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On Nonlinear Equations Involving Pseudo-A-Proper Mappings and Their Uniform Limits with Applications*

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INTRODUCTION

Let (X, Y) be a pair of real normed linear spaces and let T be in general a nonlinear mapping of $D \subseteq X$ into Y. The purpose of this paper is twofold: first, to establish two basic existence theorems for the equation

(i) $T(x) = f \ (x \in D, f \in Y)$

under various "boundary" and/or "at infinity" conditions, where T is either *pseudo-A-proper*[1] (Theorem 1) or a *uniform limit* of a special sequence of such mappings (Theorem 2); second, to apply Theorems 1 and 2 to the study of the solvability of Eq. (i) involving various special classes of mappings in Banach spaces.

Suppose (X, Y) has an oriented admissible scheme $\Gamma_n = \{X_n, Y_n, P_n, Q_n\}$. In trying (see, for example, [24, 26, 33]) to obtain the constructive existence of solutions $x \in D$ of Eq. (i) as strong limits of solutions $x_n \in D_n$ of the finite-dimensional Galerkin-type approximate equations

(ii) $T_n(x_n) = Q_n(f), \qquad (T_n = Q_n T|_{D_n}, D_n = D \cap X_n)$

the author has been led in [27–29] to the class of *A-proper* mappings [i.e., maps satisfying *condition* (H)] which later was further studied by the author [31, 34, 35], Browder and Petryshyn [8, 9], Browder [4, 6], Nussbaum [23], Wong [40], Deimling [11], Fitzpatrick [15], and others. Although the class of A-proper mappings, under suitable continuity assumptions, includes many types of mappings [e.g., compact displacements, P-compact, strongly K-monotone, $\tilde{\gamma}$-k-set-contractions, mappings of type (S) and of modified type (S)], there are existence theorems for Eq. (i) involving, for example,

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[1] For the definitions of various concepts and the precise statements of the results mentioned in the Introduction see the succeeding sections.

weakly continuous (see, for example, [38, 1, 13, 39]) or coercive demicontinuous monotone or pseudo-monotone mappings (see, for example, [21, 5, 2, 3]) for which the theory of A-proper mappings is not directly applicable.

However, a closer look at this theory suggests that if instead of its constructive we are primarily interested in its existential aspect, then the same approach can still be used to obtain existence theorems for Eq. (i) involving a much wider class of the so-called *pseudo-A-proper* mappings [or maps satisfying condition (c) for reflexive Banach spaces [26, 36] and condition (h) for general Banach spaces [30]] and their uniform limits under the "boundary" and/or the "at infinity" conditions which are more general than the K-coerciveness condition.

The basic results summarized below were announced in our note [25].

In Section 1 we introduce various concepts and establish the two basic existence theorems for Eq. (i): Theorem 1 for fa-continuous pseudo-A-proper mapping T of $D \subseteq X$ into Y which is of K-coercive type and Theorem 2 for T of X into Y which is a uniform limit on bounded sets in X of pseudo-A-proper mappings and which satisfies a condition "at infinity" which is more general than the K-coerciveness condition. As our first application, we obtain new results for weakly closed mappings T of $D \subseteq X$ into Y, which for the case when $Y = X$ (see Section 5) includes the basic fixed point theorem of Schauder [38] for weakly continuous T and its extensions [1, 13, 16].

In Section 2, assuming here and in succeeding sections that X and Y are Banach spaces with X reflexive, we apply Theorems 1 and 2 to obtain surjectivity results (Theorems 3 and 4) for Eq. (i) involving demicontinuous and weakly continuous K-monotone and K-coercive mappings T of X into Y. As special cases (see Section 5A for the case when $Y = X^*$) we obtain the basic surjectivity theorem of Minty–Browder [21, 5] for demicontinuous monotone mappings T of X into X^*, of Browder–De Figueiredo [7] for J-monotone maps as well as other results. We also discuss the conditions and give new arguments which guarantee the A-properness of certain K-monotone mappings T of X into Y which need not be bounded.

In Section 3 we first apply Theorems 1 and 2 to obtain new results for T of type (PKM) (Theorems 5 and 6) and then apply Theorem 1′ to T quasi-K-monotone (Theorem 7). As special cases (see Section 5B for the case when $Y = X^*$) we obtain the surjectivity theorem of Brezis [2] for T pseudo-monotone and of Kachurowsky [16] for T_f quasi-monotone.

In Section 4 we use Theorems 1 and 2 in the study of the solvability of Eq. (i) for the case when $T : X \to Y$ is of type (KM) (Theorems 8 and 9). As a special case of Theorem 8 when $Y = X^*$ we deduce the result of Brezis [2] for T of type (M). The result of Theorem 9 and its special cases are all new.

Note that the definitions of a pseudo-monotone and of type (M) mappings given in [2] are somewhat different from those given in this paper (cf. [20]). Our definitions involve only sequences while those of Brezis are given in terms of filters. However, since X employed in Sections 2–5 is necessarily separable and reflexive, the results of Brezis [2] are also valid for pseudo-monotone maps and maps of type (M) as defined here but without the additional condition that T be bounded. In this paper, whenever a reference is made to these maps, it is understood that they are defined in terms of sequences. We add that our arguments are different from those in [2].

In Section 5 we discuss in more detail the various special classes of mappings T of $D \subseteq X$ into X^* and T of $D \subseteq X$ into X, some of which have already been mentioned. Here we deduce a number of known results as well as some new ones from Theorems 10 and 11 in Section 5A for mappings T of $D \subseteq X$ into X^* and from Theorems 12 and 13 in Section 5B for mappings T of $D \subseteq X$ into X which are the corresponding analogoues of Theorems 1 and 2 in Section 1. In particular, from Theorem 12 we deduce the fixed point theorem for P-compact mappings established in Petryshyn [24, 33] (see also [32]) which includes the fixed point theorem of Schauder [38] and Rothe [37] for compact mappings and of Kaniel [17] for quasi-compact mappings. We also indicate the connection between the class of pseudo-A-proper mappings and the class of G-operators T of $D \subseteq X$ into X studied by De Figueiredo [13]. For other contributions see Section 5.

1. Equations Involving Fa-Continuous Pseudo-A-Proper Mappings

A pair (X, Y) of normed real spaces is said to have an admissible projectional scheme $\Gamma_n = \{X_n, Y_n, P_n, Q_n\}$ if there exist two sequences $\{X_n\} \subset X$ and $\{Y_n\} \subset Y$ of monotonically increasing finite-dimensional subspaces with $\dim X_n = \dim Y_n$ for each n and two sequences of bounded linear projections $\{P_n\}$ and $\{Q_n\}$ with $P_n(X) = X_n$ and $Q_n(Y) = Y_n$ such that $P_n(x) \to x$ and $Q_n(y) \to y$ for each x in X and y in Y (here and in what follows $\to$ denotes the strong convergence; we will also use $\rightharpoonup$ to denote the weak convergence in X and the weak* convergence in X^*, the adjoint of X).

Definition 1 [27, 29]. A mapping T of $D \subseteq X$ into Y is said to be Approximation-proper (A-proper) with respect to Γ_n if it satisfies the following condition (H): if for any sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ as $j \to \infty$ and a corresponding bounded sequence $\{x_{n_j} \mid x_{n_j} \in D_{n_j}\}$ such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, there exists a subsequence $\{x_{n_{j(k)}}\}$ and an element x in D such that $x_{n_{j(k)}} \to x$ as $k \to \infty$ and $T(x) = g$, where

$$D_n = D \cap X \qquad \text{and} \qquad T_n = Q_n T\,|_{D_n}$$

for each n.

We recall (see [26, 31]) that the equation

$$T(x) = f \qquad (x \in D, f \in Y) \tag{1}$$

is said to be *strongly* (resp. feebly) *projectionally-solvable* if there exists an integer $N \geqslant 1$ such that for each $n \geqslant N$ the approximate equation

$$T_n(x_n) = Q_n(f) \tag{2}$$

has a solution $x_n \in D_n$ such that $x_n \to x$ with x in D (resp. $x_{n_j} \to x$ as $j \to \infty$ with x in D for some subsequence $\{x_{n_j}\}$ of $\{x_n\}$) and $T(x) = f$. It has been shown in [29, 31] that, under certain conditions on T, the A-properness of T is not only sufficient be also a necessary condition for Eq. (1) to be strongly projectionally-solvable.

It is known that in order to obtain certain existence and/or approximation results for Eq. (1) we need to impose certain continuity conditions on T. The standard assumption on T is that it be either continuous, demicontinuous, hemicontinuous or weakly continuous. We recall that T is *demicontinuous at* $u \in D$ if $\{u_n\} \subset D$ and $u_n \to u$ in X imply $T(u_n) \rightharpoonup T(u)$ in Y; T is *hemicontinuous at* $u \in D$ *if* $v \in X$, $t_n > 0$, $t_n \to 0$ and $u + t_n v \in D$ imply $T(u + t_n v) \rightharpoonup T(u)$ in Y; T is *weakly continuous* at $u \in D$ if $\{u_n\} \subset D$ and $u_n \rightharpoonup u$ in X imply $T(u_n) \rightharpoonup T(u)$ in Y. Recently, considerable attention has been given to finitely continuous mappings which in our setting can be defined as follows. $T : D \subseteq X \to Y$ is *finitely continuous* if for *any* finite-dimensional subspace V of X and any sequence $\{x_k\} \subset D \cap V$ such that $x_k \to x \in D \cap V$ as $k \to \infty$ we have $(T(x_k), y^*) \to (T(x), y^*)$ for each y^* in Y^*, i.e., $T(x_k) \rightharpoonup T(x)$ in Y. In what follows we use $B(o, r)$ and $\dot{B}(o, r)$ to denote an open ball about $o \in X$ and its boundary, respectively.

In this section we consider Eq. (1) involving mappings T which are *finite approximation-continuous* (for short, *fa-continuous*) with respect to a given scheme Γ_n.

DEFINITION 2. A mapping T of $D \subseteq X$ into Y is said to be fa-continuous with respect to Γ_n if for each n the finite approximation $T_n : D_n \subseteq X_n$ into Y_n is continuous.

In [24] and in his other papers the author imposed the fa-continuity as a part of the notion of a P-compact mapping. It turns out that the fa-continuity is a rather weak assumption. In fact, the following simple observation is true.

PROPOSITION 1. *Let T be a mapping of $D \subseteq X$ into Y which is either continuous, demicontinuous, weakly continuous or finitely continuous, then T is* fa-*continuous.*

Proof. Since for finite-dimensional Banach spaces strong and weak convergence coincide, it follows that if T is either continuous, demicontinuous or weakly continuous, then it is automatically fa-continuous with respect to Γ_n. Thus, to prove Proposition 1, it suffices to show that finite continuity implies fa-continuity.

Let T be finitely continuous. For each fixed n let $\{x_k\}$ be a sequence in $D_n = X_n \cap D$ such that $x_k \to x$ as $k \to \infty$ and $x \in D_n$. To show that $T_n(x_k) \to T_n(x)$ in Y_n as $k \to \infty$, it suffices to show that $T_n(x_k) \rightharpoonup T_n(x)$ in Y_n, i.e., $(T_n(x_k), y) \to (T_n(x), y)$ for each y in Y_n^*. Since Y_n is a subspace of Y, the Hahn–Banach Theorem implies that to each y in Y_n^* there corresponds a y^* in Y^* such that $(z, y) = (z, y^*)$ for all z in Y_n and $\| y \| = \| y^* \|$. Hence, since $T_n(x_k) \in Y_n$, we have for each y in Y_n^* the existence of $y^* \in Y^*$ such that $(T_n(x_k), y) = (T_n(x_k), y^*)$. Since T is finitely continuous for every finite dimensional subspace V of X, it is so when $V = X_n$. This and the last equality imply that for each fixed n and any y in Y_n^* there exists $y^* \in Y^*$ such that as $k \to \infty$ we have

$$(T_n(x_k), y) = (T_n(x_k), y^*) = (T(x_k), Q_n^*(y^*)) \to (T(x), Q_n^*(y^*)) = (T_n(x), y).$$

Consequently, $T_n(x_k) \to T_n(x)$ in Y_n as $k \to \infty$, i.e., T is fa-continuous.

We note in passing that every linear mapping T of X into Y is fa-continuous even when T is unbounded.

For the sake of convenience and completeness we state here without proof the following result obtained by the author (see Theorem 3.1 in [31]) which, in its extended form, will play an essential role in our present discussion.

THEOREM A. *Let (X, Y) be a pair of normed linear spaces with an oriented admissible scheme Γ_n, D a bounded open convex subset of X with $o \in D$, K a (nonlinear in general) mapping of X into Y^* and K_n a (nonlinear in general) mapping of X_n into $Y_n' = R(Q_n^*) \subseteq Y^*$ such that $K(x) \neq 0$ if $x \neq 0$ and for each n*

$$(Q_n(g), K_n(x)) = (g, K(x)) \tag{+}$$

for all x in X_n and g in Y. For each n, let M_n be a linear isomorphism of X_n onto Y_n such that $(M_n(x), K_n(x)) > 0$ for all $x \neq 0$ in X_n.

If T is an fa-*continuous* A-*proper mapping of the closure $\bar{D}$ into Y and f is a given vector in Y such that*

$$(T(x), K(x)) \geqslant (f, K(x)) \tag{HA}$$

for all x on the boundary $\dot{D}$, then Eq. (1) *is feebly projectionally-solvable in $\bar{D}$ for $N = 1$, i.e., for each n, Eq.* (2) *has a solution x_n in $\bar{D}_n$ such that $x_{n_i} \to x$ in X*

for some subsequence $\{x_{n_i}\}$ *of* $\{x_n\}$ *and some element* x *in* $\bar{D}$ *with* $T(x) = f$. [*In particular, Eq.* (1) *has a solution* x *in* $\bar{D}$ *for each* f *in* Y *satisfying* (HA).] *If additionally we assume that, for* f *satisfying* (HA), *Eq.* (1) *has at most one solution in* $\bar{D}$, *then it is strongly projectionally solvable.*

It follows from the proof in [31] of Theorem A that, under its conditions on (X, Y), Γ_n, D, K, K_n and M_n, for any fa-continuous mapping T of $\bar{D}$ into Y and each given f in Y for which (HA) holds, the approximate equation (2) has a solution x_n in $\bar{D}_n$ for each n. Hence we may relax somewhat the conditions defining the A-properness of T if our primary concern is not so much construction as the existence of solutions of Eq. (1) for f satisfying (HA). Indeed, for latter purposes it suffices to assume that T satisfies condition (h) defined as follows.

DEFINITION 3. A mapping T of $\bar{D}$ into Y is said to satisfy condition (h) with respect to Γ_n on $\bar{D}$ if for any sequence $\{n_j\}$ of positive integers with $n_j \to \infty$ and a corresponding bounded sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j} \cap \bar{D}\}$ such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, there exists an element x in $\bar{D}$ such that $T(x) = g$.

The class of maps given by Definition 3 was introduced by the author (see condition (c) in [26, 36] for reflexive Banach spaces and condition (h) in [30] for general Banach spaces). Further studies of such maps were carried by the author [28, 31, 25] and Wong [40]. In what follows we shall refer to mappings satisfying condition (h) as *pseudo-A-proper*. We add in passing that for the case when $Y = X$ the concept of an fa-continuous pseudo-A-proper mapping is related to the concept of a G-operator introduced in [13]. We shall dwell on this connection more fully in Section 5B. It turns out that the existence part of Theorem A remains valid for pseudo-A-proper mappings T defined on $\bar{D} \subset X$ with D not necessarily convex. This result is contained in Theorem 1 below.

THEOREM 1. *Let* (X, Y), Γ_n, K, K_n *and* M_n *satisfy the conditions of Theorem* A *and let* D *be a bounded open subset of* X *with* $o \in D$.

(a) *If* T *is an* fa-*continuous pseudo*-A-*proper mapping of* $\bar{D}$ *into* Y *and if* f *is a given element in* Y *for which the inequality* (HA) *of Theorem* A *holds on* $\dot{D}$, *then Eq.* (1) *is solvable for each such* f *in* Y.

(b) *If* T *is an* fa-*continuous pseudo*-A-*proper mapping of* X *into* Y *which satisfies either the condition*

(i) *to each* f *in* Y *there corresponds a number* $r_f > 0$ *so that*

$$(T(x), K(x)) \geqslant (f, K(x)) \qquad \text{for all } x \text{ in } \dot{B}(o, r_f) \tag{HA}$$

or the condition.

(ii) *T is K-coercive, i.e., there exists a real-valued function $c(r)$ defined on reals R^1 such that $c(r) \to \infty$ as $r \to \infty$ and*

$$(T(x), K(x)) \geqslant c(\| x \|) \| K(x) \| \qquad \textit{for all } x \textit{ in } X, \tag{π}$$

then the mapping T is surjective, i.e., $T(X) = Y$.

Proof. For the sake of completeness we first give a short proof of the following known (see [22]) finite-dimensional fixed point theorem which plays an essential role in our proof.

LERAY–SCHAUDER THEOREM. *Let V be an oriented real finite-dimensional Banach space, D an open bounded set in V with $o \in D$ and A a continuous mapping of $\bar{D}$ into V such that*

($\pi_1^{\leqslant}$) *If $A(x) = \alpha x$ holds for some x in $\dot{D}$ then $\alpha \leqslant 1$.*

Then A has a fixed point in $\bar{D}$.

Proof. Consider the homotopy $H_t(x)$ of $\bar{D}x[0, 1]$ into V defined by $H_t(x) = x - tA(x)$ for $x \in \bar{D}$ and $t \in [0, 1]$. Without loss of generality we may assume that $H_1(x) \neq 0$ on $\dot{D}$. Then our condition ($\pi_1^{\leqslant}$) implies that

$$H_t(x) = t\left(\frac{1}{t}x - A(x)\right) \neq 0$$

for all $t \in (0, 1)$ and all $x \in \dot{D}$ while $o \in D$ implies that $H_0(x) \neq 0$ for all $x \in \dot{D}$. Thus, $H_t(x) \neq 0$ for all $x \in \dot{D}$ and $t \in [0, 1]$, under the assumption that $H_1(x) = x - Ax \neq 0$ on $\dot{D}$. Hence the Brouwer degree[2] of H_t on D over 0, $\deg(H_t, D, 0)$, is constant in $t \in [0, 1]$. Since

$$\deg(H_0, D, 0) = \deg(I, D, 0) = 1,$$

it follows that

$$1 = \deg(H_1, D, 0) = \deg(I - A, D, 0) = \deg(H_0\ D, o)$$

and, therefore, there exists $x_0 \in D$ such that $x_0 - A(x_0) = 0$, i.e., A has a fixed point in $\bar{D}$.

Proof of Theorem 1 Continued. (a) For each fixed n and every x in $\bar{D}_n$ consider the mapping $A_n(x) = T_n(x) - Q_n(f)$,

$$T_n = Q_n T|_{\bar{D}_n} : \bar{D}_n \to Y_n .$$

[2] For the definition and the properties of Brouwer degree see [10, 22].

Our conditions imply that for all x in $\dot{D}_n \subset X_n \cap \dot{D}$ and each n

$$\begin{aligned}(A_n(x), K_n(x)) &= (Q_nT(x), K_n(x)) - (Q_n(f), K_n(x)) \\ &= (T(x), K(x)) - (f, K(x)) \geqslant 0.\end{aligned} \tag{1.1}$$

Now, since M_n is a linear isomorphism of X_n onto Y_n, $V_n = M_n(D_n)$ is a bounded open set in Y_n with $o \in V_n$, $V_n \cap \dot{V}_n = \varnothing$ and M_n maps $\dot{D}_n$ homeomorphically onto $\dot{V}_n$. Let G_n be the mapping of $\bar{V}_n$ into Y_n defined by $G_n = I_n - A_nL_n$ with $L_n = M_n^{-1}$ and I_n the identity in Y_n. Since, for each fixed n, $y_n \in \bar{V}_n$ is a fixed point of G_n if and only if $x_n = L_n(y_n)$ $(\in \bar{D}_n)$ is a solution of Eq. (2) [i.e., of $T_n(x_n) = Q_n(f)$], to establish the solvability of the latter equation for each n, in view of the finite-dimensional Leray–Schauder Theorem, it suffices to show that G_n satisfies the condition $(\pi_1{}^{\leqslant})$ on $\dot{V}_n$, i.e., if $G_n(y_0) = \alpha y_0$ holds for some y_0 in $\dot{V}_n$, then $\alpha \leqslant 1$. Now, suppose that $G_n(y_0) = \alpha y_0$ for some y_0 in $\dot{V}_n$. If we let x_0 be a point in $\dot{D}_n$ so that $x_0 = L_n(y_0)$ then, by (1.1),

$$\begin{aligned}\alpha(M_n(x_0), K_n(x_0)) &= (\alpha y_0, K_n(x_0)) = (G_n(y_0), K_n(x_0)) \\ &= (y_0, K_n(x_0)) - (A_nL_n(y_0), K_n(x_0)) \\ &= (M_n(x_0), K_n(x_0)) - (A_n(x_0), K_n(x_0)) \leqslant (M_n(x_0), K_n(x_0)).\end{aligned}$$

Since $(M_n(x_0), K_n(x_0)) > 0$, it follows that $\alpha \leqslant 1$. Hence, for each n, there exists $x_n \in \bar{D}_n$ such that $T_n(x_n) = Q_n(f)$. Since $\{x_n \mid x_n \in \bar{D}_n\}$ is bounded, T is pseudo-A-proper on $\bar{D}$, and $T_n(x_n) = Q_n(f) \to f$ in Y as $n \to \infty$, there exists an element x^* in $\bar{D}$ such that $T(x^*) = f$.

(b) To prove (b) we first show that the coerciveness condition (π) implies the condition (HA). Indeed, if f is any given vector in Y then, since $c(r) \to \infty$ as $r \to \infty$, there exists a number $r_f > 0$ such that $\|f\| < c(r_f)$. Hence, for all $x \in \dot{B}(o, r_f)$ the condition (HA) holds on $\dot{B} = \dot{B}(o, r_f)$ since

$$(T(x), K(x)) - (f, K(x)) \geqslant (c(r_f) - \|f\|) \| K(x)\| > 0 \qquad \text{for all } x \in \dot{B}.$$

Now, since for each f in Y there exists $r_f > o$ such that (HA) holds on $\dot{B}$, the assertion (b) follows from (a) for $D = B(o, r_f)$. Q.E.D.

We remark in passing that Theorem 1 represents essentially a global existence result in the sense that if T is pseudo-A-proper on $\bar{D}$ and if $T_{n_j}(x_{n_j}) \to g$ in Y for some bounded sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j} \cap \bar{D}\}$, then the equation $T(x) = g$ is necessarily solvable in $\bar{D}$. Clearly if, for a given f in Y, the equation $T(x) = f$ is not solvable in $\bar{D}$ and if T is pseudo-A-proper on $\bar{D}$, then we cannot find a bounded sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j} \cap \bar{D}\}$ such that $T_{n_j}(x_{n_j}) \to f$ in Y. On the other hand, the proof of Theorem 1 suggests the possibility of obtaining local existence results for T which is only point-wise pseudo-A-proper in the following sense.

DEFINITION 3'. A mapping T of $\bar{D}$ into Y is said to be pseudo-A-proper at f in Y if for any bounded sequence $\{x_{n_j} \mid x_{n_j} \in X_{n_j} \cap \bar{D}\}$ such that $T_{n_j}(x_{n_j}) \to f$ in Y, there exists an element x in $\bar{D}$ such that $T(x) = f$.

Looking over the proof of Theorem 1 we see that it also implies the validity of the following local result.

THEOREM 1'. *Let (X, Y), Γ_n, K, K_n, M_n, and D be as in Theorem 1 and let T be an* fa-*continuous mapping of $\bar{D}$ into Y. If f is an element in Y such that T is pseudo-*A-*proper at f and*

$$(T(x), K(x)) \geqslant (f, K(x)) \qquad \textit{for } x \textit{ in } \dot{D},$$

then the equation $T(x) = f$ is solvable in $\bar{D}$.

As an illustration of the generality of Theorem 1', we shall apply it to the problem of solvability of equations involving quasi-K-monotone maps (see Theorem 7 below).

Below we apply Theorem 1 to establish a basic surjectivity theorem for an fa-continuous mapping T of X into Y which is a uniform limit on bounded subsets of X of a special sequence of pseudo-A-proper mappings and which is not K-coercive. In fact, the "at infinity" condition (c5) below is more general than the K-coerciveness condition (π). Before stating Theorem 2 we first recall that a mapping F of X into Y is said to be *bounded* if F maps bounded sets from its domain $D(F) \subseteq X$ into bounded sets in Y.

THEOREM 2. *Let (X, Y) be a pair of normed real spaces with an oriented admissible projectional scheme Γ_n, K a mapping of X into Y^*, K_n a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that for each n all $x \neq 0$ in X_n and g in Y*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \textit{and} \qquad (M_n(x), K_n(x)) > 0. \qquad (^{++})$$

Let T be an fa-*continuous mapping of X into Y and suppose also that there exists a bounded* fa-*continuous mapping F of X into Y such that*

(c1) *$T(G)$ is closed in Y if G is a bounded closed convex set in X.*

(c2) *$T_\mu = T + \mu F$ is pseudo-*A-*proper for each $\mu > o$.*

(c3) *F is positively homogeneous of order $\alpha \geqslant 1$ (i.e., $F(tx) = t^\alpha F(x)$ for all x in X, $t \geqslant o$ and some integer $\alpha \geqslant 1$).*

(c4) $(T_\mu(x) - T_\mu(o), \quad K(x)) \geqslant (\mu b \| x \|^\alpha - C) \| K(x)\|$ *for each given* $\mu > o$ *and all* x *in* X *with* $b > o$ *and* $C \geqslant o$ *some constants independent of* x *and* μ.

(c5) $\| T(x)\| \to \infty$ *as* $\| x \| \to \infty$.

Then, under the above conditions, T is surjective.

Proof. To apply Theorem 1 in the proof of Theorem 2, we first note that for each given f in Y and $\mu > o$ and all x in X we have, in view of (c4) and the equality $F(o) = o$, the relation

$$\begin{aligned}(T_\mu(x) - f, Kx) &= (T_\mu(x) - T_\mu(o), K(x)) + (T_\mu(o) - f, K(x)) \\ &\geqslant (\mu b \| x \|^\alpha - C - \| T(o) - f \|) \| K(x)\| .\end{aligned} \tag{1.2}$$

Hence, since $r^\alpha \to \infty$ as $r \to \infty$, to each given f in Y and $\mu > o$ there corresponds a positive real number $r_{\mu f} > o$ such that the right-hand side in the inequality (1.2) is positive for all x in $\dot{B}(o, r_{\mu f})$ and, therefore,

$$(T_\mu(x), K(x)) \geqslant (f, K(x)) \qquad \text{for all } x \text{ in } \dot{B}(o, r_{\mu f}).$$

Since, by our assumptions, T_μ is an fa-continuous pseudo-A-proper mapping of X into Y, Theorem 1 implies that, in view of the last inequality, to each fixed $\mu_k > o$ (with $\mu_k \to o$ as $k \to \infty$) and f in Y there exists a vector $x_k \in B(o, r_{\mu_k f})$ such that

$$T_{\mu_k}(x_k) = f \qquad \text{or} \qquad T(x_k) = f - \mu_k F(x_k). \tag{1.3}$$

The equation (1.3) and (c4) imply that for each k

$$(T_{\mu_k}(x_k) - T_{\mu_k}(o), K(x_k)) = (f - T(o), K(x_k)) \geqslant (\mu_k b \| x_k \|^\alpha - C) \| K(x_k)\| .$$

Hence, using the Schwartz–Buniakovsky inequality, we get the relation $C + \|f - T(o)\| \geqslant b(\| y_k \|^\alpha)$ with $y_k = \mu_k^{1/\alpha} x_k$ for each k from which it follows that $\{y_k\}$ is bounded. Since F is bounded and $\mu_k F(x_k) = F(y_k)$, it follows that $\{F(y_k)\}$ is also bounded and so is the sequence

$$\{T(x_k)\} = \{f - F(y_k)\}.$$

Hence (c5) implies that $\{x_k\}$ is bounded and thus there exists a ball $\bar{B}(o, d) \subset X$ such that $\{x_k\} \subset \bar{B}(o, d)$. This and the fact that F is bounded and $\mu_k \to o$ as $k \to \infty$ imply that

$$T(x_k) = f - \mu_k F(x_k) \to f \qquad \text{in } Y \text{ as } k \to \infty$$

from which, on account of (c1), it follows that there exists an x_0 in $\bar{B}(o, d)$ such that $T(x_0) = f$. Since f was arbitrary, it follows that $T(X) = Y$.

Q.E.D.

Remark 1. Note the qualitative difference between the assertion of Theorem A and that of Theorems 1 and 2. Theorem A yields an essentially constructive existence of solutions of Eq. (1) involving A-proper mappings while Theorems 1 and 2 yield only the existence of solutions of Eq. (1) involving pseudo-A-proper mappings. Nevertheless, as we shall see later, for many types of pseudo-A-proper maps our approach provides also the possibility of obtaining solutions of Eq. (1) as weak limit points of constructible sequences $\{x_n \mid x_n \in D_n\}$ with the entire sequence $\{x_n\}$ converging weakly to a solution in case of its uniqueness. In this sense our results are more constructible than at first glance they appear to be.

We continue this section with the following observation. It follows from Definition 3 that every A-proper mapping is pseudo-A-proper but the converse is not true, in general. Indeed, if (X, Y) is a pair of reflexive Banach spaces with an admissible projectional scheme Γ_n, then under certain conditions on K (see Proposition 2 below) it is not hard to show that every bounded linear mapping of X into Y is pseudo-A-proper, but examples were given by the author which show that even bounded linear mappings which satisfy the Fredholm Alternative need not be A-proper. Furthermore, under the above conditions on (X, Y) and K, every bounded nonlinear weakly closed mapping of the closure $\bar{D}$ of an open subset D of X into Y is pseudo-A-proper where $T : \bar{D} \to Y$ is said to be *weakly closed* if $\{x_n\} \subset \bar{D}$ is a sequence such that $x_n \rightharpoonup x_0$ in X and $T(x_n) \rightharpoonup h$ in Y, then x_0 lies in $\bar{D}$ and $T(x_0) = h$. If D is also assumed to be convex, then every weakly continuous mapping T of $\bar{D} \subset X$ into Y is pseudo-A-proper even when Y is not complete. This is established in the following proposition.

PROPOSITION 2. *Let (X, Y) be a pair of real normed spaces with an admissible projectional scheme Γ_n and with X reflexive, K a demicontinuous mapping of X onto Y^* and K_n a mapping of X_n into Y_n' such that for each n*

$$(Q_n(g), K_n(x)) = (g, K(x)) \tag{+}$$

for all x in X_n and g in Y.

(a) *If Y is reflexive, D an open subset of X, and T a bounded weakly closed mapping of $\bar{D}$ into Y, then T is pseudo-A-proper.*

(b) *If D is an open convex subset of X and T a weakly continuous mapping of $\bar{D}$ into Y, then T is pseudo-A-proper.*

Proof. Let $\{x_n \mid x_n \in X_n \cap \bar{D}\}$ be a bounded sequence and let $\{x_{n_j}\}$ be any of its subsequences so that $T_{n_j}(x_{n_j}) \to g$ for some g in Y.

(a) Since the sequence $\{x_{n_j}\}$ is bounded in X, T is bounded, and X and Y are reflexive, there exist a subsequence $\{x_{n_{j(k)}}\}$ and an element x_0 in X such that $x_{n_{j(k)}} \rightharpoonup x_0$ in X and $T(x_{n_{j(k)}}) \rightharpoonup h$ as $k \to \infty$ for some h in Y. Since T is weakly closed on $\bar{D}$, it follows that $x_0 \in \bar{D}$ and $T(x_0) = h$. To show that $h = g$ note that, in view of ($^+$), for any fixed x in $X_{n_{j(e)}}$ with e fixed and $n_{j(k)} > n_{j(e)}$ for $k > e$ we have

$$(T(x_{n_{j(k)}}), K(x)) = (T_{n_{j(k)}}(x_{n_{j(k)}}), K(x)).$$

Hence, since

$$T(x_{n_{j(k)}}) \rightharpoonup h \qquad \text{and} \qquad T_{n_{j(k)}}(x_{n_{j(k)}}) \to g$$

in Y, the passage to the limit in the above equality as $k \to \infty$ yields the equality $(h, K(x)) = (g, K(x))$ for all x in $X_{n_{j(e)}}$ and each fixed e. Hence, for any z in X, we have the equality $(h, KP_{n_{j(e)}}(z)) = (g, KP_{n_{j(e)}}(z))$ for each e. Since K is demicontinuous and $P_{n_{j(e)}}(z) \to z$ in X, the latter equality implies that $(h, K(z)) = (g, K(z))$ for all z in X. This and the fact that K maps X onto Y^* yields the looked for equality $h = g$.

(b) Suppose now that T is weakly continuous on $\bar{D}$ with D convex. Since $\{x_n \mid x_n \in X_n \cap D\}$ is bounded, X is reflexive and $\bar{D}$ is weakly closed in X, there exists a subsequence $\{x_{n_j}\}$ and an element x_0 in $\bar{D}$ such that $x_{n_j} \rightharpoonup x_0$ in X and $Tx_{n_j} \rightharpoonup Tx_0$ in Y by the weak continuity of T. From this, as above, we obtain the equality $Tx_0 = g$ for Y not necessarily complete, i.e., T is pseudo-A-proper. Q.E.D.

In view of Proposition 2, Theorem 1 implies the validity of the following corollary which we state here as an illustration of the generality of Theorem 1 since, as will be seen in Section 5, our Corollary 1 below includes, on the one hand, the results of Altman [1], De Figueiredo [13] and Kachurowsky [16] for weakly closed and weakly continuous mappings and, on the other hand, Corollary 1 extends to nonlinear weakly continuous mappings T of X into Y (with Y not necessarily reflexive) the Lax–Milgrain Lemma for bounded linear mappings as well as the result of Kachurowsky [16] for coercive weakly continuous mappings T of X into X^*.

COROLLARY 1. *Let (X, Y) be a pair of real normed spaces with an oriented admissible projectional scheme Γ_n and with X reflexive, K a demicontinuous mapping of X onto Y^*, K_n a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \textit{and} \qquad (M_n(x), K_n(x)) > 0 \qquad (^{++})$$

for all $x \neq 0$ in X_n and g in Y.

(a) *If Y is reflexive, D an open bounded subset of X with $o \in D$ and T an* fa-*continuous, bounded, weakly closed mapping of $\bar{D}$ into Y such that*

$(T(x), K(x)) \geqslant (f, K(x))$ *for all* x *in* $\dot{D}$ *and some* f *in* Y, *then there exists a point* $x_0 \in \bar{D}$ *such that* $T(x_0) = f$.

(b) *If* T *is a weakly continuous and K-coercive mapping of* X *into* Y, *then* T *maps* X *onto* Y.

Remark 2. It follows from the proof of Proposition 2 that in Corollary 1 we can not only assert that Eq. (1) has a solution but that it can be obtained as a weak limit point of a constructible sequence $\{x_n \mid x_n \in D_n\}$ for $n = 1, 2, 3 \ldots$. Consequently, if it is known that, for a given f, Eq. (1) has at most one solution, then the entire sequence $\{x_n\}$ converges weakly to the unique solution. In this sense our result is more constructive than it appears to be.

An easy consequence of Theorem 2 is the following new result for weakly closed mappings which will prove to be useful in various applications. Indeed, our Corollary 2 below establishes the surjectivity theorem for weakly closed mappings T of X into Y under a condition "at infinity" which is more general than the coerciveness condition.

COROLLARY 2. *Let* (X, Y) *be a pair of normed real spaces with an oriented admissible projectional scheme* Γ_n *and with* X *reflexive,* K *a mapping of* X *into* Y^*, K_n *a mapping of* X_n *into* Y_n' *and* M_n *a linear isomorphism of* X_n *onto* Y_n *such that* $(^{++})$ *of Corollary 1 holds. Let* F *be a bounded* fa-*continuous mapping of* X *into* Y *such that* F *is positively homogeneous of order* $\alpha \geqslant 1$ *and for some constant* $b > 0$

$$(F(x), K(x)) \geqslant b \| x \|^{\alpha} \| K(x) \| \tag{E1}$$

for all x *in* X.

If T *is an* fa-*continuous weakly closed mapping of* X *into* Y *such that* $T_\mu = T + \mu F$ *is pseudo-*A-*proper on* X *for each* $\mu > 0$,

$$(T(x), K(x)) \geqslant (T(o), K(x)) - |(T(o), K(x))| \tag{E2}$$

for all x *in* X, *and* $\| T(x) \| \to \infty$ *as* $\| x \| \to \infty$, *then* T *maps* X *onto* Y.

Proof. In view of our hypotheses, to prove Corollary 2, it suffices to show that our present conditions imply the validity of (c1) and (c4) of Theorem 2 since (c2), (c3) and (c5) are true by assumption. Now, the condition (c1) follows from the fact that X is reflexive and T is a weakly closed mapping of X into Y while the condition (c4) follows from (E1) and (E2) with $C = \| T(o) \|$. Q.E.D.

We add in passing that, as will be seen in Sections 2 and 3, the somewhat strange condition (E2) is, in fact, a considerable weakening of the requirement that T be a quasi-K-monotone and, in particular, a K-monotone mapping of X into Y.

In view of Theorem A and of Theorems 1 and 2, it is important to find also other conditions on T and F as well as on X and Y which would insure the A-properness or at least the pseudo-A-properness of T and/or of $T_\mu = T + \mu F$ as well as the closedness of $T(G)$ in Y for each bounded closed convex subset G of X. The next three sections are essentially devoted to this problem.

2. K-Monotone Mappings

We recall (see [18, 31]) that T, mapping $D \subseteq X$ into Y, is said to be *K-monotone* on D if

$$(T(x) - T(y), K(x - y)) \geqslant 0 \tag{K}$$

for all x and y in D.

Our first two propositions in this section provide sufficient conditions for a K-monotone mapping to satisfy the hypotheses (c1) and (c2) of Theorem 2, respectively.

Proposition 3. *Let (X, Y) be a pair of real Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K be a weakly continuous mapping of X onto Y^* such that K is positively homogeneous of order $\beta \geqslant 1$ with β an integer. Let D be an open convex subset of X and T a K-monotone mapping of D into Y which is either hemicontinuous or finitely continuous on D. Then, for every bounded closed convex subset G of D, the set $T(G)$ is closed in Y.*

Proof. Let $\{f_j\} \subset T(G)$ be a sequence so that $f_j \to f$ in Y as $j \to \infty$. To show that $f \in T(G)$, let $\{x_j\} \subset G$ be a sequence such that $T(x_j) = f_j$ for each j. Since X is reflexive and G is a closed bounded convex subset of X, without loss of generality we may assume that $x_j \rightharpoonup x_0$ in X for some x_0 in G. Since, by the K-monotonicity of T on D,

$$(T(y) - T(x_j), K(y - x_j)) > 0 \qquad \text{for all } y \in D \text{ and all } j,$$

K is weakly continuous, and $T(x_j) \to f$, the passage to the limit in the above inequality implies that

$$(T(y) - f, K(y - x_0)) \geqslant 0 \qquad \text{for all } y \in D. \tag{2.1}$$

The inequality (2.1) implies that $T(x_0) = f$. Indeed, if not, then because K is onto, there exists z in X such that $(T(x_0) - f, K(z)) < 0$. Since D is open, $G \subset D$ and x_0 lies in G, for sufficiently small $t > o$ the element

$y_t = x_0 + tz \in D$ and hence, since $K(tz) = t^{\beta}K(z)$, setting y_t for y in (2.1) yields the inequality

$$(T(x_0 + tz) - f, K(z)) \geqslant 0.$$

Because z is fixed and T is assumed to be either hemicontinuous or finitely continuous on D, the passage to the limit in the last inequality as $t \to 0$ yields the relation $(T(x_0) - f, K(z)) \geqslant 0$, contradicting the assumption that $(T(x_0) - f, K(z)) < 0$. Q.E.D.

Remark 3. Proposition 3 is certainly true when T is assumed to be either continuous, demicontinuous or weakly continuous. It follows from the proof of Proposition 3 that instead of requiring that $K(tx) = t^{\beta}K(x)$ for some $\beta \geqslant 1$ it suffices to assume that to each $t \geqslant o$ and $x \in X$ there exists $\eta_x(t) \geqslant 0$ such that $K(tx) = \eta_x(t)\, K(x)$ with $\eta_x(t) > 0$ for $t > 0$ and $\eta_x(o) = 0$.

PROPOSITION 4. *Let (X, Y) be a pair of real Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K be both a weakly continuous and a continuous mapping of X onto Y^* such that K is positively homogeneous of order $\beta \geqslant 1$ or to each $t \geqslant 0$ and x in X there exists $\eta_x(t) \geqslant 0$ such that $K(tx) = \eta_x(t)\, K(x)$ with $\eta_x(t) > 0$ for $t > 0$ and $\eta_x(o) = 0$. Let K_n be a mapping of X_n into Y_n' such that*

$$(Q_n(g), K_n(x)) = (g, K(x)) \tag{+}$$

for all x in X_n, $g \in Y$ and each n.

*If A is a demicontinuous K-monotone mapping of X into Y, then under the above conditions the mapping A is pseudo-*A*-proper.*

Proof. Let $\{x_n \mid x_n \in X_n\}$ be a bounded sequence and let $\{x_{n_i}\}$ be an arbitrary subsequence so that $A_{n_i}(x_{n_i}) \to g$ for some g in Y. Since X is reflexive and $\{x_{n_i}\}$ bounded, without loss of generality we may assume that $x_{n_i} \rightharpoonup x_0$ for some x_0 in X. Let $j > 1$ be an arbitrary but fixed integer and let y be any element in X_j. Since $\{X_n\}$ is monotonically increasing and A is K-monotone, it follows from (+) that for all $n_i > j$ we have

$$\begin{aligned}(Q_{n_i}A(x_{n_i}) &- Q_{n_i}A(y), K_{n_i}(x_{n_i} - y)) \\ &= (Q_{n_i}A(x_{n_i}) - Q_{n_i}A(y), K(x_{n_i} - y)) \geqslant 0.\end{aligned}$$

Now, since $x_{n_i} - y \rightharpoonup x_0 - y$ in X, $Q_{n_i}A(x_{n_i}) - Q_{n_i}A(y) \to g - A(y)$ in Y, and K is weakly continuous, the passage to the limit in the above inequality as $i \to \infty$ implies that

$$(g - A(y), K(x_0 - y)) \geqslant 0 \qquad \text{for all } y \text{ in } X_j \text{ and any } j.$$

Hence for any x in X we have the relation

$$(g - AP_j(x), K(x_0 - P_j(x))) \geqslant 0 \qquad \text{for all } x \text{ in } X$$

and each fixed j. Since A is demicontinuous and K is also continuous, the passage to the limit in the last inequality as $j \to \infty$ yields the relation

$$(g - A(x), K(x_0 - x)) \geqslant 0 \qquad \text{for all } x \text{ in } X.$$

From this and the properties of K, as before, we deduce the equality $A(x_0) = g$ and thus the pseudo-A-properness of A. Q.E.D.

Remark 4. If it is also assumed that K is uniformly continuous on a unit ball in X, then in view of the results of Kato [18], Proposition 4 remains also valid for hemicontinuous mappings A of X into Y.

We add in passing that Proposition 4 provides important examples of pseudo-A-proper mappings which need not be A-proper. On some occasion, however, it may be necessary to provide the constructive solvability of the equation $T_\mu(x) = T(x) + \mu F(x) = f$, where T is a given K-monotone mapping of X into Y. In this case the pseudo-A-properness of T_μ will not suffice in general. Thus, in this case, we need to find conditions which will guarantee the A-properness of T_μ for a given $\mu > 0$. Our Proposition 5 below treats this problem.

PROPOSITION 5. *Let (X, Y) be a pair of Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K be both a weakly continuous and a continuous mapping of X onto Y^* and K_n a mapping of X_n into Y_n' such that $K(o) = 0$ and*

$$(Q_n(g), K_n(x)) = (g, K(x)) \tag{+}$$

for all $x \in X_n$ and each n, and T a demicontinuous (or weakly continuous) K-monotone mapping of X into Y. Suppose that F is a demicontinuous K-monotone mapping of X into Y such that for all x and y in X

$$(F(x) - F(y), K(x - y)) \geqslant \gamma(\| x \|, \| y \|, \| x - y \|) \equiv \gamma(x, y), \tag{F}$$

where $\gamma(x, y)$ Is a continuous real-valued nonnegative function on $X \times X$ such that the following conditions (a) *and* (b) *hold:*

(a) *If $\{x_k \mid x_k \in X_k\}$ is a sequence so that $x_k \rightharpoonup x_0$ in X, then to any given $\epsilon > o$ there exists $\sigma(\epsilon) > 0$ such that if $\| y - x_0 \| < \sigma$ then*

$$| \gamma(\| x_k \|, \| y \|, \| x_k - y \|) - \gamma(\| x_k \|, \| x_0 \|, \| x_k - x_0 \|)| < \epsilon \; \forall\, k.$$

(b) *If* $\{x_k \mid x_k \in X_k\}$ *is a sequence so that* $x_k \rightharpoonup x_0$ *and*

$$\lim_k \gamma(\| x_k \|, \| x_0 \|, \| x_k - x_0 \|) = 0,$$

then $x_k \to x_0$ *in* X.

Under the above conditions, for any fixed $\mu > o$, *the mapping* $T_\mu = T + \mu F$ *is* A-*proper*.

Proof. Let $\{x_n \mid x_n \in X_n\}$ be a bounded sequence and let $\{x_m\}$ be an arbitrary subsequence of $\{x_n\}$ so that $T_m(x_m) \to g$ for some $g \in Y$. Since X is reflexive and $\{x_m\}$ bounded, without loss of generality we may assume that $x_m \rightharpoonup x_0$ in X. Let $j > o$ be an arbitrary but fixed integer and let y be any element in X_j. Since $\{X_n\}$ is monotonically increasing and T is K-monotone, the relations ($^+$) and (F) imply that for each fixed $\mu > o$, all y in X_j and $m \geqslant j$ we have

$$\begin{aligned}(Q_m T_\mu(x_m) - Q_m T_\mu(y), K_m(x_m - y))\\ = (T_\mu(x_m) - T_\mu(y), K(x_m - y)) \geqslant \mu\gamma(\| x_m \|, \| y \|, \| x_m - y \|) \geqslant 0.\end{aligned}$$

Now, since $x_m - y \rightharpoonup x_0 - y$ and $Q_m T_\mu(x_m) - Q_m T_\mu(y) \to g - T_\mu(y)$ in Y and $Q_m^2 = Q_m$, the weak continuity of K and ($^+$) imply that, as $m \to \infty$, the left-hand side of the above inequality converges to the real-valued function

$$\eta(y) = (g - T_\mu(y), K(x_0 - y)), \qquad y \in X_j \text{ for any fixed } j.$$

Consequently, to any given $\epsilon > o$ there exists $m_0 = m_0(\epsilon, y)$ such that

$$\mu\gamma(\| x_m \|, \| y \|, \| x_m - y \|) \leqslant \eta(y) + \epsilon \qquad \text{for all } m \geqslant m_0. \tag{2.2}$$

Since T is demicontinuous (or weakly continuous), F is demicontinuous, K is continuous and $P_j(z) \to z$ as $j \to \infty$ for each z in X, it follows that $\eta(y)$ is a continuous function in $y \in X$ and obviously $\eta(y) \geqslant 0$ for all y in X with $\eta(x_0) = 0$. This implies that to each given $\epsilon > o$ there exists $\sigma_0 > o$ such that $o \leqslant \eta(y) < \epsilon$ whenever $\| x_0 - y \| < \sigma_0$. Now, because Γ_n is admissible to any given $\sigma > o$ there corresponds $y \in X_j$ for j sufficiently large so that $\| x_0 - y \| < \sigma$. Since $x_m \rightharpoonup x_0$ in X, our condition (a) implies that to the same $\epsilon > o$ there corresponds $\sigma_1 > 0$ such that

$$| \gamma(\| x_m \|, \| x_0 \|, \| x_m - x_0 \|) - \gamma(\| x_m \|, \| y \|, \| x_m - y \|)| < \epsilon \ \forall\, m \tag{2.3}$$

if $\| y - x_0 \| < \delta_1$. Thus, if for a given $\epsilon > o$, we take $\delta = \min\{\delta_0, \delta_1\}$, then in addition to (2.3) we also have $o \leqslant \eta(y) < \epsilon$ whenever $\| x_0 - y \| < \delta$. Then, for this choice of y, (2.2) and (2.3) imply that for all $m \geqslant m_0(\epsilon, y)$ we have

$$\begin{aligned}0 \leqslant \mu\gamma(\| x_m \|, \| x_0 \|, \| x_m - x_0 \|) \leqslant \mu\gamma(\| x_m \|, \| y \|, \| x_m - y \|) + \epsilon\\ \leqslant \eta(y) + 2\epsilon < 3\epsilon.\end{aligned}$$

This shows that $x_m \rightharpoonup x_0$ in X and $\gamma(\| x_m \|, \| x_0 \|, \| x_m - x_0 \|) \to 0$ as $m \to \infty$ whence, in virtue of our condition (b), it follows that $x_m \to x_0$ in X.

To prove the A-properness of T_μ, it remains to show that $T_\mu(x_0) = g$. To establish this note that for each x in X and $y = P_j(x) \in X_j$ we have

$$\eta(y) = (g - T_\mu(P_j(x)), K(x_0 - P_j(x)) \geqslant 0$$

from which, on passage to the limit as $j \to \infty$, we get

$$(g - T_\mu(x), K(x_0 - x)) \geqslant 0 \qquad \text{for all } x \text{ in } X.$$

Since K is a weakly continuous mapping of X onto Y^* and T_μ is certainly hemicontinuous, the above inequality implies that $T_\mu(x_0) = g$. Hence T_μ is an A-proper mapping of X into Y. Q.E.D.

Remark 5(a). It follows from our conditions on T and F that

$$(T_\mu(x) - T_\mu(y), K(x - y)) \geqslant \mu\gamma(\| x \|, \| y \|, \| x - y \|)$$

for all x and y in X. Hence, in case T and F are continuous and

$$\gamma(\| x \|, \| y \|, \| x - y \|) = c(\| x - y \|),$$

where $c(r)$ is a continuous function of $R^+ = \{r \geqslant 0\}$ into R^+ such that $c(o) = o$, $c(r) > o$ for $r > o$ and $r_j \to o$ whenever $c(r_i) \to 0$, Proposition 5 reduces essentially to our Theorem 2.3 in [31].

(b) It follows from our proof above that the assertion of Proposition 5 concerning $T_\mu = T + \mu F$ remains valid for the case when $T = 0$ and $\mu = 1$. Consequently, every demicontinuous K-monotone mapping F of X into Y, which satisfies the inequality (F) with γ satisfying the conditions (a) and (b), is A-proper. In particular, every demicontinuous strongly K-monotone mapping of X into Y is A-proper. We add in passing that even the last assertion represents a new result since the A-properness is established without the condition that the mapping be bounded.

(c) Finally we remark that our arguments in the proof of Proposition 5 are similar to those used in [12].

We add, that, in view of Propositions 1 and 4, Theorem 1 implies the validity of the following result for K-monotone mappings T of X into Y.

THEOREM 3. *Let (X, Y) be a pair of real Banach spaces with an oriented admissible projectional scheme Γ_n and with X reflexive. Let K be both strongly and weakly continuous map of X onto Y^* such that either $K(tx) = t^\beta K(x)$ for some $\beta \geqslant 1$ or $K(tx) = \eta_x(t)\, K(x)$ with $\eta_x(t) > 0$ for $t > 0$, $x \in X$. Let K_n*

be a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that for each n and all x in X_n and g in Y

$$(Q_n(g), K_n(x)) = (g, K(x)) \quad \textit{and} \quad (M_n(x), K_n(x)) > 0 \quad \textit{for } x \neq 0. \quad (^{++})$$

If T is a demicontinuous (or a weakly continuous) K-monotone mapping of X into Y which satisfies either condition (i) *or the K-coerciveness condition* (ii) *of Theorem* 1, *then T maps X onto Y.*

We note in passing that, in view of Proposition 4, Remark 2 concerning the constructive aspect of our proof applies also to Theorem 3.

In order to obtain the surjectivity theorem for a K-monotone mapping under a condition "at infinity" which is weaker than the K-coerciveness condition (ii), we make use of Theorem 2 which, in virtue of Proposition 3 and 4, implies the validity of the following new and thus for the most general result for K-monotone mappings.

THEOREM 4. *Let (X, Y), K, K_n and M_n satisfy the conditions of Theorem* 3. *Let T be a demicontinuous K-monotone mapping of X into Y and suppose there exists a bounded demicontinuous K-monotone mapping F of X into Y such that F is positively homogeneous of order $\alpha \geqslant 1$ and*

$$(F(x), K(x)) \geqslant b(\| x \|)^\alpha \, \| K(x) \| \tag{H4}$$

for all x in X and some $b > o$.

If in addition to the above conditions we also assume that

$\| T(x) \| \to \infty$ as $\| x \| \to \infty$, then T maps X onto Y.

Proof. In view of Proposition 1 and Theorem 2, to prove Theorem 4 it suffices to show that the conditions of Theorem 4 imply the validity of the hypotheses (c1)–(c5) of Theorem 2.

Now, because T is a demicontinuous K-monotone mapping of X into Y, (c1) follows from Proposition 3 while (c2) follows from Proposition 4 since for each $\mu > o$ the mapping $A \equiv T_\mu$ is K-monotone and demicontinuous. The conditions (c3) and (c5) follow from our assumptions while (c4) with $C = 0$ follows from the K-monotonicity of T, the equality $F(o) = 0$ and the condition (H4). Q.E.D.

Remark 6. We note that Theorems 3 and 4 remain valid if instead of demicontinuity we assume that T is continuous or weakly continuous or even hemicontinuous provided that in the latter case we also assume that K is uniformly continuous on a unit ball in X.

3. Mappings of Type (PKM) and Quasi-K-Monotone Mappings

In [2] Brezis introduced and studied a class of pseudo-monotone mappings T of $D \subseteq X$ into X^* which turned out to be more general than the class of hemicontinuous monotone mappings. Further studies of bounded continuous pseudo-monotone mappings were carried out by Lions [20], Browder [6], Brezis, the author [35] and others (see [20]). In [35] the author observed that when the theory of A-proper mappings is to be applied to the study of Eq. (1) involving continuous mappings T of $D \subseteq X$ into Y, the concept of a pseudo-monotone mapping as defined in [2] is not quite satisfactory especially when T is not bounded and/or when $Y \neq X^*$. Consequently, in [35] the author introduced and studied a related class of continuous, not necessarily bounded, mappings T of $D \subseteq X$ into Y with the so-called *pm-property*, which turned out to be a more suitable condition for the applicability of the theory of A-proper mappings.

However, when T is not continuous and/or when instead of the theory of A-proper mappings we apply the theory of pseudo-A-proper mappings, then instead of mappings with the pm-property we may consider fa-continuous mappings T of D into Y which formally extend the concept of a pseudo-monotone map but defined here in terms of sequences rather than filters as in [2].

Definition 4. Let D be an open convex subset of a reflexive Banach space X. A mapping T of $\bar{D}$ into Y is said to be of type (PKM) if for any sequence $\{z_j\} \subset \bar{D}$ such that $z_j \rightharpoonup z$ in X with $z \in \bar{D}$ and

$$\limsup_j (Tz_j, K(z_j - z)) \leqslant 0,$$

we have

$$(Tz, K(z - v)) \leqslant \liminf_j (Tz_j, K(z_j - v)) \qquad \text{for all } v \in D. \qquad \text{(PKM)}$$

In [35] we referred to such a mapping as "Pseudo-K-monotone" to signify its occasional connection to a K-monotone mapping. It has been noted in [35] that if K is nonlinear, then a K-monotone mapping will not be of type (PKM). Consequently, K-monotone mappings T and mappings T of type (PKM) of X into Y have to be studied separately, which we do in this section.

We recall (see [6]) for later use that T is said to satisfy condition (S) on $\bar{D}$ if for any sequence $\{z_j\} \subset \bar{D}$ such that $z_j \rightharpoonup z$ in X with $z \in \bar{D}$ and $(Tz_j - Tz, K(z_j - z)) \to 0$ we have $z_j \to z$ in X as $j \to \infty$; a Banach space X is said to be an *π_α-space* if there exists a sequence of finite dimensional subspaces $\{X_n\}$ in X and a sequence of bounded linear projections $\{P_n\}$ such that $P_n(X) = X_n$, $X_n \subset X_{n+1}$, and $\| P_n \| \leqslant \alpha$ $(\geqslant 1)$ for each n, $\bigcup_n X_n$ is dense in X and $P_j P_n = P_j$ for $n \geqslant j$. It is easy to see that if X is an π_α-space, then the scheme $\Gamma_n = \{X_n, P_n\}$ is admissible for the pair (X, X). Moreover, it is also not hard to show, that if X is a reflexive π_α-space, then the scheme $\Gamma_n{}^* = \{X_n, X_n', P_n, P_n{}^*\}$ is admissible for the pair (X, X^*), where $P_n{}^*$ is the adjoint of P_n and

$$X_n' = R(P_n{}^*) = P_n{}^*(X^*).$$

We begin this section with the proof of the following two propositions concerning mappings of type (PKM).

PROPOSITION 6. *Let (X, Y) be a pair of real Banach spaces with an admissible scheme Γ_n and with X reflexive. Let K be a weakly continuous mapping of X onto Y^* such that either $K(tx) = t^\beta K(x)$ for all x in X, all $t \geqslant 0$ and some integer $\beta \geqslant 1$ or $K(tx) = \eta_x(t)\, K(x)$ with $\eta_x(t) > 0$ for $t > o$ and $\eta_x(o) = 0$. Let D be an open convex subset of X and T a mapping of D into Y of type* (PKM) *on D. Then, for each bounded convex closed subset G of D, the set $T(G)$ is closed in Y.*

Proof. Let $\{f_j\} \subset T(G)$ be a sequence so that $f_j \to f$ in Y. To show that $f \in T(G)$, let $\{x_j\} \subset G$ such that $T(x_j) = f_j$ for each j. Since X is reflexive and G is a bounded convex closed subset of $D \subset X$, we may assume that $x_j \rightharpoonup x_0$ with $x_0 \in G$. This, the relation $T(x_j) \to f$ and the weak continuity of K imply that

$$\lim_j \sup(T(x_j), K(x_j - x_0)) = \lim_j (T(x_j), K(x_j - x_0)) = 0.$$

Hence, since T is of type (PKM), it follows that

$$(T(x_0), K(x_0 - v)) \leqslant \lim_j \inf(T(x_j), K(x_j - v)) \qquad \forall v \in D$$

from which, since

$$\lim_j \inf(T(x_j), K(x_j - v)) = \lim_j (T(x_j), K(x_j - v)) = (f, K(x_0 - v)),$$

it follows that

$$(T(x_0), K(x_0 - v)) \leqslant (f, K(x_0 - v)) \qquad \forall v \in D. \tag{3.1}$$

Since D is open and $x_0 \in D$, it follows from the above inequality and the assumed properties of K and K_n that $T(x_0) = f$. Indeed, if $T(x_0) \neq f$, then since K is onto there would exist a vector z in X such that

$$(T(x_0) - f, K(z)) > 0.$$

Since (3.1) is true for all v in the open set D and $x_0 \in D$, for each $t > o$ and sufficiently small, the vector $v_t = x_0 - tz \in D$ and, by (3.1),

$$(T(x_0), K(tz)) \leqslant (f, K(tz))$$

or, by the assumed properties of K, $(T(x_0) - f, K(z)) \leqslant 0$, in contradiction to our assumption on z. Hence, $T(G)$ is a closed set in Y. Q.E.D.

PROPOSITION 7. *Let (X, Y) be a pair of Banach spaces with an admissible scheme Γ_n and with X reflexive. Let K be a positively homogeneous mapping of order $\beta \geqslant 1$ of X onto Y^* which is both weakly continuous on X and uniformly continuous on each bounded set in X. Let K_n be a mapping of X_n into Y_n such that for each n*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \textit{for all } x \textit{ in } X_n \textit{ and } g \textit{ in } Y. \qquad (^+)$$

If T is a bounded mapping of type (PKM) *of X into Y, then T is pseudo-*A-*proper.*

Proof. Let $\{x_n \mid x_n \in X_n\}$ be a bounded sequence and let $\{x_{n_j}\}$ be an arbitrary subsequence such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y. Since X is reflexive and $\{x_n\}$ is bounded, we may assume that $\{x_{n_j}\} \subset B(o, r)$ for some $r > o$ and $x_{n_j} \rightharpoonup x_0$ in X with $x_0 \in \bar{B}(o, r)$. Without loss of generality we may assume that $x_0 \neq 0$. To prove Proposition 7, we first show that $(T(x_{n_j}), K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$. To obtain this, note first that if for each $s > o$ we define the function $\psi_r(s)$ by

$$\psi_r(s) = \sup\{\| K(x) - K(y)\| \mid \| x - y \| \leqslant s \text{ for } x, y \in \bar{B}(o, r)\},$$

then, since K is uniformly continuous on $\bar{B}(o, r)$, the function $\psi_r(s)$ is non-decreasing in s, $\psi_r(s) \to 0$ as $s \to 0$ and

$$\| K(x) - K(y)\| \leqslant \psi_r(\| x - y \|) \qquad \text{for all } x, y \in \bar{B}(o, r). \qquad (3.2)$$

Since $K(tx) = t^\beta K(x)$ for all x in X, all $t \geqslant 0$ and some integer $\beta \geqslant 1$, we have

$$\begin{aligned}(T(x_{n_j}), K(x_{n_j} - x_0)) &= (T(x_{n_j}), K(x_{n_j} - y_{n_j})) \\ &\quad + 2^\beta\{(T(x_{n_j}), K(\tfrac{1}{2}(x_{n_j} - x_0)) - K(\tfrac{1}{2}(x_{n_j} - y_{n_j})))\},\end{aligned} \qquad (3.3)$$

where

$$y_{n_j} = \| x_0 \| \, \| P_{n_j}(x_0)\|^{-1} P_{n_j}(x_0) \in X_{n_j} \cap \bar{B}(o, r)$$

with $y_{n_j} \to x_0$ as $j \to \infty$. Since

$$z_{n_j}^0 \equiv \tfrac{1}{2}(x_{n_j} - x_0) \qquad \text{and} \qquad z_{n_j} \equiv \tfrac{1}{2}(x_{n_j} - y_{n_j})$$

lie in $\bar{B}(o, r)$ for each j, $\{T(x_{n_j})\}$ is bounded by some $M > 0$, and

$$\| z_{n_j}^0 - z_{n_j} \| = \tfrac{1}{2} \| y_{n_j} - x_0 \| \to 0$$

as $j \to \infty$, in view of (3.2), it follows

$$|(T(x_{n_j}), K(z_{n_j}^0) - K(z_{n_j}))| \leqslant M\psi_r(\| z_{n_j}^0 - z_{n_j} \|) \to 0$$

as $j \to \infty$. On the other hand, by our conditions on K and K_n,

$$\begin{aligned}(T(x_{n_j}), K(x_{n_j} - y_{n_j})) &= (T_{n_j}(x_{n_j}), K_{n_j}(x_{n_j} - y_{n_j})) \\ &= (T_{n_j}(x_{n_j}), K(x_{n_j} - y_{n_j}))\end{aligned}$$

and, therefore, since $T_{n_j}(x_{n_j}) \to g$ in Y and $K(x_{n_j} - y_{n_j}) \rightharpoonup 0$ in Y^*, it follows that $(T(x_{n_j}), K(x_{n_j} - y_{n_j})) \to 0$ as $j \to \infty$. In virtue of the equality (3.3), the above observations imply that

$$\lim_j \sup(T(x_{n_j}), K(x_{n_j} - x_0)) = \lim_j (T(x_{n_j}), K(x_{n_j} - x_0)) = 0$$

as $j \to \infty$. Since T is of type (PKM), it follows that

$$(T(x_0), K(x_0 - v)) \leqslant \lim_j \inf(T(x_{n_j}), K(x_{n_j} - v)) \qquad \text{for all } v \text{ in } X. \tag{3.4}$$

Let v be any element in $B(o, d)$ with $d > r$ and set

$$v_{n_j} = \| v \| \| P_{n_j}(v)\|^{-1} P_{n_j}(v)$$

for each j. Since

$$w_{n_j}^0 \equiv \tfrac{1}{2}(x_{n_j} - v) \qquad \text{and} \qquad w_{n_j} \equiv \tfrac{1}{2}(x_{n_j} - v_{n_j})$$

lie in $\bar{B}(o, d)$ for each j and

$$\| w_{n_j}^0 - w_{n_j} \| = \tfrac{1}{2} \| v_{n_j} - v \| \to 0$$

as $j \to \infty$, an argument similar to the one used above shows that

$$\begin{aligned}&|(T(x_{n_j}), K(x_{n_j} - v)) - (T(x_{n_j}), K(x_{n_j} - v_{n_j}))| \\ &\quad \leqslant 2^\beta |(T(x_{n_j}), K(\omega_{n_j}^0) - K(\omega_{n_j}))| \leqslant 2^\beta M\psi_d(\| \omega_{n_j}^0 - \omega_{n_j} \|) \to 0\end{aligned} \tag{3.5}$$

as $j \to \infty$. Since $x_{n_j} - v_{n_j} \rightharpoonup x_0 - v$ in X, $T_{n_j}(x_{n_j}) \to g$ in Y and $K(x_{n_j} - v_{n_j}) \rightharpoonup K(x_0 - v)$, the equality $(^+)$ satisfied by K and K_n imply that for each v in $B(o, d)$

$$(T(x_{n_j}), K(x_{n_j} - v_{n_j})) \to (g, K(x_0 - v)) \qquad \text{as} \qquad j \to \infty.$$

This and the relation (3.5) imply that for each v in $B(o, d)$

$$\liminf_j (T(x_{n_j}), K(x_{n_j} - v)) = \lim_j (T(x_{n_j}), K(x_{n_j} - v)) = (g, K(x_0 - v))$$

whence, in view of (3.4), it follows that

$$(T(x_0), K(x_0 - v)) \leqslant (g, K(x_0 - v)) \qquad \text{for all } v \text{ in } B(o, d).$$

From this, as in the proof Proposition 6, we obtain the equality $T(x_0) = g$, i.e., T is pseudo-A-proper. Q.E.D.

In virtue of Propositions 6 and 7, Theorem 1 implies the validity of the following theorem for mappings of type (PKM) which is analogous to Theorem 3 for K-monotone mappings.

THEOREM 5. *Let (X, Y) be a pair of Banach spaces with an oriented admissible scheme Γ_n and with X reflexive. Let K be a positively homogeneous mapping of order $\beta \geqslant 1$ of X onto Y^* which is both weakly continuous on X and uniformly continuous on each bounded set in X. Let K_n be a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that for each n and all x in X_n and g in Y*

$$(Q_n(g), K_n(x)) = (g, K(x)) \quad \text{and} \quad (M_n(x), K_n(x)) > 0 \quad \text{for } x \neq 0. \quad (^{++})$$

If T is a bounded fa-*continuous mapping of type* (PKM) *of X into Y such that either the condition* (i) *or the K-coerciveness condition* (ii) *of Theorem* 1 *holds, then for each f in Y the equation $T(x) = f$ has a solution x in X which can be obtained as a weak limit point of a constructable sequence $\{x_n \mid x_n \in X_n\}$.*

To obtain a result for mappings of type (PKM) which would be analogous to Theorem 4 for K-monotone mappings we first have to establish the following propositions.

PROPOSITION 8. *Let (X, Y) be a pair of Banach spaces with an admissible scheme Γ_n and with X reflexive. Let K be a weakly continuous mapping of X onto Y^* such that K is positively homogeneous on X and uniformly continuous on each bounded set of X. Let K_n be a mapping of X_n into Y_n' such that* ($^+$) *holds, i.e.,* $(Q_n(g), K_n(x)) = (g, K(x))$ *for all x in X_n, $g \in Y$ and each n. Let T be a bounded mapping of X into Y which is of type* (PKM) *and let F be a bounded* K-*monotone mapping of X into Y which is also of type* (PKM). *Then, for each real $\mu > o$, the mapping $T_\mu = T + \mu F$ is pseudo-*A-*proper on X.*

Proof. Let $\{x_n \mid x_n \in X_n\}$ a bounded sequence and let $\{x_{n_j}\}$ be any subsequence of $\{x_n\}$ so that $T_{\mu n_j}(x_{n_j}) \to g$ for some g in Y with $\mu > o$ any fixed number. Since X is reflexive and $\{x_{n_j}\}$ is bounded we may assume that $\{x_{n_j}\} \subset B(o, r)$ for some $r > o$ and $x_{n_j} \rightharpoonup x_0$ in X with $x_0 \in \bar{B}(o, r)$.

To prove Proposition 8 we first note that, under our conditions, the same arguments as those used in the proof of Proposition 7 show that,

$$(T_\mu(x_{n_j}) - T_\mu(x_0), K(x_{n_j} - x_0)) \to 0 \quad \text{as} \quad j \to \infty \tag{3.6}$$

for each fixed $\mu > 0$.

Now, since $\mu > o$, F is K-monotone and $T_\mu = T + \mu F$, it follows that for each j we have

$$(T_\mu(x_{n_j}) - T_\mu(x_0), K(x_{n_j} - x_0)) \geqslant (T(x_{n_j}) - T(x_0), K(x_{n_j} - x_0)).$$

In virtue of the relation (3.6), the latter inequality implies that

$$\limsup_j (T(x_{n_j}) - T(x_0), K(x_{n_j} - x_0)) \leqslant 0. \tag{3.7}$$

Because $(T(x_0), K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$, the last inequality implies that

$$\limsup_j (T(x_{n_j}), K(x_{n_j} - x_0)) \leqslant 0,$$

from which, since T is of type (PKM), it follows that

$$(T(x_0), K(x_0 - v)) \leqslant \liminf_j (T(x_{n_j}), K(x_{n_j} - v)) \qquad \text{for all } v \in X. \tag{3.8}$$

Again, since $(T(x_0), K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$, the relation (3.8) for $v = x_0$ implies that

$$\begin{aligned} &\liminf_j (T(x_{n_j}) - T(x_0), K(x_{n_j} - x_0)) \\ &\qquad = \liminf_j (T(x_{n_j}), K(x_{n_j} - x_0)) \geqslant (T(x_0), K(o)) = 0. \end{aligned}$$

This together with (3.7) shows that

$$\lim_j (T(x_{n_j}) - T(x_0), K(x_{n_j} - x_0)) = 0.$$

The latter relation together with (3.6) imply that

$$(F(x_{n_j}) - F(x_0), K(x_{n_j} - x_0)) \to 0 \quad \text{as} \quad j \to \infty$$

from which, since K is weakly continuous and $x_{n_j} - x_0 \rightharpoonup 0$, we get

$$(F(x_0), K(x_{n_j} - x_0)) \to 0$$

and

$$\lim_j \sup(F(x_{n_j}), K(x_{n_j} - x_0)) = \lim_j (F(x_{n_j}), K(x_{n_j} - x_0)) = 0.$$

In view of this and the assumption that F is of type (PKM) it follows that

$$(F(x_0), K(x_0 - v)) \leqslant \lim_j \inf(F(x_{n_j}), K(x_{n_j} - v)) \qquad \text{for all } v \in X. \tag{3.9}$$

Multiplying (3.9) by $\mu > o$ and then adding the corresponding sides of (3.8) and (3.9) and using certain properties of lim inf's we obtain the looked for inequality

$$\begin{aligned}(T_\mu(x_0), K(x_0 - v)) &\leqslant \lim_j \inf(T(x_{n_j}), K(x_{n_j} - v)) \\ &\quad + \lim_j \inf(\mu F(x_{n_j}), K(x_{n_j} - v)) \\ &\leqslant \lim_j \inf(T_\mu(x_{n_j}), K(x_{n_j} - v)) \qquad \text{for all } v \in X.\end{aligned} \tag{3.10}$$

Since $T_{\mu n_j}(x_{n_j}) \to g$ in Y and K is both weakly continuous on X and uniformly continuous on $B(o, d)$ for each $d > o$, the same arguments as those used in the proof of Proposition 7 show that for each v in $B(o, d)$ with $d > r$ we have the equality

$$\lim_j \inf(T_\mu(x_{n_j}), K(x_{n_j} - v)) = \lim_j(T_\mu(x_{n_j}), K(x_{n_j} - v)) = (g, K(x_0 - v).$$

It follows from this and (3.10) that

$$(T_\mu(x_0), K(x_0 - v)) \leqslant (g, K(x_0 - v)) \qquad \text{for all } v \text{ in } B(o, d)$$

whence, as before, we obtain the equality $T_\mu(x_0) = g$. Hence $T_\mu = T + \mu F$ is pseudo-A-proper for each $\mu > o$. Q.E.D.

It turns out that if we strengthen somewhat the conditions on F, then the mapping $T_\mu = T + \mu F$ will actually be A-proper. Indeed, the following result holds.

Proposition 9. *Let (X, Y), K and K_n be as in Proposition* 8. *Let T be a bounded mapping of type* (PKM) *of X into Y. Let F be a bounded* K*-monotone mapping of X into Y which satisfies condition* (S) *on X. Suppose further that either F is demicontinuous and $Q_n^*(y^*) \to y^*$ in Y^* for each y^* in Y^* or F is continuous. Then, for each real $\mu > o$, $T_\mu = T + \mu F$ is* A*-proper.*

Proof. Let $\{x_n \mid x_n \in X_n\}$ be a bounded sequence and let $\{x_{n_j}\}$ be any subsequence of $\{x_n\}$ so that $T_{\mu n_j}(x_{n_j}) \to g$ for some g in Y. It was shown in the proof of Proposition 8 that, under our conditions, we have the relation $x_{n_j} \rightharpoonup x_0$ in X with $x_0 \in \bar{B}(o, r)$, the inequality

$$(T(x_0), K(x_0 - v)) \leqslant \liminf_j (T(x_{n_j}), K(x_{n_j} - v)) \qquad \text{for all } v \text{ in } X, \qquad (3.11)$$

and the equality

$$\lim_j (F(x_{n_j}) - F(x_0), K(x_{n_j} - x_0)) = 0.$$

In view of the condition (S) satisfied by F, the last equality implies that $x_{n_j} \to x_0$ in X as $j \to \infty$.

Finally, to show that $T_\mu(x_0) = g$, assume first that F is demicontinuous and $Q_n{}^*(y^*) \to y^*$ in Y^* for each y^* in Y^*. Since $x_{n_j} \to x_0$ in X, the preceding assumption implies that $F(x_{n_j}) \rightharpoonup F(x_0)$ and $Q_{n_j} F(x_{n_j}) \rightharpoonup F(x_0)$ in Y. Consequently,

$$T_{n_j}(x_{n_j}) = T_{\mu n_j}(x_{n_j}) - \mu F_{n_j}(x_{n_j}) \rightharpoonup g - \mu F(x_0)$$

in Y. Let v be any element in $B(o, d)$ with $d > r$ and set

$$v_{n_j} = \| v \| \, \| P_{n_j} v \|^{-1} P_{n_j}(v)$$

for each j. Since $x_{n_j} - v_{n_j} \to x_0 - v$ in X, $K(x_{n_j} - v_{n_j}) \to K(x_0 - v)$ in Y^* and $T_{n_j}(x_{n_j}) \rightharpoonup g - \mu F(x_0)$ in Y, the properties of K and K_n imply that

$$(T(x_{n_j}), K(x_{n_j} - v_{n_j})) \to (g - \mu F(x_0), K(x_0 - v))$$

for each v in $B(o, d)$. This and the properties of K and K_n , as above imply that for each v in $B(o, d)$ we have the equality

$$\begin{aligned} \liminf_j (T(x_{n_j}), K(x_{n_j} - v)) &= \lim_j (T(x_{n_j}), K(x_{n_j} - v_{n_j})) \\ &= (g - \mu F(x_0), K(x_0 - v)) \end{aligned}$$

whence, in virtue of (3.11), it follows that

$$(T(x_0), K(x_0 - v)) \leqslant (g - \mu F(x_0), K(x_0 - v)) \qquad \forall v \in B(o, d). \qquad (3.12)$$

Hence, the same arguments as before show that $T(x_0) = g - \mu F(x_0)$, i.e., T_μ is A-proper.

Suppose now that F is continuous. Then, since $x_{n_j} \to x_0$ in X, $F(x_{n_j}) \to F(x_0)$ and, therefore, $Q_{n_j} F(x_{n_j}) \to F(x_0)$ in Y. Consequently,

$$T_{n_j}(x_{n_j}) = T_{\mu n_j}(x_{n_j}) - \mu F_{n_j}(x_{n_j}) \to g - \mu F(x_0) \text{ in } Y.$$

As before, this implies that for each v in $B(o, d)$

$$\lim_j (T(x_{n_j}), K(x_{n_j} - v))$$
$$= \lim_j (T_{n_j}(x_{n_j}), K_{n_j}(x_{n_j} - v_{n_j})) \to (g - \mu F(x_0), K(x_0 - v)).$$

In view of (3.11), the last relation implies also in this case the validity of (3.12) and, thus, the equality $T_\mu(x_0) = g$, i.e., T_μ is also A-proper. Q.E.D.

In virtue of Propositions 6, 7 and 8, 9, Theorem 2 implies the validity of the following new result for the mappings of type (PKM).

THEOREM 6. *Let (X, Y) be a pair of Banach spaces with an oriented admissible projectional scheme Γ_n and with X reflexive. Let K be a weakly continuous mapping of X onto Y^* such that $K(tx) = t^\beta K(x)$ for all x in X, all $t \geqslant 0$ and some integer $\beta \geqslant 1$ and such that K is uniformly continuous on bounded sets in X. Let K_n be a mapping of X_n into $Y_n' = R(Q_n^*)$ and M_n a linear isomorphism of X_n onto Y_n such that*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \text{and} \qquad (M_n(x), K_n(x)) > 0 \qquad (^{++})$$

for $x \neq 0$ in X_n and g in Y. Let F be a bounded K-monotone mapping of X into Y such that F is positively homogeneous of order $\alpha \geqslant 1$. Suppose further that F satisfies either one of the following two conditions:

(I) *F is an* fa-*continuous mapping of type* (PKM) *on X.*

(II) *F satisfies condition* (S) *on X and either F is demicontinuous and $Q_n^*(y^*) \to y^*$ in Y^* for each y^* in Y^* or F is continuous on X.*

If, under the above condtions, T is a bounded fa-*continuous mapping of type* (PKM) *of X into Y such that for each fixed $\mu > o$ and some constants $b > o$ and $C \geqslant 0$*

(c4) $(T_\mu(x) - T_\mu(o), K(x)) \geqslant (\mu b \| x \|^\alpha - C) \| K(x) \|$ *for all x in X*

and

(c5) $\| T(x) \| \to \infty$ *as* $\| x \| \to \infty$,

then T maps X onto Y.

Remark 7. The condition (c4) holds, in particular, if

(c4a) $(Tx, Kx) \geqslant (T(o), Kx) - |(T(o), Kx)|$ for all x in X

and

(c4b) $(Fx, Kx) \geqslant b(\| x \|)^\alpha \| Kx \|$ for all x in X and some $b > o$.

We complete this section by deducing from our Theorem 1′ a generalization of Kachurowsky's result (see Theorem 4 in [16]) stated in [16] without proof for bounded finitely-continuous coercive mappings T of X into X^* such that

$$(T(x) - T(y), x - y) + |(T(y), x - y)| \geqslant 0 \qquad \text{for all } x \text{ and } y \text{ in } X.$$

In what follows we shall say that a mapping T of X into Y is *quasi*-K-*monotone*[3] if for all x and y in X

$$(T(x) - T(y), K(x - y)) + |(T(y), K(x - y))| \geqslant 0. \qquad \text{(qm)}$$

It follows that every K-monotone mapping is also quasi-K-monotone but the converse is not true in general (see [16]).

PROPOSITION 10. *Let (X, Y) be a pair of real Banach spaces with an oriented admissible projectional scheme Γ_n and with X reflexive. Let K be a weakly continuous and positively homogeneous of order $\beta \geqslant 1$ mapping of X onto Y^* and let K_n be a mapping of X_n into Y_n' such that* (+) *of Proposition* 8 *holds. If T is a mapping of X into Y such that $T_f = T - f$ is quasi*-K-*monotone for given f in Y, then T is pseudo*-A-*proper at f provided it satisfies any one of the following conditions:*

(a) *T is continuous.*

(b) *T is demicontinuous and K is also continuous.*

(c) *T is bounded and finitely continuous* (*or hemicontinuous*) *and K is also uniformly continuous on bounded sets in X.*

Proof. Let $\{x_n \mid x_n \in X_n\}$ be a bounded sequence so that $T_{n_j}(x_{n_j}) \to f$ in Y for given f in Y. To establish the existence of an element x_0 in X such that $T(x_0) = f$, we note first that since X is reflexive, $\{x_{n_j}\}$ is bounded and $Q_n(f) \to f$, we may assume that $x_{n_j} \rightharpoonup x_0$ for some x_0 in X and

$$T_{fn_j}(x_{n_j}) \equiv T_{n_j}(x_{n_j}) - Q_{n_j}(f) \to 0 \qquad \text{as} \qquad j \to \infty.$$

(a) Suppose that T is continuous. Then since $x_{n_j} \rightharpoonup x_0$ in X, $T_{fn_j}(x_{n_j}) \to 0$ in Y, $Q_{n_j} T_f P_{n_j}(x) \to T_f(x)$ for each x in X, K is weakly continuous and T_f is

[3] We refer to mappings T satisfying the inequality (qm) for $Y = X^*$ and $K = I$ as quasi-monotone instead of pseudo-monotone, as referred to by Kachurowsky, to distinguish them from pseudo-monotone mappings as defined and used by Brezis, Browder, Lions, the author, and others.

quasi-K-monotone, it follows from this and (+) that for each x in X we have the inequality

$$(Q_{n_j} T_f P_{n_j}(x) - T_{fn_j}(x_{n_j}), K_{n_j}(P_{n_j}(x) - x_{n_j})) + |(T_{fn_j}(x_{n_j}), K_{n_j}(P_{n_j}(x) - x_{n_j}))| \geqslant 0$$

and hence the passage to the limit as $j \to \infty$ in the above inequality yields the relation

$$(T_f(x), K(x - x_0)) \geqslant 0 \qquad \text{for all } x \text{ in } X.$$

For any z in X, letting $x = x_0 + tz$ with $t > o$ and using the positive homogeneity of K we get $(T_f(x_0 + tz), K(z)) \geqslant 0$ for each $t > o$ and each z in X. Taking the limit as $t \to 0$ we obtain the inequality $(T_f(x_0), K(z)) \geqslant 0$ for each z in X from which it follows that $T_f(x_0) = 0$, i.e., $T(x_0) = f$.

(b) Let T be demicontinuous and K also continuous. Let k be an arbitrary but fixed integer and let x be any element in X_{n_k}. Since $\{X_{n_j}\}$ is monotonically increasing and T is quasi-K-monotone, it follows from (+) that for all $n_j > n_k$ we have

$$(Q_{n_j} T_f(x) - T_{fn_j}(x_{n_j}), K_{n_j}(x - x_{n_j})) + |(T_{fn_j}, K_{n_j}(x - x_{n_j}))| \geqslant 0$$

from which, on the passage to the limit as $j \to \infty$, we obtain

$$(T_f(x), K(x - x_0)) \geqslant 0 \qquad \text{for all } x \in X_{n_k} \tag{3.13}$$

and each fixed k. Since $P_{n_k}(y) \to y$ for each y in X as $k \to \infty$, T_f is demicontinuous and K is continuous, the inequality (3.13) also holds for all x in X and consequently we get the equality $T(x_0) = f$.

(c) Suppose now that T is bounded and finitely continuous (or hemicontinuous) and that K is uniformly continuous on bounded sets in X. Since

$$(T_{fn_j}(x_{n_j}), K_{n_j}(P_{n_j}(x) - x_{n_j})) \to 0$$

for every fixed x in X and

$$|(T_f(x_{n_j}), K(x - x_{n_j})) - (T_{fn_j}(x_{n_j}), K_{n_j}(P_{n_j}(x) - x_{n_j}))| \leqslant 2^\beta M\psi(\| \tfrac{1}{2} (P_{n_j}(x) - x)\|) \to 0$$

as $j \to \infty$, it follows that $(T_f(x_{n_j}), K(x - x_{n_j})) \to 0$ for each fixed x in X. In view of the above relation and the inequality

$$(T_f(x) - T_f(x_{n_j}), K(x - x_{n_j})) + |(T_f(x_{n_j}), K(x - x_{n_j}))| \geqslant 0, \tag{3.14}$$

the passage to the limit in (3.14) as $j \to \infty$ yields the relation

$$(T_f(x), K(x - x_0)) \geqslant 0$$

for each x in X. From the latter inequality we derive, as before, the equation $T(x_0) = f$. Thus, in each case, T is pseudo-A-proper at f. Q.E.D.

In virtue of Proposition 10, Theorem 1′ implies the validity of the following theorem for quasi-K-monotone mappings which is a generalization of Theorem 3 for K-monotone mappings.

THEOREM 7. *Let (X, Y), K and K_n satisfy the conditions of Proposition* 10 *and let M_n be a linear isomorphism of X_n onto Y_n such that* $(^{++})$ *of Theorem* 6 *holds. For a given f in Y let T be a mapping of X into Y such that $T_f = T - f$ is quasi-*K*-monotone, T satisfies any one of the conditions* (a), (b) *or* (c) *of Proposition* 10, *and T is* K*-coercive. Then the equation $T(x) = f$ is solvable in X.*

Proof. In virtue of the hypotheses of Theorem 7, Propositions 1 and 10 imply that in either case T is a fa-continuous mapping of X into Y which is pseudo-A-proper at f. Since f is a fixed element of Y and T is K-coercive, there exists a number $r > 0$ such that

$$(T(x), K(x)) \geqslant (f, K(x)) \qquad \text{for all } x \text{ in } \dot{B}(o, r).$$

Hence, by Theorem 1′, there exists an element x_0 in $\bar{B}(o, r)$, such that $T(x_0) = f$. Q.E.D.

Since, for each f in Y, $T_f = T - f$ is quasi-K-monotone if T is a K-monotone mapping of X into Y, we see that Theorem 7 is indeed a generalization of Theorem 3. We add that Theorem 4 in [16] is deduced from Theorem 7 by setting $Y = X^*$ and $K = I$.

4. MAPPINGS OF TYPE (KM)

In this section we show that mappings T of $D \subseteq X$ into Y of type (KM) defined below and, in particular, the mappings T of X into X^* of type (M), defined here in terms of sequences rather than filters as in Brezis [2], form a subclass of the class of pseudo-A-proper mappings. Consequently, Theorems 1 and 2 of Section 1 are applicable to mappings of type (KM). In particular, Brezis' basic existence results (Theorem 10 and Corollary 14 in [2]) for mappings T of type (M) defined in terms of sequences follow from our Theorem 1 for pseudo-A-proper mappings or more precisely they will be deduced as special cases of our Theorem 8. Theorem 9 below is new even for the mappings of type (M). We add in passing that, unless K is linear, a mapping of type (PKM) need not be of type (KM) when $Y \neq X^*$ and K is nonlinear. Consequently, the mappings T of $D \subseteq X$ into Y of type (KM) have to be discussed as a separate subclass of pseudo-A-proper mappings.

DEFINITION 5. Let D be an open convex subset of a reflexive Banach space X. A mapping T of $\bar{D}$ into Y is said to be of type (KM) if for any sequence $\{z_j\} \subset \bar{D}$ such that $z_j \rightharpoonup z$ in $\bar{D}$, $T(z_j) \rightharpoonup f$ for some f in Y and $\lim_j \sup(T(z_j), K(z_j)) \leqslant (f, K(z))$ we have $T(z) = f$.

When $Y = X^*$ and $K = I$, a map of type (KM) becomes, except for the continuity assumption, a map of type (M) studied in [2] if fillers are replaced by sequences. We add in passing that, as was shown in [2], in general a sum of two mappings of type (M) need not be a mapping of type (M). We first prove the following two useful propositions for mappings of type (KM).

PROPOSITION 11. *Let (X, Y) be a pair of real Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K be a weakly continuous mapping of X into Y^*, D an open convex subset of X and T a mapping of D into Y of type* (KM) *on D. Then, for each bounded convex closed subset G of D, the set $T(G)$ is closed in Y.*

Proof. Let $\{x_j\} \subset G$ be a sequence so that $T(x_j) \to f$ for some f in Y. To show that $f \in T(G)$, note that since X is reflexive and $\{x_j\}$ is bounded, without loss of generality we may assume that $x_j \rightharpoonup x_0$ with $x_0 \in G$. Thus, since K is weakly continuous and $x_j \rightharpoonup x_0$ in X, $T(x_j) \to f$ in Y and $K(x_j) \rightharpoonup Kx_0$ in Y^* and, therefore,

$$\lim_j \sup(T(x_j), K(x_j)) = \lim_j (T(x_j), K(x_j)) = (f, K(x_0)).$$

Since T is of type (KM), it follows that $T(x_0) = f$, i.e., $T(G)$ is a closed set in Y. Q.E.D.

To establish the pseudo-A-properness of T we assume additionally that T is bounded.

PROPOSITION 12. *Let (X, Y) be a pair of Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K be a mapping of X onto Y^* which is both continuous and weakly continuous. Let K_n be a mapping of X_n into Y_n' such that for each n*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \text{for all } x \text{ in } X_n,\ g \text{ in } Y. \tag{+}$$

If D is a convex subset of X and if T is a bounded mapping of type (KM) *of $\bar{D}$ into Y, then T is pseudo-A-proper.*

Proof. Let $\{x_n \mid x_n \in \bar{D}_n\}$ be a bounded sequence and let $\{x_{n_j}\}$ be an arbitrary subsequence such that $T_{n_j}(x_{n_j}) \to g$ for some $g \in Y$. Since X is reflexive, $\{x_{n_j}\} \subset \bar{D}$ is bounded and $\bar{D}$ is weakly closed, we may assume that $x_{n_j} \rightharpoonup x_0$ in X with x_0 in $\bar{D}$. It is easy to see that since K is onto, T is bounded and $T_{n_j}(x_{n_j}) \to g$, we have the relation $T(x_{n_j}) \rightharpoonup g$ in Y, i.e., for each y^* in Y^* we have $(T(x_{n_j}), y^*) \to (g, y^*)$. Indeed, note first that since K is onto, to any y^* in Y^* there exists an y in X such that $y^* = K(y)$. Now, consider the equality

$$\begin{aligned}(T(x_{n_j}), K(y)) - (g, K(y)) &= (T(x_{n_j}), K(y)) - (T(x_{n_j}), K(P_{n_j}(y))) \\ &\quad + (T(x_{n_j}), KP_{n_j}(y)) - (g, K(y)).\end{aligned}$$

Since $P_{n_j}(y) \to y$ in X, $\{T(x_{n_j})\}$ is bounded and, by ($^+$) and the continuity of K,

$$(T(x_{n_j}), K(y)) - (T(x_{n_j}), K(P_{n_j}(y)) \to 0 \qquad \text{as} \qquad j \to \infty$$

and

$$(T(x_{n_j}), KP_{n_j}(x)) = (T_{n_j}(x_{n_j}), KP_{n_j}(x)) \to (g, K(x)),$$

it follows from the above equality that for each y^* in Y^*

$$(T(x_{n_j}), y^*) = (T(x_{n_j}), K(y)) \to (g, K(y)) = (g, y^*).$$

Furthermore, since K is weakly continuous, $x_{n_j} \rightharpoonup x_0$ in X and $T_{n_j}(x_{n_j}) \to g$, the relation ($^+$) also implies that

$$\begin{aligned}\limsup_j (T(x_{n_j}), K(x_{n_j})) &= \limsup_j (T_{n_j}(x_{n_j}), K(x_{n_j})) \\ &= \lim_j (T_{n_j}(x_{n_j}), K(x_{n_j})) = (g, K(x_0)).\end{aligned}$$

In view of the above discussion and the fact that T is of type (KM), it follows that $T(x_0) = g$, i.e., T is pseudo-A-proper. Q.E.D.

In view of Proposition 12, Theorem 1 implies the validity of the following theorem for fa-continuous bounded mappings of type (KM).

THEOREM 8. *Let (X, Y) be a pair of Banach spaces with an oriented admissible projectional scheme Γ_n and with X reflexive, D a bounded open convex subset of X with $o \in D$, K a mapping of X onto Y^* which is both continuous and weakly continuous, K_n a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that for each n and all x in X_n and g in Y.*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \textit{and} \qquad (M_n(x), K_n(x)) > 0, \forall x \neq 0.$$

(a) *If T is a bounded* fa-*continuous mapping of $\bar{D}$ into Y of type* (KM) *and if f is a given element in Y such that $(T(x), K(x)) \geqslant (f, K(x))$ for all x in $\dot{D}$, then Eq.* (1) *has a solution in $\bar{D}$.*

(b) *If T is a bounded* fa-*continuous mapping of X into Y of type* (KM) *such that either to each f in Y there corresponds an $r_f > 0$ so that*

$$(T(x), K(x)) \geqslant (f, K(x)) \qquad \text{for all } x \text{ in } \dot{B}(o, r_f)$$

or T is K-*coercive, then T maps X onto Y.*

In view of Proposition 12, Remark 2 concerning the constructive aspect of our proof applies also to Theorem 8.

To obtain an analogue of Theorem 2 for mappings T of type (KM) we have first to establish the pseudo-A-properness of the mapping $T_\mu = T + \mu F$ of X into Y for a suitable mapping F of X into Y and each $\mu > 0$. This we do in Proposition 13 under rather restrictive conditions on F.

PROPOSITION 13. *Let (X, Y) be a pair of real Banach spaces with an admissible projectional scheme Γ_n and with X reflexive. Let K and K_n be as in Proposition* 12. *If T is a bounded mapping of type* (KM) *of X into Y and if F is a weakly continuous mapping of X into Y such that the functional $f(x) = (F(x), K(x))$ of X into R^1 is weakly lower semicontinuous, then $T_\mu = T + \mu F$ is pseudo-*A-*proper for each $\mu > 0$.*

Proof. In view of Proposition 12, it suffices to show that T_μ is of type (KM) for each fixed $\mu > o$. To prove the latter, let $\mu > o$ be any fixed number and let $\{x_j\} \subset X$ be any sequence so that

$$x_j \rightharpoonup x_0 \quad \text{in} \quad X, \qquad T_\mu(x_j) \rightharpoonup g \quad \text{in} \quad Y$$

and

$$\lim_j \sup(T_\mu(x_j), K(x_j)) \leqslant (g, K(x_0)) \qquad \text{for some } x_0 \text{ in } X \text{ and } g \text{ in } Y.$$

Hence it follows from the weak continuity of F, the weak lower semicontinuity of $f(x) = (F(x), K(x))$ and the equality

$$(T(x_j), K(x_j)) = (T_\mu(x_j), K(x_j)) - \mu(F(x_j), K(x_j))$$

that

$$T(x_j) = T_\mu(x_j) - \mu F(x_j) \rightharpoonup g - \mu F(x_0)$$

and

$$\begin{aligned} \lim_j \sup(T(x_j), K(x_j)) &\leqslant \lim_j \sup(T_\mu(x_j), K(x_j)) - \mu \lim_j \inf(F(x_j), K(x_j)), \\ &\leqslant (g, K(x_0)) - \mu(F(x_0), K(x_0)) \\ &= (g - \mu F(x_0), K(x_0)). \end{aligned}$$

Since T is of type (KM), it follows from the above that $T(x_0) = g - \mu F(x_0)$ or $T_\mu(x_0) = g$, i.e., T_μ is pseudo-A-proper. Q.E.D.

In virtue of Propositions 11 and 13, Theorem 2 implies the validity of the following new result for mappings of type (KM) and, in particular, for mappings of type (M).

THEOREM 9. *Let (X, Y) be a pair of Banach spaces with Γ_n an oriented admissible scheme and with X reflexive, K a mapping of X onto Y^* which is both continuous and weakly continuous, K_n a mapping of X_n into Y_n' and M_n a linear isomorphism of X_n onto Y_n such that for x in X_n and g in Y.*

$$(Q_n(g), K_n(x)) = (g, K(x)) \qquad \textit{and} \qquad (M_n(x), K_n(x)) > 0, \forall x \neq 0. \quad (^{++})$$

If T is a bounded fa*-continuous mapping of X into Y of type* (KM) *and F is a weakly continuous mapping of X into Y such that the functional $f(x) = (F(x), K(x))$ of X into R^1 is weakly lower semicontinuous and F is positively homogeneous of order $\alpha \geqslant 1$ and if the conditions* (c4) *and* (c5) *of Theorem* 2 *hold, then T maps X onto Y.*

5. Special Cases

In this section we discuss the applicability of the preceeding theorems to various classes of mappings T of $D \subseteq X$ into X^* and to mappings T of $D \subseteq X$ into X by specifying Y, Y_n, Q_n, K, K_n, M_n, and F. From our results for fa-continuous pseudo-A-proper mappings (Theorems 10 and 12 below) and their extensions (Theorem 11 and 13 below) we deduce most of the known fixed point and surjectivity theorems as well as some new ones for various classes of mappings such as P-compact, weakly closed, monotone, pseudo-monotone, J-monotone as well as mappings of types (JPM), (M), (JM) and others. In case T is continuous and A-proper some of the results or their variants were obtained by the author in [35].

A. *Existence Theorems for Mappings T from X to X^**

Let X be a real reflexive Banach space with a Schauder basis

$$\{\psi_1, \psi_2, \psi_3, \ldots\} \subset X, \qquad X_n = \operatorname{span}\{\psi_1, \ldots, \psi_n\},$$

and P_n the projection of X onto X_n. To deduce the corresponding results for mappings T of $D \subseteq X$ into X^* from those obtained in the preceeding sections, we set $Y = X^*$ and $Q_n = P_n^*$ with $Y_n = R(P_n^*) \equiv X_n' \subset X^*$ and observe that in this case it is known that $\Gamma_n = \{X_n, X_n', P_n, P_n^*\}$ is an admissible projectional scheme for the pair (X, X^*); we recall that $P_n^* : X^* \to X_n'$ is the adjoint of the linear mapping P_n. Since X is reflexive and $Y = X^*$, it follows that $Y^* = X$ and hence the simplest choice for $K : X \to Y^* = X$ and $K_n : X_n \to Y_n' = R(Q_n^*) = R(P_n) = X_n$ is to take $K = I$ and $K_n = I_n$, where I and I_n denote the identities in X and X_n, respectively. To construct a suitable linear isomorphism M_n of X_n onto $Y_n = X_n'$, let $\{f_j\}$ be the sequence in X^* which satisfies the biorthogonality relation $(f_i, \psi_j) = \delta_{ij}$ $(i, j = 1, 2, 3,...)$. Then, for each n,

$$X_n' = \operatorname{span}\{f_1, f_2, ..., f_n\}$$

and, therefore, $M_n : X_n \to X_n'$ defined by $M_n(x) = \sum_{i=1}^n f_i(x) f_i$ for each x in X_n is linear, one-to-one, onto, and such that for each n

$$(M_n(x), K_n(x)) = (M_n(x), x) = \sum_{i=1}^{n} f_i^2(x) > 0 \qquad \text{for each } x \in X_n$$

with $x \neq 0$. Clearly, the mappings K, K_n and M_n thus chosen satisfy all the corresponding conditions used in Sections 1–4. Consequently, for mappings T of X into X^*, Theorem 1 reduces to the following new result for pseudo-A-proper mappings T of $D \subset X$ into X^*.

THEOREM 10. *Let X be a real reflexive Banach space with a Schauder basis and D a bounded open subset of X with $o \in D$.*

(a) *If T is an* fa-*continuous pseudo*-A-*proper mapping of $\bar{D}$ into X^* such that $(T(x), x) \geqslant (f, x)$ for all x in $\dot{D}$ and some f in X^*, then the equation $T(x) = f$ has a solution in $\bar{D}$.*

(b) *If T is an* fa-*continuous pseudo*-A-*proper map of X into X^* such that either* (i): *to each f in X^* there corresponds $r_f > o$ such that $(T(x), x) \geqslant (f, x)$ for all x in $\dot{B}(o, r_f)$ or* (ii): *T is coercive (i.e., $(T(x), x) \geqslant c(\| x \|) \| x \|$ for all x in X with $c(r) \to \infty$ as $r \to \infty$), then T maps X onto X^*.*

Now, for the case when $Y = X^*$ and $K = I$, the class of K-monotone mappings reduces to the class of monotone mappings T of X into X^*, the class of mappings T of type (PKM) reduces to the class of pseudo-monotone mappings T of X into X^*, while the class of fa-continuous mappings of type (KM) reduces to maps of type (M) in the sense of Brezis if sequences replace filters. Furthermore, by Propositions 1 and 4, every demicontinuous monotone map T of X into X^* is an fa-continuous pseudo-A-proper map while, by Propositions 7 and 12, every bounded pseudo-monotone and every bounded of type (M) map T of X into X^* is a bounded pseudo-A-proper map which is also fa-continuous by the results of Brezis [2]. Consequently, as a corollary of our Theorem 10(b) (or a special case of Theorems 3, 5 and 8, respectively) we obtain the following basic existence results for the three special classes of mappings.

COROLLARY 3. *If X is a real reflexive Banach space with a Schauder basis and T is a mapping of X into X^* which satisfies either the condition* (i) *or the coerciveness condition* (ii) *of Theorem* 10, *then the following three assertions are valid:*

(A) *If T is also demicontinuous* (*or weakly continuous*) *and monotone, then $T(X) = X^*$.*

(B) *If T is also bounded and pseudo-monotone, then $T(X) = X^*$.*

(C) *If T is also bounded and of type* (M) *then $T(X) = X^*$.*

Remark 9. Corollary 3(A), which is the basic surjectivity theorem for monotone mappings T of X into X^*, has been obtained independently by Minty [21] and Browder [5] for T demicontinuous and coercive (see also [41]) and by Kachurowsky [16] for T weakly continuous and coercive. Corollaries 3(B) and (C) have been established by Brezis [2]. The above authors used different methods and obtained their results for reflexive spaces not necessarily having Schauder bases.

To obtain an analogue of Theorem 2 for mappings T of X into X^* we assume additionally that X^* is strictly convex and then take $F : X \to X^*$ to be the duality mapping $F = J_0 : X \to X^*$ defined as follows:

$$J_0(o) = o$$

and

$$J_0(x) = \{\omega \mid \omega \in X^*, (\omega, x) = \| \omega \| \| x \| , \| \omega \| = \psi(\| x \|)\}, \tag{J1}$$

where $\psi(r) = r^\alpha$ for $r \geqslant o$ with α some positive integer. Since X is reflexive and X^* strictly convex, it is known [6] that J_0 is a single-valued demicontinuous mapping of X onto X^* which is clearly bounded and positively homogeneous of order $\alpha \geqslant 1$; furthermore, J_0 is monotone and, in fact,

$$(J_0(x) - J_0(y), x - y) \geqslant (\psi(\| x \|) - \psi(\| y \|)) (\| x \| - \| y \|) \qquad \text{for } x, y \text{ in } X. \tag{J2}$$

Now, Theorem 2 yields the validity of the following new result for a mapping T of X into X^* which is a uniform limit of pseudo-A-proper mappings on bounded subsets of X and which satisfies a condition "at infinity" that is more general than the coerciveness condition (ii) of Theorem 10.

THEOREM 11. *Let X be a real reflexive Banach space with a Schauder basis and with X^* strictly convex. Let T be an* fa-*continuous mapping of X into X^* such that*

(c1A) $T(G)$ *is closed in X^* for each bounded closed convex set G in X.*

(c2A) $T_\mu = T + \mu J_0$ *is pseudo-A-proper for each $\mu > o$.*

(c4A) $(T(x), x) \geqslant - C \| x \| + (T(o), x)$ *for all $x \in X$ and some $C \geqslant 0$.*

(c5A) $\| T(x) \| \to \infty$ *as* $\| x \| \to \infty$.

Then T is surjective, i.e., $T(X) = X^*$.

Now, in case T is a hemicontinuous monotone mapping of X into X^*, the conditions (c1A) and (c4A) follow from Proposition 3 and the monotonicity of T with $C = 0$ while, since J_0 is monotone and $\mu > o$, (c2A) follows from Proposition 4 for $A = T_\mu$. Consequently, from Theorem 11 (or from Theorem 4 for $Y = X^*$ and $K = I$) we deduce the validity of the following general result for monotone mappings which was essentially obtained in [5] under the additional condition that X is locally uniformly convex.

COROLLARY 4. *Let X be a reflexive Banach space with a Schauder basis and with X^* strictly convex. If T is a hemicontinuous monotone mapping of X into X^* such that $\| T(x) \| \to \infty$ as $\| x \| \to \infty$, then T maps X onto X^*.*

Remark 10. If, as in [14], we assume the existence of a function $c(r)$ of R^+ to R^+ with $c(r) \to \infty$ as $r \to \infty$ such that for some $r > o$

$$\| T(x) - tT(-x) \| \geqslant c(\| x \|) \qquad \text{for all } t \text{ in } [0, 1] \text{ and } \| x \| \geqslant r,$$

then, for $t = 0$, $\| T(x) \| \geqslant c(\| x \|)$ and, consequently, Theorem 2 in [14] follows from our Corollary 4 without the assumption that T is bounded or even continuous.

If T is a bounded pseudo-montone map of X into X^*, then Theorem 11 (or Theorem 6 for $Y = X^*$ and $K = I$) is also applicable. Thus, if for F in Theorem 6 we take the duality mapping $F = J_0$, then F thus chosen is bounded, demicontinuous, positively homogeneous of order $\alpha \geqslant 1$ and monotone. Hence, the results in [2] imply that F satisfies the condition (I) of Theorem 6. In view of the above remarks, Theorem 6 or 11 yields the validity of the following new surjectivity theorem for bounded pseudo-monotone mappings.

COROLLARY 5. *Let X be a reflexive Banach space with a Schauder basis and with X^* is strictly convex. If T is a bounded pseudo-monotone map of X into X^* which satisfies the conditions* (c4A) *and* (c5A) *of Theorem 11, then T maps X onto X^*.*

We note that Theorem 11 (or Theorem 9 for $Y = X^*$ and $K = I$) is also applicable to mappings of type (M) if we assume additionally that $J_0 : X \to X^*$ is also weakly continuous, set $F = J_0$, and observe that the functional $f(x) = (J(x), x)$ is weakly lower semicontinuous. Thus under the above more restrictive condition on F we have the following new result for mappings of type (M).

COROLLARY 6. *Let X be a reflexive Banach space with a Schauder basis such that X^* is strictly convex and the duality mapping $J_0 : X \to X^*$ is also weakly continuous. If T is a bounded* fa*-continuous mapping of X into X^* of type* (M) *which satisfies the conditions* (c4A) *and* (c5A) *of Theorem* 11, *then T maps X onto X^*.*

We conclude Section 5A by establishing the A-properness of certain mappings T of X into X^* by utilizing the assertion of Proposition 5. We first recall that X is said to have Property (H) if X is strictly convex and if $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x$ in X and $\| x_n \| \to \| x \|$, then $x_n \to x$ in X.

PROPOSITION 14. *Let X be a real reflexive Banach space with a Schauder basis and such that X has Property* (H) *and X^* is strictly convex. Let $J_\psi : X \to X^*$ be the duality map corresponding to a given strictly increasing real-valued function $\psi(r)$ of R^+ into R^+ such that $\psi(o) = o$ and $\psi(r) \to \infty$ as $r \to \infty$. If T is a demicontinuous* (*or a weakly continuous*) *monotone mapping of X into X^*, then $T_\mu = T + \mu J_\psi$ is an* A-*proper mapping of X into X^* for each fixed $\mu > o$.*

Proof. If in Proposition 5 for the case when $Y = X^*$, $K = I$, and $K_n = I_n$ we choose $F = J_\psi$, then since J_ψ is a demicontinuous mapping of X into X^* which satisfies the inequality (F) of Proposition 5 in the form

$$(J_\psi(x) - J_\psi(y), x - y) \geqslant (\psi(\| x \|) - \psi(\| y \|)) (\| x \| - \| y \|)$$

for all x and y in X, it suffices to show that the function γ, defined in our case by

$$\gamma(\| x \|, \| y \|) = (\psi(\| x \|) - \psi(\| y \|)) (\| x \| - \| y \|) \qquad (x, y \in X),$$

is nonnegative and satisfies conditions (a) and (b) of Proposition 5. To simplify the notation we set $t = \| x \|$, $s = \| y \|$, $t_k = \| x_k \|$ for $k = 0, 1, 2,\ldots$, and $\gamma(t, s) = (\psi(t) - \psi(s)\ (t - s)$. Now, since ψ is strictly increasing, $\gamma(t, s) \geqslant o$ for all $t \geqslant o$ and $s \geqslant o$, i.e., for all x and y in X. Condition (a) follows from the easily established equality

$$\begin{aligned}\gamma(t, s) - \gamma(t_1, t_0) = \psi(t)\,(t_0 - s) + t(\psi(t_0) - \psi(s)) + \psi(s)\,(s - t_0) \\ + t_0(\psi(t_0) - \psi(s)).\end{aligned} \tag{5.1}$$

Indeed, to prove (a) we have to show that if $\{x_k \mid x_k \in X_k\}$ is a sequence so that $x_k \rightharpoonup x_0$, in X, then to any given $\epsilon > o$ there exists $\delta = \delta(\epsilon) > o$ such that

$$|\gamma(t_k, s) - \gamma(t_k, t_0)| = |\gamma(\|x_k\|, \|y\|) - \gamma(\|x_k\|, \|x_0\|)| < \epsilon \qquad (5.2)$$

for all k if $\|x_0 - y\| < \delta$. Since $\{\|x_k\|\}$ and $\{\psi(\|x_k\|)\}$ are bounded by some $M > o$, it follows from (5.1) that for all k and each $s \geqslant o$

$$|\gamma(t_k, s) - \gamma(t_k, t_0)| \leqslant (M + \psi(s))| t_0 - s | + (M + t_0) | \psi(t_0) - \psi(s)| .$$

Let $B(o, d_0)$ be a ball in X such that $\{x_k\}$ and x_0 lie in $\bar{B}(o, d)$. It suffices to restrict our attention to y in X such that $\|x_0 - y\| \leqslant 1$. Hence we may assume that all elements under consideration belong to the ball $B(o, d)$ for $d = d_0 + 1$. Let $\psi(\|y\|) \leqslant M_1$ for all y in $\bar{B}(o, d)$ and let

$$C = \max\{M, \|x_0\|, M_1\}.$$

Then for any $\{x_k\}$ with $x_k \rightharpoonup x_0$ and all y in $B(o, d)$

$$|\gamma(t_k, s) - \gamma(t_k, t_0)| \leqslant 2C | t_0 - s | + 2C | \psi(t_0) - \psi(s)| \qquad \text{for all } k.$$

Now, to any given $\epsilon > o$ there exists $\delta_1 = \delta_1(\epsilon) > o$ such that

$$| \psi(t_0) - \psi(s)| < \frac{\epsilon}{4C} \qquad \text{if} \qquad | t_0 - s | < \delta_1 .$$

On the other hand, the function $\eta(s) = | t_0 - s |$ is continuous, $\eta(t_0) = o$ and $\eta(s) \geqslant o$. Hence to the same $\epsilon > o$, there corresponds a $\delta_2 = \delta_2(\epsilon) > o$ such that

$$o \leqslant \eta(s) = | \eta(t_0) - \eta(s)| \leqslant \frac{\epsilon}{4C} \qquad \text{if} \qquad | t_0 - s | < \delta_2 .$$

Hence, since $| \|x_0\| - \|y\| | \leqslant \|x_0 - y\|$, to any given $\epsilon > o$ there corresponds a $\delta = \min(\delta_1, \delta_2)$ such that

$$|\gamma(t_k, s) - \gamma(t_k, t_0)| \leqslant 2C\frac{\epsilon}{4C} + 2C\frac{\epsilon}{4C} = \epsilon \qquad \text{if} \qquad \|x_0 - y\| < \delta,$$

i.e., (5.2) and, consequently, (a) holds.

To verify (b), suppose $x_k \rightharpoonup x_0$ in X and

$$\lim_k \gamma(\|x_k\|, \|x_0\|) = \lim_k (\psi(\|x_k\|) - \psi(\|x_0\|)) (\|x_k\| - \|x_0\|) = 0.$$

Since ψ is strictly increasing, it follows from the above equality and Lemma 2.1 in [3] that $\|x_k\| \to \|x_0\|$ whence, since $x_k \rightharpoonup x_0$ in X and X has Property (H), it follows that $x_k \to x_0$ in X, i.e., (b) holds.

Remark 11. In case T and J_ψ are continuous, Proposition 14 reduces essentially to our Theorem 2.3 in [31].

B. *Existence Theorems for Mappings T of X into X*

Let X be a real Banach π_1-space with X^* strictly convex. In this case $\Gamma_n = \{X_n, P_n\}$ is an admissible projectional scheme for the pair (X, X). To deduce the corresponding results for mappings T of X into X from those obtained in Sections 1–4 for mappings T of X into Y, we set $Y = X$, $Y_n = X_n$, and $Q_n = P_n$ and note that, since $Y^* = X^*$, one of the simplest possible choices for

$$K : X \to X^*, \qquad K_n : X_n \to X_n' = R(P_n^*) \subset X^*$$

and

$$M_n : X_n \to X_n$$

is to take

$$K = J : X \to X^*, \qquad K_n = P_n^* J\,|_{X_n} : X_n \to X_n'$$

and

$$M_n = I_n : X_n \to X_n ,$$

where I_n is an identity on X_n and J is the single-valued duality map of X into X^* (with either $J = J_0$ or $J = J_\psi$) which is known [6] to be continuous from the strong topology of X to the weak* topology of X^* and for which

$$P_n^* J(x) = J(x) \qquad \text{for all } x \text{ in } X_n \text{ and each } n \text{ [13]}.$$

It follows that for this choice of K, K_n and M_n we have

$$(P_n(g), K_n(x)) = (P_n(g), P_n^* J(x)) = (g, J(x)) \qquad \forall x \in X_n , \quad g \in X,$$

and for all x in X_n with $x \neq o$

$$(M_n(x), K_n(x)) = (x, P_n^* J(x)) = (x, J(x)) = \| x \| \| J(x) \| > 0.$$

Consequently, Theorem 1 reduces to the following new existence theorems for pseudo-A-proper mappings T of D into X.

THEOREM 12. *Let X be a real Banach π_1-space with X^* strictly convex, J a duality mapping of X into X^*, and D a bounded open subset of X with $o \in D$.*

(a) *If T is an* fa-*continuous pseudo-*A-*proper mapping of $\bar{D}$ into X and f an element in X such that*

$$(T(x), J(x)) \geqslant (f, J(x)) \qquad \textit{for all } x \textit{ in } \dot{D},$$

then Eq. (1), *$T(x) = f$, has a solution in $\bar{D}$.*

(b) *If T is an* fa-*continuous pseudo*-A-*proper mapping of X into X such that either the condition* (i) *or the coerciveness condition* (ii) *of Theorem* 1(b) *holds, then $T(X) = X$.*

We observe that if D is also assumed to be convex and $f = o$, then Theorem 12(a) reduces essentially to the fixed point theorem established in [13] for G-operators since it is easy to see in this case that if T is an fa-continuous pseudo-A-proper map of $\bar{D}$ into X such that $(T(x), J(x)) \geqslant 0$ for all x in $\dot{D}$, then the map $\tilde{T} = I - T$ is a G-operator of $\bar{D}$ into X such that $(\tilde{T}(x), J(x)) \leqslant (x, J(x))$ for all x in $\dot{D}$ and consequently $\tilde{T}$ (see [13]) has a fixed point in $\bar{D}$ or equivalently the equation $T(x) = 0$ has a solution in $\bar{D}$. We recall (see [13]) that $T : \bar{D} \to X$ is said to be a G-*operator* if T is fa-continuous and if T has a fixed point in $\bar{D}$ whenever T_n has a fixed point in $\bar{D}_n$ for each n.

An immediate consequence of Theorem 12(a) is the following fixed point theorem for projectionally-compact (P-compact) operators established in Petryshyn [24] for $D = B(o, r)$ (see also Petryshyn–Tucker [32] and Browder–Petryshyn [9] for a slightly more general result). We recall (see [24]) that $T : \bar{D} \to X$ is said to be P-*compact* if T is fa-continuous on $\bar{D}$ and $T_d = T - dI$ is A-proper for each $d > o$.

COROLLARY 7. *Let X be a Banach π_1-space with X^* strictly convex and with J a given duality mapping of X into X^*. If D is a bounded open subset of X and T a* P-*compact map of $\bar{D}$ into X which satisfies the boundary condition $(T(x), J(x)) \leqslant (x, J(x))$ for all x in $\dot{D}$, then T has a fixed point in $\bar{D}$. The above conclusion holds, in particular, when $D = B(o, r)$ and $T(\dot{B}) \subseteq \bar{B}$.*

Proof. Since $\tilde{T} = I - T$ is obviously an fa-continuous A-proper map of $\bar{D}$ into X, and in particular, pseudo-A-proper and $(\tilde{T}(x), J(x)) \geqslant 0$ for all x in $\dot{D}$, the first part of Corollary 7 follows from Theorem 12(a). To prove the second part, it suffices to show that $T(\dot{B}) \subset \bar{B}$ implies the relation $(\tilde{T}(x), J(x)) \geqslant 0$ for all x in $\dot{B}$. Now, since $T(x) \in \bar{B}$ for each x in $\dot{B}$, it follows that $\| T(x)\| \leqslant r = \| x \|$ and, therefore,

$$(T(x), J(x)) \leqslant \| T(x)\| \, \| J(x)\| \leqslant \| x \| \, \| J(x)\| = (x, J(x)) \qquad \text{for all } x \text{ in } \dot{B},$$

i.e.,

$$(\tilde{T}(x), J(x)) \geqslant 0 \qquad \text{for all } x \text{ in } \dot{B}.$$

Q.E.D.

Since every compact mapping T of $\bar{D}$ into X as well as every quasi-compact mapping in the sense of Kaniel [17] is P-compact, the fixed point theorem of Schauder [38], Rothe [37] and Kaniel for $D = B(o, r)$ follow from Corollary 7 and thus from Theorem 12(a).

It is easy to see that if T is a weakly closed map of $\bar{D}$ into X with X reflexive, then $\tilde{T} = I - T$ is also weakly closed; moreover, if D is also assumed to be convex and T is a weakly continuous map of $\bar{D}$ into X, then T is also weakly closed. Hence, in view of Proposition 2, Theorem 12(a) implies also the validity of the following corollary whose special cases are due to Schauder [38] for T a weakly continuous map of $\bar{B}(o, r)$ into $\bar{B}(o, r)$, to Altman [1] for X a Hilbert space and T a bounded weakly closed map of $\bar{B}$ into X with $(Tx, x) \leqslant \| x \|^2$ for all x in $\dot{B}$ (see also Shinbrot [39]), and to De Figueiredo [13] for D a bounded open convex subset of X with $o \in D$ and T a weakly continuous map of $\bar{D}$ into X with

$$(T(x), J(x)) \leqslant (x, J(x)) \qquad \text{for all } x \in \dot{D}.$$

COROLLARY 8. *Let X be a reflexive π_1-space with X^* strictly convex. If D is a bounded open subset of X and T a bounded* fa-*continuous, and weakly closed map of $\bar{D}$ into X such that $(T(x), J(x)) \leqslant (x, J(x))$ for all x in $\dot{D}$, then T has a fixed point in $\bar{D}$.*

If the duality mapping J of X into X^* is assumed to be both strongly and weakly continuous, then it follows from Proposition 4 for the case when $Y = X$ and $K = J$ that every demicontinuous J-monotone map of X into X is pseudo-A-proper. Consequently, Theorem 12 implies the validity of the following corollary which includes Theorems 1, 2, and 3 established in Browder–De Figueiredo [7].

COROLLARY 9. *Let X be a reflexive π_1-space with X^* strictly convex and with a duality map J of X into X^* which is both strongly and weakly continuous. Let T be a demicontinuous J-monotone mapping of X into X.*

(a) *If D is a bounded open subset of X with $o \in D$ such that $(T(x), J(x)) \geqslant 0$ for all x in $\dot{D}$, then $T(x) = 0$ has a solution in $\bar{D}$.*

(b) *If T is* J-*coercive, then T maps X onto X.*

We note in passing that for the mapping J to be strongly continuous it suffices to assume that X^* has Property (H). Hilbert spaces, uniformly convex and locally uniformly convex Banach spaces are examples of such spaces. It is also known that Hilbert spaces and Banach 1_p spaces with $1 < p < \infty$ possess duality mappings which are both strongly and weakly continuous. Consequently, Corollary 9 is certainly true for the latter spaces.

If we strengthen further the conditions on J, then the assertion analogous to Corollary 9 also holds for mappings of type (PJM). Thus we get the following new result.

Corollary 10. *Let X be a reflexive π_1-space with X^* strictly convex and with a duality map J of X into X^* which is both weakly continuous on X and uniformly continuous on bounded subsets of X. Let T be a bounded* fa-*continuous mapping of type* (PJM) *of X into X.*

(a) *If D is a bounded open subset of X with $o \in D$ such that*

$$(T(x), J(x)) \geqslant 0 \qquad \textit{for all } x \textit{ in } \dot{D},$$

then $T(x) = 0$ has a solution in $\bar{D}$.

(b) *If T is* J-*coercive, then T maps X onto X.*

For the sake of completeness we also state the following corollary for mappings T of type (JM) as a special case of Theorem 8 or 12.

Corollary 11. *Suppose that X, X^* and J satisfy the conditions of Corollary* 9.

(a) *If D is a bounded open convex subset of X with $o \in D$ and T is a bounded* fa-*continuous mapping of $\bar{D}$ into X of type* (JM) *such that $(T(x), J(x)) \geqslant 0$ for all x in $\dot{D}$, then $T(x) = 0$ has a solution in $\bar{D}$.*

(b) *If T is a bounded* fa-*continuous map of X into X of type* (JM) *such that T is* J-*coercive, then T maps X onto X.*

To obtain an analogue of Theorem 2 for a mapping T of X into X which on bounded sets in X is a uniform limit of a special sequence of pseudo-A-proper mappings and which satisfies the "at infinity" condition that is more general than the coerciveness condition (ii), we take for $F : X \to X$ the identity $I (= F)$ on X and observe that in this case the corresponding conditions (c3) and (c4) of Theorem 2 are certainly satisfied, in view of our choice of $K = J$, if we assume that $(T(x), J(x)) \geqslant - C \| J(x) \| + (T(o), J(x))$ for all x in X and some $C \geqslant 0$. Consequently, we have the following new result for mappings T of X into X.

Theorem 13. *Let X be a Banach π_1-space with X^* strictly convex and with $J : X \to X^*$ a given duality mapping. Let T be an* fa-*continuous mapping of X into X such that*

(c1B) *$T(G)$ is closed in X whenever G is a bounded closed convex set in X.*

(c2B) *$T_\mu = T + \mu I$ is pseudo-*A-*proper for each $\mu > o$.*

(c4B) *$(T(x), J(x)) \geqslant - C \| J(x) \| + (T(o), J(x))$ for all x in X and some $C \geqslant o$.*

(c5B) *$\| T(x) \| \to \infty$ as $\| x \| \to \infty$.*

Then T maps X onto X.

Now Theorem 13 (or Theorem 4 for $Y = X$, $K = J$ and $F = I$) yields immediately the validity of the following new and to our knowledge the most general result for J-monotone mappings.

COROLLARY 12. *Let X be a reflexive π_1-space with X^* strictly convex and with the duality mapping J of X into X^* which is both strongly and weakly continuous. If T is a demicontinuous* J*-monotone map of X into X such that* (c5B) *of Theorem* 13 *holds, then T maps X onto X.*

To obtain a similar result for mappings T of type (PJM) we need to strengthen somewhat the conditions on X and J.

COROLLARY 13. *Let X be a reflexive π_1-space with Property* (H) *and with X^* strictly convex. Suppose that the duality map J of X into X^* given by* (J1) *is both weakly continuous on X and uniformly continuous on bounded sets in X. If T is a bounded* fa*-continuous mapping of type* (PJM) *of X into X such that* (c4B) *and* (c5B) *of Theorem* 13 *hold, then T maps X onto X.*

Proof. Since $K = J : X \to X^*$, to prove Corollary 13, it suffices to verify that, under our conditions on X and J, the mapping $F = I$ satisfies all the conditions of Theorem 6 for $Y = X$. Since $F = I$ is obviously J-monotone, bounded, and positively homogeneous of order $\alpha = 1$, we need only to verify either the condition (I) or the condition (II).

We shall show that when $F = I$, then F satisfies condition (S) on X, i.e., we verify (II). Let $\{x_n\}$ be any sequence in X such that $x_n \rightharpoonup x_0$ in X and $(x_n - x_0, J(x_n - x_0)) \to 0$ as $n \to \infty$. In view of the inequality (J2) satisfied by J, it follows that $\| x_n \| \to \| x_0 \|$ as $n \to \infty$ from which, since $x_n \rightharpoonup x_0$ in X with X having Property (H), it follows that $x_n \to x_0$, in X, i.e., $F = I$ satisfies condition (S) on X and, therefore, Corollary 13 follows from Theorem 6 or 13. Q.E.D.

We add in passing that in view of Proposition 9 we have in effect shown that $T_\mu = T + \mu I$ is an A-proper map of X into X for each $\mu > o$. We remark also that, using essentially the arguments of Kato [19], it has been shown by the author in [34] that a sufficient condition for J given by (J1) to be uniformly continuous on bounded subsets of X is that X^* be uniformly convex.

As our final application of Theorem 13 (or Theorem 9 for $Y = X$, $K = J$, and $F = I$) under the assumption that J is both strongly and weakly continuous on X we obtain the following new result for mappings T of type (JM).

COROLLARY 14. *Suppose that X, X^* and J satisfy the conditions of Corollary* 12. *If T is a bounded* fa*-continuous mapping of type* (JM) *of X into X such that the conditions* (c4B) *and* (c5B) *of Theorem* 13 *hold, then T maps X onto X.*

We continue this section by deducing from Proposition 5 for a J-monotone mapping T of X into X the analogue of Proposition 14.

PROPOSITION 15. *Let X be a reflexive Banach π_1-space with X^* strictly convex and let J_ψ be the duality mapping of X into X^* corresponding to the gauge function $\psi(r)$ which is both strongly and weakly continuous on X. If T is a demicontinuous (or a weakly continuous)* J*-monotone mapping of X into X, then the mapping $T_\mu = T + \mu I$ is an* A*-proper mapping for each $\mu > o$ (i.e., $-T$ is* P*-compact).*

Proof. To deduce Proposition 15 from Proposition 5 for the case when $Y = X$, $K = J$ and $K_n = P_n^* : J|_{X_n}$, it suffices to show that the J-monotone map $F = I$ of X into X which satisfies the inequality (F) of Proposition 5 in the form

$$\begin{aligned}(F(x) - F(y), J(x - y)) &= (x - y, J(x - y)) \\ &= \| x - y \| \psi(\| x - y \|) \qquad \forall x, y \in X\end{aligned}$$

is such that the nonnegative function γ defined by

$$\gamma(\| x \|, \| y \|, \| x - y \|) = \| x - y \| \psi(\| x - y \|)$$

satisfies the conditions (a) and (b) of Proposition 5. But the latter fact has been established in [12] and so we omit its proof here.

Remark 12. Some special cases of Proposition 15 have been proved earlier by the writer in [30] for T assumed also bounded and in [31] for T continuous. Proposition 15 for unbounded demicontinuous T as stated here follows from Lemma 2 in [12].

We conclude this section by observing that, in view of Theorem A, Theorem 3 in [33] obtained there for Hilbert spaces remains also valid for Banach spaces.

THEOREM 14. *Let X be a Banach π_1-space with X^* strictly convex and with J a duality map of X into X^*. If T is an* fa*-continuous* P*-compact mapping of X into X such that*

$$(T(x), J(x)) \leqslant (T(o), J(x)) \qquad \textit{for all } x \textit{ in } X, \tag{k}$$

then for any $\mu > o$ the equation $\mu x - T(x) = f$ is feebly projectionally-solvable for each f in X and, in particular, $(\mu I - T)$ is onto.

Proof. Since T is P-compact, $T_\mu = \mu I - T$ is A-proper for each fixed

$\mu > o$. Let f be any given element in X. Then it follows from the inequality (k) that for each given $\mu > o$ and all x in X

$$\begin{aligned}(T_\mu(x), J(x)) &- (f, J(x)) \\ &= \mu(x, J(x)) - (T(x) - T(o), J(x)) + (f + T(o), J(x)) \\ &\geqslant \mu(x, J(x)) - (f + T(o), J(x)) \\ &\geqslant (\mu \| x \| - \|f + T(o)\|) \| J(x)\| .\end{aligned}$$

This shows that if we take

$$r_{\mu_f} > \frac{\|f + T(o)\|}{\mu},$$

then

$$(T_\mu(x), J(x)) \geqslant (f, J(x)) \qquad \text{for all } x \in \dot{B}(o, r_{\mu_f})$$

and consequently the conclusion of Theorem 14 follows from Theorem A for the case when

$$Y = X, \qquad K = J, \qquad K_n = P_n{}^*J\,|_{X_n}, \qquad \text{and} \qquad M_n = I_n .$$

Q.E.D.

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