Available online at www.sciencedirect.com
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An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 49 (2005) 1101–1112
www.elsevier.com/locate/camwa

The Tanh and the Sine-Cosine Methods for the Complex Modified K dV and the Generalized K dV Equations

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(Received and accepted August 2004)

Abstract—The complex modified K dV (CMK dV) equation and the generalized K dV equation are investigated by using the tanh method and the sine-cosine method. A variety of exact travelling wave solutions with compact and noncompact structures are formally obtained for each equation. The study reveals the power of the two schemes where each method complements the other. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Complex modified K dV equation, Modified K dV equation, Compactons, Kink solitons, Periodic solutions, Tanh method, Sine-cosine method.

1. INTRODUCTION

In this work, we aim to cast light on the complex modified K dV equation,

$$w_t + w_{xxx} + \alpha\left(|w|^2 w\right)_x = 0, \tag{1}$$

where w is a complex valued function of the spatial coordinate x and the time t, α is a real parameter. This equation has been proposed as a model for the nonlinear evolution of plasma waves [1–5]. The physical model (1) incorporates the propagation of transverse waves in a molecular chain model [1–5], and in a generalized elastic solid.

The two-dimensional steady-state distribution of lower-hybrid waves [1] is governed by the CMK dV equation (1). In [1], two types of solitary waves are obtained: one is a constant phase pulse, whereas the other is an envelope solitary wave.

The CMK dV equation (1) is completely integrable by the inverse scattering method and it admits sech-shaped soliton solutions whose amplitudes and velocities are free parameters [5]. Yang [5] handled this equation by using a soliton perturbation theory which shows that a continuous family of sech-shaped embedded solitons exist and are nonlinearly stable. The results obtained in [5] showed that embedded solitons can be robust despite being in resonance with the linear spectrum.

We also aim to investigate generalized forms of the modified K dV equation given by

$$\phi_t + n\alpha\phi^{n-1}\phi_x + \phi_{xxx} = 0, \tag{2}$$

0898-1221/05/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2004.08.013

and

$$\phi_t + n\alpha\phi^{-n-1}\phi_x + \phi_{xxx} = 0, \tag{3}$$

where ϕ is a real valued function. For $n = 2$, equation (2) becomes the famous K dV equation from soliton theory [6],

$$\phi_t + \alpha\left(\phi^2\right)_x + \phi_{xxx} = 0, \tag{4}$$

that describes long nonlinear waves of small amplitude on the surface of inviscid ideal fluid. The K dV equation is known to have infinitely many polynomial conservation laws. The K dV equation (4) is integrable by the inverse scattering transform and gives rise to solitons, that exist due to the balance between the weak nonlinearity and dispersion of that equation. Soliton is a localized wave that has an infinite support or a localized wave with exponential tails.

Karney *et al.* [1] formally examined the close relation between the complex modified K dV equation and the modified K dV equation. They showed that (4) is closely related to (1) by rewriting the CMK dV equation in two ways and then examined the phase variation of w.

There has been an enormous number of examples of solitons equations, verifying that the K dV is not just a freak equation [5–15]. The complexity of the nonlinear wave equations made it impossible to establish one unified method to find all solutions of these equations. Several methods, analytical and numerical, such as Backlund transformation, the inverse scattering method, bilinear transformation, the tanh method [16–18], the homogeneous balance method, and the sine-cosine ansatz, are used to treat these topics.

The $K(n,n)$ equation,

$$u_t + a\left(u^n\right)_x + \left(u^n\right)_{xxx} = 0, \qquad n > 1, \tag{5}$$

introduced in [13], gives rise to the so-called *compactons*: solitons with the absence of infinite wings. The delicate interaction between nonlinear convection $(u^n)_x$ with genuine nonlinear dispersion $(u^n)_{xxx}$ in the $K(n,n)$ equation (5) generates solitary waves with exact compact support that are termed *compactons*. Unlike the K dV equation, the $K(n,n)$ equations have only a finite number of local conservation laws. Solitons and compactons have been receiving considerable attention in mathematical physics.

For more details about solitons and compactons phenomena, the reader is advised to read the works in [1–12] for solitons, and the works in [13–15,19–34] for compactons.

The aim of the present work is to obtain travelling wave solutions the CMK dV equation (1). Two strategies will be pursued to achieve our goal, namely, the tanh method [16–18] and the sine-cosine method [19–34].

In what follows,the sine-cosine ansatz and the tanh method will be reviewed briefly because details can be found in [16–18] and in [19–34].

2. THE TWO METHODS

2.1. Review of the Sine-Cosine Method

The features of this method can be summarized as follows. A PDE,

$$P\left(u, u_t, u_x, u_{xx}, u_{xxx}, \dots\right) = 0, \tag{6}$$

can be converted to an ODE,

$$Q\left(u, u', u'', u''', \dots\right) = 0, \tag{7}$$

upon using a wave variable $\xi = (x - ct)$. Then, equation (7) is integrated as long as all terms contain derivatives where integration constants are considered zeros. The solutions of the reduced ODE equation can be expressed in the form,

$$u(x,t) = \begin{cases} \left\{\lambda \cos^{\beta}(\mu\xi)\right\}, & |\xi| \le \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

or in the form,

$$u(x,t) = \begin{cases} \{\lambda \sin^{\beta}(\mu\xi)\}, & |\xi| \le \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where λ, μ, and β are parameters that will be determined, μ and c are the wave number and the wave speed, respectively. These assumptions give

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi) \tag{10}$$

and

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi). \tag{11}$$

Using (8)–(11) into the reduced ODE gives a trigonometric equation of $\cos^R(\mu\xi)$ or $\sin^R(\mu\xi)$ terms. Then, the parameters are determined by first balancing the exponents of each pair of cosine or sine to determine R. Next, we collect all coefficients of the same power in $\cos^k(\mu\xi)$ or $\sin^k(\mu\xi)$, where these coefficients have to vanish. This gives a system of algebraic equations among the unknowns β, λ and μ that will be determined. The solutions proposed in (8) and (9) follow immediately.

2.2. Review of the Tanh Method

The tanh method is developed by Malfliet [16–18]. Malfliet [16–18] used the tanh technique by introducing tanh as a new variable, since all derivatives of a tanh are represented by a tanh itself.

Introducing a new independent variable,

$$Y = \tanh(\mu\xi), \tag{12}$$

leads to the change of derivatives,

$$\begin{aligned} \frac{d}{d\xi} &= \mu\left(1-Y^2\right)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= \mu^2\left(1-Y^2\right)\left(-2Y\frac{d}{dY} + \left(1-Y^2\right)\frac{d^2}{dY^2}\right). \end{aligned} \tag{13}$$

Then, we apply the following series expansion,

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{14}$$

where M is a positive integer, in most cases, that will be determined. However, if M is not an integer, a transformation formula is usually used to overcome this difficulty. Substituting (13) and (14) into the simplified ODE results in an equation in powers of Y.

To determine the parameter M, we usually balance the linear terms of highest-order in the resulting equation with the highest-order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k $(k = 0, \ldots, M)$, μ, and c. Having determined these parameters, knowing that M is a positive integer in most cases, and using (14), we obtain an analytic solution $u(x,t)$ in a closed form.

3. THE COMPLEX MODIFIED K dV EQUATION

In this section, the CMK dV equation,

$$w_t + w_{xxx} + \alpha\left(|w|^2 w\right)_x = 0, \tag{15}$$

will be investigated by using the two schemes presented before.

3.1. Using the Sine-Cosine Method

We begin our analysis by decomposing w into its real and imaginary parts, where we set

$$w = u + iv, \qquad i^2 = -1, \tag{16}$$

to obtain the coupled pair of the modified K dV (MK dV) equations

$$\begin{aligned} u_t + u_{xxx} + \alpha\left[\left(u^2+v^2\right)u\right]_x &= 0, \\ v_t + v_{xxx} + \alpha\left[\left(u^2+v^2\right)v\right]_x &= 0. \end{aligned} \tag{17}$$

These two coupled nonlinear equations describe the interaction of two orthogonally polarized transverse waves [3], where u and v represent y-polarized and z-polarized transverse waves respectively, propagating in the x-direction in an xyz coordinate system.

Using the wave variable $\xi = x - ct$ into system (17) and integrating, we obtain

$$\begin{aligned} -cu + \alpha u^3 + \alpha u^2 v + u'' &= 0, \\ -cv + \alpha v^3 + \alpha u v^2 + v'' &= 0. \end{aligned} \tag{18}$$

Then, we use (8) in the form,

$$\begin{aligned} u(x,t) &= \lambda \cos^{\beta}(\mu\xi), \\ v(x,t) &= \tilde{\lambda} \cos^{\tilde{\beta}}(\mu\xi), \end{aligned} \tag{19}$$

into (18) to get

$$\begin{aligned} &-c\lambda\cos^{\beta}(\mu\xi) + \alpha\lambda^3\cos^{3\beta}(\mu\xi) + \alpha\lambda^2\tilde{\lambda}\cos^{2\beta}(\mu\xi)\cos^{\tilde{\beta}}(\mu\xi) \\ &\quad -\mu^2\lambda\beta^2\cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0, \\ &-c\tilde{\lambda}\cos^{\tilde{\beta}}(\mu\xi) + \alpha\tilde{\lambda}^3\cos^{3\tilde{\beta}}(\mu\xi) + \alpha\lambda\tilde{\lambda}^2\cos^{2\tilde{\beta}}(\mu\xi)\cos^{\beta}(\mu\xi) \\ &\quad -\mu^2\tilde{\lambda}\tilde{\beta}^2\cos^{\tilde{\beta}}(\mu\xi) + \mu^2\tilde{\lambda}\tilde{\beta}\left(\tilde{\beta}-1\right)\cos^{\tilde{\beta}-2}(\mu\xi) = 0. \end{aligned} \tag{20}$$

Using the balance method, by equating the exponents and the coefficients of $\cos^j$, we get

$$\begin{aligned} \beta - 1 &\neq 0, & \tilde{\beta} - 1 &\neq 0, \\ 3\beta = 2\beta + \tilde{\beta} &= \beta - 2, & 3\tilde{\beta} = \beta + 2\tilde{\beta} &= \tilde{\beta} - 2, \\ \mu^2\beta^2 &= -c, & \mu^2\tilde{\beta}^2 &= -c, \\ \alpha\lambda^3 + \alpha\lambda^2\tilde{\lambda} &= -\lambda\mu^2\beta(\beta-1), & \alpha\tilde{\lambda}^3 + \alpha\lambda\tilde{\lambda}^2 &= -\tilde{\lambda}\mu^2\tilde{\beta}\left(\tilde{\beta}-1\right). \end{aligned} \tag{21}$$

Solving system (21) leads to the results,

$$\begin{aligned} &\beta = \tilde{\beta} = -1, \\ &\mu = \sqrt{-c}, \qquad c < 0, \\ &\lambda = \tilde{\lambda} = \sqrt{\frac{c}{\alpha}}, \end{aligned} \tag{22}$$

consequently, for $c < 0$, we obtain the following periodic solutions,

$$u(x,t) = v(x,t) = \sqrt{\frac{c}{\alpha}}\csc\left(\sqrt{-c}\,(x-ct)\right), \qquad 0 < \mu(x-ct) < \pi, \tag{23}$$

and

$$u(x,t) = v(x,t) = \sqrt{\frac{c}{\alpha}} \sec\left(\sqrt{-c}(x - ct)\right), \qquad |\mu(x - ct)| < \frac{\pi}{2}. \tag{24}$$

Noting that $w(x,t) = u(x,t) + iv(x,t)$, the solutions of the CMK dV equation read

$$w(x,t) = (1+i)\sqrt{\frac{c}{\alpha}} \csc\left(\sqrt{-c}(x - ct)\right), \qquad 0 < \mu(x - ct) < \pi, \tag{25}$$

and

$$w(x,t) = (1+i)\sqrt{\frac{c}{\alpha}} \sec\left(\sqrt{-c}(x - ct)\right), \qquad |\mu(x - ct)| < \frac{\pi}{2}. \tag{26}$$

However, for $c > 0$, we obtain the complex solution,

$$w(x,t) = (1+i)\sqrt{-\frac{c}{\alpha}} \operatorname{csch}\left(\sqrt{c}(x - ct)\right) \tag{27}$$

and

$$w(x,t) = (1+i)\sqrt{\frac{c}{\alpha}} \operatorname{sech}\left(\sqrt{c}(x - ct)\right). \tag{28}$$

3.2. Using the tanh Method

In this section, we will use the tanh method as presented by Malfliet [16–18] to handle the (CMK dV) equation. It was shown before that the CMK dV equation takes the form of two coupled nonlinear equations of the system,

$$\begin{aligned} u_t + u_{xxx} + \alpha\left[\left(u^2 + v^2\right)u\right]_x &= 0, \\ v_t + v_{xxx} + \alpha\left[\left(u^2 + v^2\right)v\right]_x &= 0. \end{aligned} \tag{29}$$

The tanh method admits the use of finite series

$$\begin{aligned} u(x,t) = S(Y) &= \sum_{m=0}^{M} a_m Y^m, \\ v(x,t) = \tilde{S}(Y) &= \sum_{m=0}^{M_1} b_m Y^m, \end{aligned} \tag{30}$$

to express the solutions $u(x,t)$ and $v(x,t)$, where $Y = \tanh(\mu\xi)$. Substituting (30) into the ODE (29) gives

$$\begin{aligned} -cS + \alpha S^3 + \alpha S\tilde{S}^2 + \mu^2\left(1 - Y^2\right)\left(-2YS' + \left(1 - Y^2\right)S''\right) &= 0, \\ -c\tilde{S} + \alpha\tilde{S}^3 + \alpha S^2\tilde{S} + \mu^2\left(1 - Y^2\right)\left(-2Y\tilde{S}' + \left(1 - Y^2\right)\tilde{S}''\right) &= 0. \end{aligned} \tag{31}$$

Balancing the linear term of highest order with the nonlinear term in both equations, we find

$$\begin{aligned} 3M &= M + 2M_1 = 4 + M - 2, \\ 3M_1 &= 2M + M_1 = 4 + M_1 - 2, \end{aligned} \tag{32}$$

which gives $M = M_1 = 1$. This means that

$$\begin{aligned} u(x,t) &= a_0 + a_1 Y, \\ v(x,t) &= b_0 + b_1 Y. \end{aligned} \tag{33}$$

Substituting (33) into the two components of (31), and collecting the coefficients of Y gives the following two system of algebraic equations for a_0, a_1, b_0, b_1, and μ,

$$\begin{aligned} &Y^3 \text{ coeff.}: && 2\mu^2 a_1 + \alpha a_1^3 + \alpha a_1 b_1^2 = 0, \\ &Y^2 \text{ coeff.}: && 2\alpha a_1 b_0 b_1 + 3\alpha a_0 a_1^2 + \alpha a_0 b_1^2 = 0, \\ &Y^1 \text{ coeff.}: && -ca_1 + 3\alpha a_0^2 a_1 + \alpha a_1 b_0^2 + 2\alpha a_0 b_0 b_1 - 2\mu^2 a_1 = 0, \\ &Y^0 \text{ coeff.}: && \alpha a_0^3 + \alpha a_0 b_0^2 - ca_0 = 0, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &Y^3 \text{ coeff.}: && 2\mu^2 b_1 + \alpha b_1^3 + \alpha b_1 a_1^2 = 0, \\ &Y^2 \text{ coeff.}: && 2\alpha b_1 a_0 a_1 + 3\alpha b_0 b_1^2 + \alpha b_0 a_1^2 = 0, \\ &Y^1 \text{ coeff.}: && -cb_1 + 3\alpha b_0^2 b_1 + \alpha b_1 a_0^2 + 2\alpha b_0 a_0 a_1 - 2\mu^2 b_1 = 0, \\ &Y^0 \text{ coeff.}: && \alpha b_0^3 + \alpha b_0 a_0^2 - cb_0 = 0. \end{aligned} \tag{35}$$

Solving these systems gives

$$\begin{aligned} a_0 &= b_0 = 0, \\ a_1 &= b_1 = \sqrt{c/2\alpha}, \\ \mu &= \sqrt{-c/2}, c < 0. \end{aligned} \tag{36}$$

The kink solitons solutions for $c < 0$ take the forms,

$$u(x,t) = v(x,t) = \sqrt{c/2\alpha} \tanh\left(\sqrt{-c/2}\,(x - ct)\right) \tag{37}$$

and

$$u(x,t) = v(x,t) = \sqrt{c/2\alpha} \coth\left(\sqrt{-c/2}\,(x - ct)\right). \tag{38}$$

This means that the solutions of the CMK dV equation take the forms,

$$w(x,t) = (1+i)\left(\sqrt{c/2\alpha} \tanh\left(\sqrt{-c/2}\,(x - ct)\right)\right) \tag{39}$$

and the form

$$w(x,t) = (1+i)\left(\sqrt{c/2\alpha} \coth\left(\sqrt{-c/2}\,(x - ct)\right)\right), \tag{40}$$

However, for $c > 0$, we find the periodic solutions,

$$u(x,t) = v(x,t) = \sqrt{-c/2\alpha} \tan\left(\sqrt{c/2}\,(x - ct)\right) \tag{41}$$

and

$$u(x,t) = v(x,t) = \sqrt{-c/2\alpha} \cot\left(\sqrt{c/2}\,(x - ct)\right), \tag{42}$$

and as a result, we get

$$w(x,t) = (1+i)\left(\sqrt{-c/2\alpha} \tan\left(\sqrt{c/2}\,(x - ct)\right)\right) \tag{43}$$

and the form

$$w(x,t) = (1+i)\left(\sqrt{-c/2\alpha} \cot\left(\sqrt{c/2}\,(x - ct)\right)\right). \tag{44}$$

4. THE GENERALIZED K dV EQUATION

In this section, the generalized K dV equation,

$$u_t + n\alpha u^{n-1} u_x + u_{xxx} = 0, \tag{45}$$

will be investigated by using the two proposed methods.

4.1 Using the Sine-Cosine Method

Using the wave variable $\xi = x - ct$ into system (45) and integrating, we obtain

$$-cu + \alpha u^n + u^{''} = 0. \tag{46}$$

Substituting (8) into (46) gives

$$-c\lambda \cos^{\beta}(\mu\xi) + \alpha\lambda^n \cos^{n\beta}(\mu\xi) - \mu^2\lambda\beta^2 \cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0. \tag{47}$$

Using the balance method as applied before yields

$$\begin{aligned} \beta - 1 &\neq 0, \\ n\beta &= \beta - 2, \\ \mu^2\beta^2 &= -c, \\ \alpha\lambda^n &= -\lambda\mu^2\beta(\beta-1). \end{aligned} \tag{48}$$

Solving this system, we find

$$\begin{aligned} \beta &= -\frac{2}{n-1}, \\ \mu &= \frac{n-1}{2}\sqrt{-c}, \qquad c < 0, \\ \lambda &= \left(\frac{c(n+1)}{2\alpha}\right)^{1/n-1}. \end{aligned} \tag{49}$$

Consequently, for $c < 0$, we obtain the following periodic solutions

$$u(x,t) = \left(\frac{c(n+1)}{2\alpha}\csc^2\left(\frac{n-1}{2}\sqrt{-c}(x-ct)\right)\right)^{1/n-1}, \qquad 0 < \mu(x-ct) < \pi, \tag{50}$$

and

$$u(x,t) = \left(\frac{c(n+1)}{2\alpha}\sec^2\left(\frac{n-1}{2}\sqrt{-c}(x-ct)\right)\right)^{1/n-1}, \qquad |\mu(x-ct)| < \frac{\pi}{2}. \tag{51}$$

However, for $c > 0$, we obtain the solitons solutions,

$$u(x,t) = \left(-\frac{c(n+1)}{2\alpha}\operatorname{csch}^2\left(\frac{n-1}{2}\sqrt{c}(x-ct)\right)\right)^{1/n-1} \tag{52}$$

and

$$u(x,t) = \left(\frac{c(n+1)}{2\alpha}\operatorname{sech}^2\left(\frac{n-1}{2}\sqrt{-c}(x-ct)\right)\right)^{1/n-1}. \tag{53}$$

4.2 Using the Tanh Method

It remains to use the tanh method to investigate the generalized K dV equation,

$$u_t + n\alpha u^{n-1}u_x + u_{xxx} = 0, \tag{54}$$

that will be carried into

$$-cu + \alpha u^n + u^{''} = 0. \tag{55}$$

Using the tanh method

$$u(x,t) = S(Y) = \sum_{m=0}^{M} a_m Y^m \tag{56}$$

gives

$$-cS + \alpha S^n + \mu^2 (1 - Y^2) \left(-2YS^{'} + (1 - Y^2)S^{''}\right) = 0. \tag{57}$$

Balancing the linear term of highest-order with the nonlinear term in both equations, we find

$$nM = 4 + M - 2. \tag{58}$$

which gives

$$M = \frac{2}{n-1}. \tag{59}$$

To obtain a closed form solution, then M should be an integer. This means that $M = 1$ for $n = 3$ and $M = 2$ for $n = 2$. This means that

$$\begin{aligned} u(x,t) &= S(Y) = a_0 + a_1 Y, \\ u(x,t) &= S(Y) = b_0 + b_1 Y + b_2 Y^2. \end{aligned} \tag{60}$$

CASE I FOR $M = 1$. We first substitute $S = a_0 + a_1 Y$, $n = 3$ into (57) and collect the coefficients of Y to find

$$\begin{aligned} Y^3 \text{ coeff.}: &\quad \alpha a1^3 + 2\mu^2 a1 = 0, \\ Y^2 \text{ coeff.}: &\quad 3\alpha a_0 a1^2 = 0, \\ Y^1 \text{ coeff.}: &\quad -2\mu^2 a1 - ca1 + 3\alpha a_0^2 a1 = 0, \\ Y^0 \text{ coeff.}: &\quad -ca_0 + \alpha a_0^3 = 0. \end{aligned} \tag{61}$$

Solving this system gives

$$\begin{aligned} a_0 &= 0, \\ a_1 &= \sqrt{c/\alpha}, \\ \mu &= \sqrt{-c/2}. \end{aligned} \tag{62}$$

Consequently, for $c < 0$, we obtain the solutions,

$$u(x,t) = \sqrt{c/\alpha} \tanh\left(\sqrt{-c/2}\,(x - ct)\right) \tag{63}$$

and

$$u(x,t) = \sqrt{c/\alpha} \coth\left(\sqrt{-c/2}\,(x - ct)\right). \tag{64}$$

However, for $c > 0$ we find the solutions,

$$u(x,t) = \sqrt{-c/\alpha} \tan\left(\sqrt{c/2}\,(x - ct)\right) \tag{65}$$

and

$$u(x,t) = \sqrt{-c/\alpha} \cot\left(\sqrt{c/2}\,(x - ct)\right). \tag{66}$$

CASE II FOR $M = 2$. We first substitute $S = b_0 + b_1 Y + b_2 Y^2, n = 2$ into (57) and collect the coefficients of Y to find

$$\begin{aligned} Y^4 \text{ coeff.}: &\quad 6\mu^2 b_2 + \alpha b_2^2 = 0, \\ Y^3 \text{ coeff.}: &\quad 2\alpha b_1 b_2 + 2\mu^2 b_1 = 0, \\ Y^2 \text{ coeff.}: &\quad 2\alpha b_0 b_2 - c b_2 + \alpha b_1^2 - 8\mu^2 b_2 = 0, \\ Y^1 \text{ coeff.}: &\quad -c b_1 - 2\mu^2 b_1 + 2\alpha b_0 b_1 = 0, \\ Y^0 \text{ coeff.}: &\quad \alpha b_0^2 + 2\mu^2 b_2 - c b_0 = 0. \end{aligned} \tag{67}$$

Solving this system gives two sets of solutions,

$$\begin{aligned} b_0 &= \frac{3c}{2\alpha}, \\ b_1 &= 0, \\ b_2 &= -\frac{3c}{2\alpha}, \\ \mu &= \frac{\sqrt{c}}{2}, \qquad c > 0, \end{aligned} \tag{68}$$

and

$$\begin{aligned} b_0 &= -\frac{c}{2\alpha}, \\ b_1 &= 0, \\ b_2 &= \frac{3c}{2\alpha}, \\ \mu &= \frac{\sqrt{-c}}{2}, \qquad c < 0. \end{aligned} \tag{69}$$

The first set gives, for $c > 0$, the solitons solutions,

$$u(x,t) = -\frac{3c}{2\alpha}\operatorname{csch}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{70}$$

$$u(x,t) = \frac{3c}{2\alpha}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{71}$$

and, for $c < 0$, we find the periodic solutions,

$$u(x,t) = \frac{3c}{2\alpha}\csc^2\left(\frac{\sqrt{-c}}{2}(x - ct)\right), \tag{72}$$

$$u(x,t) = \frac{3c}{2\alpha}\sec^2\left(\frac{\sqrt{-c}}{2}(x - ct)\right). \tag{73}$$

The second set, for $c < 0$, gives the solutions,

$$u(x,t) = -\frac{c}{2\alpha}\left(1 - 3\tanh^2\left(\frac{\sqrt{-c}}{2}(x - ct)\right)\right) \tag{74}$$

and

$$u(x,t) = -\frac{c}{2\alpha}\left(1 - 3\coth^2\left(\frac{\sqrt{-c}}{2}(x - ct)\right)\right), \tag{75}$$

and for $c > 0$, the solutions,

$$u(x,t) = -\frac{c}{2\alpha}\left(1 + 3\tan^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)\right) \tag{76}$$

and

$$u(x,t) = -\frac{c}{2\alpha}\left(1 + 3\cot^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)\right). \tag{77}$$

5. THE GENERALIZED K dV WITH NEGATIVE EXPONENT

We now consider the generalized K dV equation with negative exponent,

$$u_t + n\alpha u^{-n-1}u_x + u_{xxx} = 0, \qquad n > 1. \tag{78}$$

This equation will be examined only by using the sine-cosine method. Using the tanh method to handle this equation requires a transformation formula, therefore, it will be examined in a forthcoming work.

Proceeding as before, we get

$$-cu - \alpha u^{-n} + u'' = 0. \tag{79}$$

Substituting (8) into (79) gives

$$-c\lambda \cos^{\beta}(\mu\xi) - \alpha\lambda^{-n}\cos^{-n\beta}(\mu\xi) - \mu^2\lambda\beta^2\cos^{\beta}(\mu\xi) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0. \tag{80}$$

The balance technique gives the system,

$$\begin{aligned} &\beta - 1 \neq 0, \\ &-n\beta = \beta - 2, \\ &\mu^2\beta^2 = -c, \\ &\alpha\lambda^{-n} = \lambda\mu^2\beta(\beta-1), \end{aligned} \tag{81}$$

from which we find

$$\begin{aligned} \beta &= \frac{2}{n+1}, \\ \mu &= \frac{n+1}{2}\sqrt{-c}, c < 0, \\ \lambda &= \left(\frac{2\alpha}{c(n-1)}\right)^{1/n+1}. \end{aligned}$$

Consequently, for $c < 0$, we obtain the following compactons solutions,

$$u(x,t) = \begin{cases} \left\{\dfrac{2\alpha}{c(n-1)}\sin^2\left(\dfrac{n+1}{2}\sqrt{-c}(x-ct)\right)\right\}^{1/n+1}, & |\mu\xi| < \pi, \quad c < 0, \\ 0, & \text{otherwise}, \end{cases} \tag{83}$$

and

$$u(x,t) = \begin{cases} \left\{\dfrac{2\alpha}{c(n-1)}\cos^2\left(\dfrac{n+1}{2}\sqrt{-c}(x-ct)\right)\right\}^{1/n+1}, & |\mu\xi| < \dfrac{\pi}{2}, \quad c < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{84}$$

However, for $c > 0$, we obtain the solitary patterns solutions,

$$u(x,t) = \left\{-\frac{2\alpha}{c(n-1)}\sinh^2\left(\frac{n+1}{2}\sqrt{c}(x-ct)\right)\right\}^{1/n+1} \tag{85}$$

and

$$u(x,t) = \left\{\frac{2\alpha}{c(n-1)}\cosh^2\left(\frac{n+1}{2}\sqrt{c}(x-ct)\right)\right\}^{1/n+1}. \tag{86}$$

It is interesting to point out that compactons are generated as a result of the interaction between nonlinear convection and the nonlinear dispersion. From the previous results it is clear that compactons solutions (83) and (84) were generated although the dispersion is linear. The physical explanation of this result is beyond the scope of this work.

6. DISCUSSION

In this work, we have extended the well-known works in [1–5] to obtain a variety of exact travelling wave solutions for the CMK dV and the generalized K dV equations. Our analysis rests mainly on the sine-cosine method and the tanh method. The performance of the two schemes has been monitored in that some of the results are in agreement with other results reported in the literature, and new results were formally established. The study revealed that the two methods are reliable and one complements the other. The two strategies can be applied to a large number of models.

Using the tanh method for the case where M is not an integer, a transformation formula is needed. This is the case, in addition to the case of negative exponent, that will be pursued in a further work.

Many types of exact solutions with distinct physical structures have been found. Periodic solutions, solitons solutions, kink solitons, and compactons were established. The results demonstrate that, contrary to previous conclusions, compactons solutions can be generated despite dispersion being linear as shown in the last section.

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