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Approximation properties of Gamma operators ☆

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Abstract

In this paper the approximation properties of Gamma operators G_n are studied to the locally bounded functions and the absolutely continuous functions, respectively. Firstly, in Section 2 of the paper a quantitative form of the central limit theorem in probability theory is used to derive an asymptotic formula on approximation of Gamma operators G_n for sign function. And then, this asymptotic formula combining with a metric form $\Omega_x(f,\lambda)$ is used to derive an asymptotic estimate on the rate of convergence of Gamma operators G_n for the locally bounded functions. Next, in Section 3 of the paper the optimal estimate on the first order absolute moment of the Gamma operators $G_n(|t-x|,x)$ is obtained by direct computations. And then, this estimate and Bojanic–Khan–Cheng's method combining with analysis techniques are used to derive an asymptotically optimal estimate on the rate of convergence of Gamma operators G_n for the absolutely continuous functions.

Keywords: Approximation properties; Locally bounded functions; Absolutely continuous functions; Gamma operators; Probabilistic methods

1. Introduction and definitions

Let f be a function defined on $[0,\infty)$ and satisfying the following growth condition:

$$\left|f(t)\right| \leqslant Me^{\beta t} \quad (M>0;\ \beta\geqslant 0;\ t\to\infty). \tag{1}$$

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Then, the Gamma operator G_n applied to f is

$$G_n(f,x) = \frac{1}{x^n \Gamma(n)} \int_0^{+\infty} f(t/n) t^{n-1} e^{-t/x} \, dt. \tag{2}$$

In this paper the properties of pointwise approximation of Gamma operators G_n will be studied to the class of locally bounded functions Φ_B and the class of absolutely continuous functions Φ_{DB}, respectively. The two classes of functions Φ_B and Φ_{DB} are defined as follows:

$$\Phi_B = \{ f \mid f \text{ is bounded on every finite subinterval of } [0,\infty) \},$$

$$\Phi_{DB} = \left\{ f \;\middle|\; f(x) - f(0) = \int_0^x h(t)\,dt;\ x \geqslant 0; \right.$$
$$\left. h \text{ is bounded on every finite subinterval of } [0,\infty) \right\}.$$

Furthermore, for a function $f \in \Phi_B$, we introduce the following metric form:

$$\Omega_x(f,\lambda) = \sup_{t \in [x-\lambda, x+\lambda]} \bigl| f(t) - f(x) \bigr|,$$

where $x \in [0,\infty)$ is fixed, $\lambda \geqslant 0$.

It is clear that

(i) $\Omega_x(f,\lambda)$ is monotone non-decreasing with respect to λ.
(ii) $\lim_{\lambda \to 0} \Omega_x(f,\lambda) = 0$, if f is continuous at the point x.
(iii) If f is bounded variation on $[a,b]$, and $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$, then $\Omega_x(f,\lambda) \leqslant \bigvee_{x-\lambda}^{x+\lambda}(f)$.

The main contents of this article are organized as follows. In Section 2 a quantitative form of the central limit theorem in probability theory is used to derive an asymptotic formula on approximation of Gamma operators G_n for sign function. And then, this asymptotic formula combining with the metric form $\Omega_x(f,\lambda)$ is used to derive an asymptotic estimate on the rate of convergence of Gamma operators G_n for the locally bounded function $f \in \Phi_B$ at the point x where $f(x+)$ and $f(x-)$ exist. In Section 3 the first order absolute moment of the Gamma operators $G_n(|t-x|,x)$ is estimated to get

$$\left| G_n\bigl(|t-x|,x\bigr) - \sqrt{\frac{2}{n\pi}}\,x \right| \leqslant \frac{x}{15 n^{3/2}}. \tag{3}$$

Estimate (3) is the asymptotically optimal and it is better than a result of Bojanic and Khan [1, Section 3.7] that

$$G_n\bigl(|t-x|,x\bigr) = \sqrt{\frac{2}{n\pi}}\,x + O(n^{-1}). \tag{4}$$

And then, the estimate (3) and Bojanic–Khan–Cheng's method combining with analysis techniques are used to derive an estimate on the rate of convergence of Gamma operators G_n for absolutely continuous function $f \in \Phi_{DB}$. This estimate is the asymptotically optimal.

2. Approximation for locally bounded functions

In this section we study the rate of convergence of Gamma operators G_n for function $f \in \Phi_B$. The main result of this section is as follows:

Theorem 1. *Let $f \in \Phi_B$ and let $f(t) = O(e^{\beta t})$ for some $\beta \geqslant 0$ as $t \to \infty$. If $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0, \infty)$, then for $n > 4\beta x$ we have*

$$\left| G_n(f, x) - \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}} \right| \leqslant \frac{5}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1}), \tag{5}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases} \tag{6}$$

We point out that Theorem 1 subsumes the case of approximation of functions of bounded variation, from Theorem 1 we get immediately

Corollary 1. *Let f be a function of bounded variation on every subinterval of $[0, \infty)$ and let $f(t) = O(e^{\beta t})$ for some $\beta \geqslant 0$ as $t \to \infty$. Then for $x \in (0, \infty)$ and $n > 4\beta x$ we have*

$$\left| G_n(f, x) - \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}} \right| \leqslant \frac{5}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}) + O(n^{-1}) \leqslant \frac{5}{n} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-1}). \tag{7}$$

Corollary 2. *Under the conditions of Theorem 1, if $\Omega_x(g_x, \lambda) = o(\lambda)$, then*

$$G_n(f, x) = \frac{f(x+) + f(x-)}{2} - \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}} + o(n^{-1/2}). \tag{8}$$

To prove Theorem 1, we need some preliminary results.

Lemma 1. *For $x \in (0, \infty)$, $k = 0, 1, 2, \ldots,$ there holds*

$$G_n(t^k, x) = \frac{(n + k - 1)!}{(n - 1)!n^k} x^k. \tag{9}$$

Proof. Direct computation gives

$$G_n(1,x)=1, \qquad G_n(t,x)=x.$$

Suppose that (9) holds for some non-negative integer k, then

$$\begin{aligned}
G_n(t^{k+1},x) &= \frac{1}{x^n\Gamma(n)}\int_0^{+\infty}(t/n)^{k+1}t^{n-1}e^{-t/x}\,dt\\
&= \frac{x(n+1)^k}{n^k x^{n+1}\Gamma(n+1)}\int_0^{+\infty}\left(\frac{t}{n+1}\right)^k t^{n+1-1}e^{-t/x}\,dt\\
&= \frac{x(n+1)^k}{n^k}G_{n+1}(t^k,x)\\
&= \frac{x(n+1)^k}{n^k}\frac{(n+1+k-1)!}{(n+1-1)!(n+1)^k}x^k\\
&= \frac{(n+k+1-1)!}{(n-1)!\,n^{k+1}}x^{k+1}.
\end{aligned}$$

Lemma 1 is proved. □

Lemma 2. *For $x\in(0,\infty)$ there holds*

$$G_n\big((t-x)^2,x\big)=\frac{x^2}{n}; \tag{10}$$

$$\sqrt{G_n((t-x)^4,x)}\leqslant\frac{3}{n}x^2; \tag{11}$$

$$\sqrt{G_n((t-x)^6,x)}\leqslant\frac{17}{n^{3/2}}x^3; \tag{12}$$

$$G_n(e^{2\beta t},x)\leqslant(2e)^{2\beta x}\quad \textit{for } n>4\beta x. \tag{13}$$

Proof. By Lemma 1 and direct computations, we get

$$\begin{aligned}
G_n\big((t-x)^2,x\big)&=\frac{x^2}{n},\\
G_n\big((t-x)^4,x\big)&=\frac{3n+6}{n^3}x^4,\\
G_n\big((t-x)^6,x\big)&=\frac{15n^2+130n+120}{n^5}x^6,
\end{aligned}$$

which derive Eqs. (10)–(12). On the other hand, if $n>4\beta x$, putting $t=\frac{nx}{n-2\beta x}u$, we have

$$G_n(e^{2\beta t},x)=\frac{1}{x^n\Gamma(n)}\int_0^{+\infty}e^{2\beta t/n}t^{n-1}e^{-t/x}\,dt$$

$$= \frac{(1/x - 2\beta/n)^{-n} x^{-n}}{\Gamma(n)} \int_0^{+\infty} u^{n-1} e^{-u}\, du$$
$$= (1/x - 2\beta/n)^{-n} x^{-n} = \left(\frac{n}{n - 2\beta x}\right)^n$$
$$= \left(1 + \frac{2\beta x}{n - 2\beta x}\right)^n \leqslant (2e)^{2\beta x}. \qquad \square$$

The following Lemma 3 is an asymptotic form of the central limit theorem in probability theory. Its proof can be found in Feller [2, pp. 540–542].

Lemma 3. *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E\xi_1 = a_1$, the variance $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E(\xi_1 - a_1)^4 < \infty$, and let F_n stand for the distribution function of $\sum_{k=1}^{n}(\xi_k - a_1)/\sigma\sqrt{n}$. If F_n is not a lattice distribution, then the following equation holds for all $t \in (-\infty, +\infty)$:*

$$F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du = \frac{E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}}(1 - t^2)\frac{1}{\sqrt{2\pi}} e^{-t^2/2} + O(n^{-1}). \tag{14}$$

Proof of Theorem 1. Let f satisfy the conditions of Theorem 1, then f can be expressed as

$$\begin{aligned} f(t) = {} & \frac{f(x+) + f(x-)}{2} + g_x(t) + \frac{f(x+) - f(x-)}{2}\operatorname{sign}(t - x) \\ & + \delta_x(t)\left[f(x) - \frac{f(x+) + f(x-)}{2}\right], \end{aligned} \tag{15}$$

where $g_x(t)$ is defined in (6), $\operatorname{sign}(t)$ is sign function and

$$\delta_x(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

Obviously,

$$G_n(\delta_x, x) = 0. \tag{16}$$

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same Gamma distribution and their probability density functions are

$$P_{\xi_i}(t) = \begin{cases} \frac{1}{x}\exp(-t/x), & \text{if } t > 0, \\ 0, & \text{if } t \leqslant 0, \end{cases}$$

where $x \in (0, \infty)$ is a parameter. Then by direct computation we get

$$E(\xi_1) = x, \qquad E(\xi_1 - E\xi_1)^2 = \sigma^2 = x^2, \tag{17}$$

$$E(\xi_1 - E\xi_1)^3 = 2x^3, \qquad E(\xi_1 - E\xi_1)^4 = 9x^4 < \infty. \tag{18}$$

Let $\eta_n = \sum_{i=1}^{n} \xi_i$ and F_n stand for the distribution function of $\sum_{i=1}^{n}(\xi_i - E\xi_i)/\sigma\sqrt{n}$. Then the probability distribution of the random variable η_n is

$$P(\eta_n \leqslant y) = \frac{1}{\Gamma(n)x^n} \int_0^y t^{n-1} e^{-t/x}\, dt.$$

Thus

$$\begin{aligned} G_n\bigl(\operatorname{sign}(t-x), x\bigr) &= \frac{1}{\Gamma(n)x^n} \int_{nx}^{+\infty} t^{n-1} e^{-t/x}\, dt - \frac{1}{\Gamma(n)x^n} \int_0^{nx} t^{n-1} e^{-t/x}\, dt \\ &= 1 - 2P(\eta_n \leqslant nx) = 1 - 2F_n(0). \end{aligned} \tag{19}$$

By Lemma 3, (17), (18) combining with simple computations, we obtain

$$1 - 2F_n(0) = -\frac{2E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}} \frac{1}{\sqrt{2\pi}} + O(n^{-1}) = \frac{-2}{3\sqrt{2\pi n}} + O(n^{-1}). \tag{20}$$

It follows from (15), (16), (19) and (20) that

$$\left| G_n(f,x) - \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{3\sqrt{2\pi n}} \right| \leqslant \bigl|G_n(g_x, x)\bigr| + O(n^{-1}). \tag{21}$$

We need to estimate $|G_n(g_x, x)|$. Let

$$K_n(x,t) = P(\eta_n \leqslant t) = \frac{1}{x^n \Gamma(n)} \int_0^t v^{n-1} e^{-v/x}\, dv.$$

Then

$$G_n(g_x, x) = \int_0^{+\infty} g_x(t/n)\, d_t K_n(x,t). \tag{22}$$

Suppose $0 \leqslant v \leqslant t < nx$, then, noting that $E(\eta_n - E\eta_n)^2 = nx^2$ and by Chebyshev inequality, we have

$$K_n(x,t) = P(\eta_n \leqslant t) = P\bigl(|\eta_n - nx| \geqslant nx - t\bigr) \leqslant \frac{nx^2}{(nx-t)^2}. \tag{23}$$

Decompose the integral of (22) into four parts as

$$\int_0^{+\infty} g_x(t/n)\, d_t K_n(x,t) = \Delta_{1,n}(g_x) + \Delta_{2,n}(g_x) + \Delta_{3,n}(g_x) + \Delta_{4,n}(g_x),$$

where

$$\Delta_{1,n}(g_x) = \int_0^{nx-\sqrt{n}x} g_x(t/n)\, d_t K_n(x,t), \qquad \Delta_{2,n}(g_x) = \int_{nx-\sqrt{n}x}^{nx+\sqrt{n}x} g_x(t/n)\, d_t K_n(x,t),$$

$$\Delta_{3,n}(g_x) = \int_{nx+\sqrt{n}x}^{2nx} g_x(t/n)\,d_t K_n(x,t), \qquad \Delta_{4,n}(g_x) = \int_{2nx}^{+\infty} g_x(t/n)\,d_t K_n(x,t).$$

We will evaluate $\Delta_{1,n}(g_x)$, $\Delta_{2,n}(g_x)$, $\Delta_{3,n}(g_x)$ and $\Delta_{4,n}(g_x)$. First, for $\Delta_{2,n}(g_x)$, noting that $g_x(x) = 0$, we have

$$\left|\Delta_{2,n}(g_x)\right| \leqslant \int_{nx-\sqrt{n}x}^{nx+\sqrt{n}x} \left|g_x(t/n) - g_x(x)\right| d_t K_n(x,t) \leqslant \Omega_x(g_x, x/\sqrt{n}). \tag{24}$$

To estimate $|\Delta_{1,n}(g_x)|$, note that $\Omega_x(g_x,\lambda)$ is monotone non-decreasing with respect to λ, thus it follows that

$$\left|\Delta_{1,n}(g_x)\right| = \left|\int_0^{nx-\sqrt{n}x} g_x(t/n)\,d_t K_n(x,t)\right| \leqslant \int_0^{nx-\sqrt{n}x} \Omega_x(g_x, x-t/n)\,d_t K_n(x,t).$$

Using integration by parts with $y = nx - \sqrt{n}\,x$, we have

$$\int_0^{nx-\sqrt{n}x} \Omega_x(g_x, x-t/n)\,d_t K_n(x,t) \leqslant \Omega_x(g_x, x-y/n) K_n(x,y) + \int_0^y K_n(x,t)\,d_t\bigl(-\Omega_x(g_x, x-t/n)\bigr). \tag{25}$$

From (25) and using inequality (23), we get

$$\left|\Delta_{1,n}(g_x)\right| \leqslant \Omega_x(g_x, x-y/n)\frac{nx^2}{(nx-y)^2} + \int_0^y \frac{nx^2}{(nx-t)^2} d_t\bigl(-\Omega_x(g_x, x-t/n)\bigr). \tag{26}$$

Since

$$\int_0^y \frac{d_t(-\Omega_x(g_x, x-t/n))}{(nx-t)^2} = \frac{-\Omega_x(g_x, x-y/n)}{(nx-y)^2} + \frac{\Omega_x(g_x,x)}{(nx)^2} + \int_0^y 2\frac{\Omega_x(g_x, x-t/n)}{(nx-t)^3}\,dt,$$

from (25), (26) it follows that

$$\left|\Delta_{1,n}(g_x)\right| \leqslant \frac{1}{n}\Omega_x(g_x,x) + 2nx^2 \int_0^{nx-\sqrt{n}x} \frac{\Omega_x(g_x, x-t/n)}{(nx-t)^3}\,dt.$$

Putting $t = x - x/\sqrt{u}$ for the last integral, we get

$$\int_0^{nx-\sqrt{n}x} \frac{\Omega_x(g_x, x - t/n)}{(nx-t)^3}\,dt = \frac{1}{2(nx)^2}\int_1^n \Omega_x(g_x, x/\sqrt{u})\,du.$$

Consequently

$$\left|\Delta_{1,n}(g_x)\right| \leqslant \frac{1}{n}\left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{u})\,du\right). \tag{27}$$

Using the similar method to estimate $|\Delta_{3,n}(g_x)|$, we obtain

$$\left|\Delta_{3,n}(g_x)\right| \leqslant \frac{1}{n}\left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{u})\,du\right). \tag{28}$$

Finally, by assumption that $g_x(t) \leqslant M(e^{\beta t})$ as $t \to \infty$, using Hölder inequality and the inequality (11), (13), for $n \geqslant 4\beta x$ we have

$$\begin{aligned}
\left|\Delta_{4,n}(g_x)\right| &\leqslant M\int_{2nx}^{+\infty} e^{\beta t/n} d_t K_n(x,t) \\
&\leqslant \frac{M}{x^2}\int_0^{+\infty} (t/n - x)^2 e^{\beta t/n} d_t K_n(x,t) \\
&\leqslant \frac{M}{x^2}\left(\int_0^{+\infty} (t/n - x)^4 d_t K_n(x,t)\right)^{1/2}\left(\int_0^{+\infty} e^{2\beta t/n} d_t K_n(x,t)\right)^{1/2} \\
&\leqslant \frac{3M(2e)^{\beta x}}{n}.
\end{aligned} \tag{29}$$

Equations (24), (27)–(29) derive

$$\begin{aligned}
\left|G_n(g_x, x)\right| &\leqslant \left|\Delta_{1,n}(g_x)\right| + \left|\Delta_{2,n}(g_x)\right| + \left|\Delta_{3,n}(g_x)\right| + \left|\Delta_{4,n}(g_x)\right| \\
&\leqslant \Omega_x(g_x, x/\sqrt{n}) + \frac{2}{n}\left(\Omega_x(g_x, x) + \int_1^n \Omega_x(g_x, x/\sqrt{u})\,du\right) \\
&\quad + \frac{3M(2e)^{\beta x}}{n} \\
&\leqslant \frac{5}{n}\sum_{k=1}^n \Omega_x(g_x, x/\sqrt{k}) + \frac{3M(2e)^{\beta x}}{n}.
\end{aligned} \tag{30}$$

Theorem 1 now follows from (21) and (30). $\square$

3. Approximation for absolutely continuous functions

In this section we study the rate of convergence of Gamma operators G_n for function $f \in \Phi_{DB}$. The main result of this section is as follows:

Theorem 2. *Let f be a function in Φ_{DB} and let $f(t) \leqslant Me^{\beta t}$ for some $M > 0$ and $\beta \geqslant 0$ as $t \to \infty$. If $h(x+)$ and $h(x-)$ exist at a fixed point $x \in (0, \infty)$, then for $n > 4\beta x$ we have*

$$\left| G_n(f, x) - f(x) - \frac{\tau x}{\sqrt{2\pi n}} \right| \leqslant \frac{5x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x, x/k) + \frac{|\tau|x + 17M(2e)^{\beta x}}{n^{3/2}}, \tag{31}$$

where $\tau = h(x+) - h(x-)$, and

$$\phi_x(t) = \begin{cases} h(t) - h(x+), & x < t < \infty; \\ 0, & t = x; \\ h(t) - h(x-), & 0 \leqslant t < x. \end{cases} \tag{32}$$

Remark 1. Theorems 1 and 2 need condition $n > 4\beta x$ because Theorems 1 and 2 consider the approximation function f that satisfies the growth condition: $f(t) \leqslant Me^{\beta t}$ for some $M > 0$ and $\beta \geqslant 0$ as $t \to \infty$. In particular, if $\beta = 0$, that is, f is bounded on $[0, \infty)$. Then condition $n > 4\beta x$ becomes $n = 1, 2, 3, \ldots$. From viewpoint of approximation, an approximation process is true for $n = 1, 2, 3, \ldots$ is better than that for n sufficiently large. It should be point out that references [1,3–5,7–9] obtained a lot of approximation results (include approximation of Gamma operators) which are true for $n = 1, 2, 3, \ldots$.

Remark 2. If f is a function with derivative of bounded variation, then $f \in \Phi_{DB}$. Thus the approximation of functions with derivatives of bounded variation is a special case of Theorem 2. In this special case Theorem 2 is better than a result of [1]. More important, the estimate of Theorem 2 has been the asymptotic optimal.

To prove Theorem 2, we need to estimate the first order absolute moment of the Gamma operators: $G_n(|t - x|, x)$. As concerns this research, Bojanic and Khan [1] proved that

$$G_n\big(|t - x|, x\big) = \sqrt{\frac{2}{n\pi}}x + O(n^{-1}). \tag{33}$$

Hereinbelow, we will present an optimal estimate to $G_n(|t - x|, x)$.

Lemma 4. *For $x \in (0, \infty)$, there holds*

$$G_n\big(|t - x|, x\big) = \frac{2xn^n e^{-n}}{n!}. \tag{34}$$

Proof. By the fact that $G_n(t, x) = x$, we have

$$G_n\big(|t - x|, x\big) = \frac{1}{x^n \Gamma(n)} \int_0^{+\infty} |t/n - x| t^{n-1} e^{-t/x}\, dt$$

$$= \frac{1}{x^n \Gamma(n)} \left(\int_0^{nx} (x - t/n) t^{n-1} e^{-t/x}\, dt - \int_{nx}^{+\infty} (x - t/n) t^{n-1} e^{-t/x}\, dt \right)$$
$$= \frac{2}{x^n \Gamma(n)} \int_0^{nx} (x - t/n) t^{n-1} e^{-t/x}\, dt$$
$$= \frac{2x}{\Gamma(n)} \int_0^{n} u^{n-1} e^{-u}\, du - \frac{2x}{\Gamma(n+1)} \int_0^{n} u^{n} e^{-u}\, du.$$

But

$$\int_0^{n} u^{n-1} e^{-u}\, du = n^{n-1} e^{-n} + \frac{1}{n} \int_0^{n} u^{n} e^{-u}\, du.$$

Thus

$$G_n\big(|t-x|, x\big) = \frac{2x n^{n-1} e^{-n}}{\Gamma(n)} = \frac{2x n^n e^{-n}}{n!}. \qquad \square$$

From Lemma 4 and Stirling ’s formula we get immediately

Corollary 3. *For $x \in (0, \infty)$, there holds*

$$\left| G_n\big(|t-x|, x\big) - \sqrt{\frac{2}{n\pi}}\, x \right| \leqslant \frac{x}{15 n^{3/2}}. \tag{35}$$

Estimation (35) is the best possible, that is to say, it cannot be asymptotically improved.

Proof. By Lemma 4 and using Stirling’s formula (cf. [6]):

$$n! = \sqrt{2\pi n}\,(n/e)^n e^{c_n}, \quad (12n+1)^{-1} < c_n < (12n)^{-1},$$

we have

$$\sqrt{\frac{2}{n\pi}}\, x - G_n\big(|t-x|, x\big) = \sqrt{\frac{2}{n\pi}}\, x\big(1 - e^{-c_n}\big),$$

and a simple calculation derives

$$\sqrt{2/\pi}\, \frac{x}{15 n^{3/2}} \leqslant \sqrt{\frac{2}{n\pi}}\, x\big(1 - e^{-c_n}\big) \leqslant \frac{x}{15 n^{3/2}}. \tag{36}$$

Proof of Theorem 2. By direct computation, we find that

$$\begin{aligned} G_n(f, x) - f(x) = {} & \frac{h(x+) - h(x-)}{2} G_n\big(|t-x|, x\big) - L_{n,x}(\phi_x) \\ & + R_{n,x}(\phi_x) + T_{n,x}(\phi_x), \end{aligned} \tag{37}$$

where

$$L_{n,x}(\phi_x)=\int_0^{nx}\left(\int_{t/n}^{x}\phi_x(u)\,du\right)d_tK_n(x,t),$$

$$R_{n,x}(\phi_x)=\int_{nx}^{2nx}\left(\int_{x}^{t/n}\phi_x(u)\,du\right)d_tK_n(x,t),$$

$$T_{n,x}(\phi_x)=\int_{2nx}^{+\infty}\left(\int_{x}^{t/n}\phi_x(u)\,du\right)d_tK_n(x,t).$$

Integration by parts derives

$$\begin{aligned}L_{n,x}(\phi_x)&=\int_0^{nx}\left(\int_{t/n}^{x}\phi_x(u)\,du\right)d_tK_n(x,t)\\&=\int_{t/n}^{x}\phi_x(u)\,du\,K_n(x,t)\Big|_0^{nx}+\frac{1}{n}\int_0^{nx}K_n(x,t)\phi_x(t/n)\,dt\\&=\int_0^{x}K_n(x,nv)\phi_x(v)\,dv\\&=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x}\right)K_n(x,nv)\phi_x(v)\,dv.\end{aligned}$$

Note that $K_n(x,nv)\leqslant 1$ and $\phi_x(x)=0$, it follows that

$$\left|\int_{x-x/\sqrt{n}}^{x}K_n(x,nv)\phi_x(v)\,dv\right|\leqslant\frac{x}{\sqrt{n}}\Omega_x\left(\phi_x,\frac{x}{\sqrt{n}}\right)\leqslant\frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\phi_x,x/k).$$

On the other hand, by inequality (23) and using change of variable $t=x-x/u$, we have

$$\begin{aligned}\left|\int_0^{x-x/\sqrt{n}}K_n(x,nv)\phi_x(v)\,dv\right|&\leqslant\frac{x^2}{n}\int_0^{x-x/\sqrt{n}}\frac{\Omega_x(\phi_x,x-v)}{(x-v)^2}\,dv\\&=\frac{x}{n}\int_1^{\sqrt{n}}\Omega_x(\phi_x,x/u)\,du\leqslant\frac{x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\phi_x,x/k).\end{aligned}$$

Thus, it follows that

$$\left|L_{n,x}(\phi_x)\right|\leqslant\frac{3x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\phi_x,x/k).\tag{38}$$

A similar evaluation gives

$$\left|R_{n,x}(\phi_x)\right| \leqslant \frac{3x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega_x(\phi_x, x/k). \tag{39}$$

Finally, by the assumption that $f(t) \leqslant Me^{\beta t}$ $(M > 0,\ \beta \geqslant 0)$, and using inequality (12) and (13) we have

$$\begin{aligned}
\left|T_{n,x}(\phi_x)\right| &\leqslant M \int_{2nx}^{+\infty} e^{\beta t/n} d_t K_n(x,t) \\
&\leqslant \frac{M}{x^3} \int_{2nx}^{+\infty} (t/n - x)^3 e^{\beta t/n} d_t K_n(x,t) \\
&\leqslant \frac{M}{x^3} \left(\int_0^{+\infty} (t/n - x)^6 d_t K_n(x,t) \right)^{1/2} \left(\int_0^{+\infty} e^{2\beta t/n} d_t K_n(x,t) \right)^{1/2} \\
&\leqslant \frac{17M(2e)^{\beta x}}{n^{3/2}}.
\end{aligned} \tag{40}$$

Theorem 2 now follows from (35), (37)–(40) combining with a simple calculation.

In the final paragraph we show that the estimate of Theorem 2 is asymptotically optimal. By direct computation, we find that

$$|t - x| - |0 - x| = \int_0^t \operatorname{sign}(u - x)\, du, \quad t \in [0, \infty).$$

In Theorem 2, taking $f(t) = |t - x|$, then $h(t) = \operatorname{sign}(t - x)$, $\tau = h(x+) - h(x-) = 2$, $\phi_x \equiv 0$. Hence from (31), (36) and by simple computation, we obtain

$$\sqrt{2/\pi}\, \frac{x}{15n^{3/2}} \leqslant \left| G_n\big(|t - x|, x\big) - \sqrt{\frac{2}{n\pi}}\, x \right| \leqslant \frac{2x + 17M(2e)^{\beta x}}{n^{3/2}}. \tag{41}$$

Therefore (31) cannot be any further asymptotically improved. $\square$

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Further reading

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