - A_0A_{n-1} + A_{n-1}^2) + \cdots \\ &\quad + 2^n(A_{n-1}^2 - A_{n-1}A_0 + A_0^2)] + A_n^3 \\ &= 2^n(A_0^2 + \cdots + A_{n-1}^2) - 2^{n-1}(A_0A_{n-1} + \cdots + A_{n-1}A_0) + A_n^3 \\ &= 2^n(A_0^2 + \cdots + A_n^2) - 2^{n-1}(A_0A_{n-1} + \cdots + A_{n-1}A_0) \\ &= 2^nS_2 - 2^{n-1}P_1 = 2^n\left[(n+2)2^{2n-1} - \frac{1}{2}n\binom{2n}{n}\right] - 2^{n-1}\frac{1}{2}n\binom{2n}{n} \\ &= (n+2)2^{3n-1} - 3 \times 2^{n-2}n\binom{2n}{n}. \end{aligned}$$

3. The general formulae

Suppose $p > 0$. Then

$$
\begin{aligned}
P_p &= A_0^p A_{n-1}^p + \cdots + A_{n-1}^p A_0^p \\
&= A_0^p(2^n - A_0)^p + \cdots + A_{n-1}^p(2^n - A_{n-1})^p + A_n^p(2^n - A_n)^p \\
&= A_0^p \left(2^{pn} - \binom{p}{1} 2^{(p-1)n} A_0 + - \cdots + (-1)^p \binom{p}{p} A_0^p\right) + \cdots \\
&\quad + A_n^p \left(2^{pn} - \binom{p}{1} 2^{(p-1)n} A_n + - \cdots + (-1)^p \binom{p}{p} A_n^p\right) \\
&= 2^{pn} S_p - \binom{p}{1} 2^{(p-1)n} S_{p+1} + - \cdots + (-1)^p \binom{p}{p} S_{2p},
\end{aligned}
$$

so

$$
\begin{aligned}
S_{2p} &= \binom{p}{1} 2^n S_{2p-1} - \binom{p}{2} 2^{2n} S_{2p-2} + - \cdots + (-1)^{p-1} \binom{p}{p} 2^{pn} S_p \\
&\quad + (-1)^p P_p.
\end{aligned}
$$

Also,

$$
\begin{aligned}
S_{2p+1} &= A_0^{2p+1} + \cdots + A_n^{2p+1} \\
&= \tfrac{1}{2}[(A_0^{2p+1} + A_{n-1}^{2p+1}) + \cdots + (A_{n-1}^{2p+1} + A_0^{2p+1})] + A_n^{2p+1} \\
&= \tfrac{1}{2}[(A_0 + A_{n-1})(A_0^{2p} - + \cdots + (-1)^p A_0^p A_{n-1}^p + \cdots + A_{n-1}^{2p}) + \cdots \\
&\quad + (A_{n-1} + A_0)(A_{n-1}^{2p} - + \cdots + (-1)^p A_{n-1}^p A_0^p + \cdots + A_0^{2p})] \\
&\quad + A_n^{2p+1} \\
&= \tfrac{1}{2} 2^n [(A_0^{2p} - + \cdots + (-1)^p A_0^p A_{n-1}^p + \cdots + A_{n-1}^{2p}) + \cdots \\
&\quad + (A_{n-1}^{2p} - + \cdots + (-1)^p A_{n-1}^p A_0^p + \cdots + A_0^{2p})] + A_n^{2p+1} \\
&= \tfrac{1}{2} 2^n [2(A_0^{2p} + \cdots + A_{n-1}^{2p}) - 2(A_0^{2p-1} A_{n-1} + \cdots + A_{n-1}^{2p-1} A_0) + \cdots \\
&\quad + (-1)^{p-1} 2(A_0^{p+1} A_{n-1}^{p-1} + \cdots + A_{n-1}^{p+1} A_0^{p-1}) \\
&\quad + (-1)^p (A_0^p A_{n-1}^p + \cdots + A_{n-1}^p A_0^p)] + A_n^{2p+1}
\end{aligned}
$$

$$= 2^n(A_0^{2p} + \cdots + A_n^{2p})$$

$$-2^n(A_0^{2p-1}(2^n - A_0) + \cdots + A_{n-1}^{2p-1}(2^n - A_{n-1}) + A_n^{2p-1}(2^n - A_n))$$

$$+2^n(A_0^{2p-2}(2^n - A_0)^2 + \cdots + A_{n-1}^{2p-2}(2^n - A_{n-1})^2$$

$$+A_n^{2p-2}(2^n - A_n)^2) - + \cdots$$

$$+(-1)^{p-1}2^n(A_0^{p+1}(2^n - A_0)^{p-1} + \cdots$$

$$+A_{n-1}^{p+1}(2^n - A_{n-1})^{p-1} + A_n^{p+1}(2^n - A_n)^{p-1})$$

$$+(-1)^p 2^{n-1} P_p$$

$$2^n(A_0^{2p} + \cdots + A_n^{2p})$$

$$+2^n\left(\left(A_0^{2p} - \binom{1}{1} 2^n A_0^{2p-1}\right) + \cdots + \left(A_n^{2p} - \binom{1}{1} 2^n A_n^{2p-1}\right)\right)$$

$$+2^n\left(\left(A_0^{2p} - \binom{2}{1} 2^n A_0^{2p-1}\right.\right.$$

$$\left. + \binom{2}{2} 2^{2n} A_0^{2p-2}\right) + \cdots$$

$$\left.+ \left(A_n^{2p} - \binom{2}{1} 2^n A_n^{2p-1} + \binom{2}{2} 2^{2n} A_n^{2p-2}\right)\right)$$

$$+2^n\left(\left(A_0^{2p} - \binom{3}{1} 2^n A_0^{2p-1} + \binom{3}{2} 2^{2n} A_0^{2p-2} - \binom{3}{3} 2^{3n} A_0^{2p-3}\right) + \cdots\right.$$

$$\left. + \left(A_n^{2p} - \binom{3}{1} 2^n A_n^{2p-1} + \binom{3}{2} 2^{2n} A_n^{2p-2} - \binom{3}{3} 2^{3n} A_n^{2p-3}\right)\right) + \cdots$$

$$+2^n\left(\left(A_0^{2p} - \binom{p-1}{1} 2^n A_0^{2p-1} + - \cdots\right.\right.$$

$$\left. +(-1)^{p-1}\binom{p-1}{p-1} 2^{(p-1)n} A_0^{p+1}\right) + \cdots$$

$$+ \left(A_n^{2p} - \binom{p-1}{1} 2^n A_n^{2p-1} + - \cdots\right.$$

$$+(-1)^{p-1}$$

$$\binom{p-1}{p-1} 2^{(p-1)n} A_n^{p+1}\bigg)\bigg)$$

$$+(-1)^p 2^{n-1} P_p$$

$$= p2^n S_{2p} - \left(\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{p-1}{1}\right) 2^{2n} S_{2p-1}$$

$$+ \left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{p-1}{2}\right) 2^{3n} S_{2p-2} - + \cdots$$

$$+(-1)^{p-1} \binom{p-1}{p-1} 2^{pn} S_{p+1} + (-1)^p 2^{n-1} P_p$$

$$= \binom{p}{1} 2^n S_{2p} - \binom{p}{2} 2^{2n} S_{2p-1} + \binom{p}{3} 2^{3n} S_{2p-2} - + \cdots$$

$$+(-1)^{p-1} \binom{p}{p} 2^{pn} S_{p+1} + (-1)^p 2^{n-1} P_p.$$

References

[1] N.J. Calkin, A curious binomial identity, Discrete Math. 131 (1994) 335–337.