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On the Simultaneous Approximation of Derivatives by Lagrange and Hermite Interpolation

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We investigate here, for a positive integer q, simultaneous approximation of the first q derivatives of a function by the derivatives of its Lagrange interpolant, and then we augment this procedure by Hermite interpolation at the endpoints of the interval, obtaining a great improvement in the quality of approximation. In both cases, we estimate the quality of simultaneous approximation in terms of the norm of an associated Lagrange interpolation, and the estimates are thus valid for any sequence of interpolations by polynomials of successively higher degree. This communication continues work begun by K. Balázs and generalizes a recent work of Muneer Yousif Elnour, who treats simultaneous approximation with nodes at the zeroes of the Tchebycheff polynomials. Our efforts to obtain results which are independent of the choice of nodes have also led to some interesting consequences of a theorem of Gopengauz on simultaneous approximation.

PREFACE

With few exceptions, existing results on simultaneous approximation of a function and its derivatives by interpolation depend on a system of nodes generated by some particular method, such as placement at the zeroes of a sequence of orthogonal polynomials. While such procedures can give good results, they are very inflexible, revealing little about what happens on other systems of nodes. We investigate here, for a positive integer q, simultaneous approximation of the first q derivatives of a function by

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the derivatives of its Lagrange interpolant, and then we augment this procedure using Hermite interpolation at the endpoints of the interval, obtaining a great improvement in the quality of approximation. In both cases, we estimate the quality of simultaneous approximation in terms of the norm of an associated Lagrange interpolation, and the estimates are thus valid for any sequence of interpolations by polynomials of successively higher degree.

We mention four recent contributions to our topic. This communication first of all continues work of generalization begun in K. Balázs [1]. Second Y. E. Muneer [7] has recently treated simultaneous approximation with nodes at the zeroes of the Tchebycheff polynomials. His analysis of the augmented interpolation especially is quite serious and has challenged us to undertake its generalization. Our efforts to obtain results which are independent of the choice of nodes have also led to some interesting consequences of a theorem of Gopengauz [4] on simultaneous approximation.

The third recent contributor is J. Szabados [9], who has created a system of nodes with some good convergence properties for simultaneous Lagrange interpolation. His interesting construction has been a great impetus for additional work on problems relating to simultaneous approximation. Most recently, P. Runck and P. Vértesi [8] have discovered a class of nodes with good convergence properties. We will further describe these results at an appropriate point in our exposition.

Introduction

For Lagrange interpolation on the interval $[-1, 1]$, we will assume that nodes $x_1, ..., x_n$ are given satisfying $-1 \leqslant x_1 < ... < x_n \leqslant 1$. When such a set of nodes is chosen by some prearranged scheme for each n, $n = 1, 2, ...$, we use the term *system* of nodes. The fundamental polynomials of degree $n-1$ are $l_1, ..., l_n$, satisfying $l_i(x_j) = \delta_{ij}$ (Kronecker delta). A standard construction for the polynomials l_i is to define

$$W_n(x) = (x-x_1)(x-x_2)\cdots(x-x_n) \tag{1}$$

and for $i = 1, ..., n$ to set

$$l_i(x) = W_n(x)\,[(x-x_i)\,W'_n(x_i)]^{-1}. \tag{2}$$

Lagrange interpolation is then defined by

$$L_n f(x) = \sum_{i=1}^{n} f(x_i)\,l_i(x), \tag{3}$$

for f in $C[-1, 1]$. It is easily seen that

$$\|L_n\| = \left\| \sum_{i=1}^{n} |l_i(x)| \right\|,$$

the usual sup norm of $C[-1, 1]$ being used on the right. We also define, for k a non-negative integer,

$$L_n^{(k)} f(x) = \sum_{i=1}^{n} f(x_i)\, l_i^{(k)}(x), \tag{4}$$

where (k) signifies the kth derivative. We record the useful observation

$$\left| \sum_{i=1}^{n} \operatorname{sgn}[l_i^{(k)}(x)]\, l_i(x) \right| \leqslant \sum_{i=1}^{n} |l_i(x)| \leqslant \|L_n\|. \tag{5}$$

Quality of simultaneous approximation by any L_n of the form (3) must satisfy the following theorem, in which the value of $\|L_n\|$ depends, of course, on the nodes $x_1, ..., x_n$. We note that a system of nodes defines a sequence of interpolation operators L_n, and vice versa. For a function g which is continuous on $[-1, 1]$, the notation $\omega(g; h)$ denotes the *modulus of continuity* of g and is described by

$$\omega(g; h) = \sup_{|x-y| \leqslant h} |g(x) - g(y)|.$$

We are now ready to state two theorems, which taken together will describe the convergence properties of an arbitrary sequence of successive Lagrange interpolations. The first theorem appears in Balázs [1].

THEOREM. *Let q be a fixed positive integer, f in $C^q[-1, 1]$, and $\{L_n\}$ a sequence of Lagrange interpolation operators, each into space of polynomials of degree $n-1$ or less. Then for $-1 < x < 1$ and for $i = 0, ..., q$*

$$|f^{(i)}(x) - L_n^{(i)} f(x)| = O(n^{i-q})(1-x^2)^{-i/2}\, \omega(f^{(q)}; 1/n)\, \|L_n\|,$$

and consequently

$$|f^{(i)}(x) - L_n^{(i)} f(x)| = O(n^{i-q})\, (f^{(q)}; 1/n)\, \|L_n\|$$

holds on any compact subset of $(-1, 1)$.

To this result, one may add the following:

THEOREM 1. *Let q be a fixed positive integer, f in $C^q[-1, 1]$, and $\{L_n\}$ a sequence of Lagrange interpolation operators, each respectively into the space of polynomials of degree $n-1$ or less. Then for $i = 0, ..., q$ we have*

$$|f^{(i)}(x) - L_n^{(i)} f(x)| = O(n^{2i-2q})\, \omega(f^{(2q)}; 1/n)\, \|L_n\|,$$

whence $L_n^{(i)} f$ converges uniformly to $f^{(i)}$ on $[-1, 1]$ if $n^{2i-2q}\omega(f^{(q)}; 1/n)\, \|L_n\| \to 0$.

This estimate is in particular valid at $x = 1$ and at $x = -1$.

Remarks. 1. The best possible choices of the nodes $x_1, ..., x_n$ for Lagrange interpolation lead to $\|L_n\| = O(\log n)$ (see Brutman [2] or Vértesi [11] for some good estimates) as do other, near-optimal choices, such as the zeroes of the Tchebycheff polynomial $T_n(x) = \cos(n \arccos x)$, and thus Muneer [7, Theorem 1.1] follows immediately, inserting $\log n$ in place of $\|L_n\|$. We remark that, more generally, if a system of nodes is constructed by taking for each n the zeroes of the orthogonal polynomial of degree n associated with a weight function $\omega(x) \geq m > 0$, then $\|L_n\| = O(n)$ (Grünwald and Turán [5]). If $\omega(x) = (1-x)^\alpha (1+x)^\beta$, for $\alpha, \beta > -1$, the orthogonal polynomials thus generated are the classical Jacobi polynomials, and one obtains on the associated system of nodes $\|L_n\| = O(\log n)$ if $\gamma = \max(\alpha, \beta) \leq -\frac{1}{2}$, and $\|L_n\| = O(n^{\gamma+1/2})$ if $\gamma = \max(\alpha, \beta) > \frac{1}{2}$ (Szegö [10, p. 338]).

2. The result of Szabados [9] is that, on a set of nodes specially constructed, it is possible to obtain

$$\|f^{(q)} - L_n^{(q)} f\| = O(1)\, \omega(f^{(q)}; 1/n) \log n \qquad \text{as} \quad n \to \infty.$$

3. The result of Runck and Vértesi [8] improves our Theorem 1 considerably for certain classes of specially chosen nodes, for which they have demonstrated

$$\|f^{(i)} - L_n^{(i)} f\| = O(n^{i-q})\, \omega(f^{(q)}; 1/n) \log n \qquad \text{for} \quad i = 0, ..., q.$$

A DISCUSSION OF THE LAGRANGE–HERMITE INTERPOLATION

We may continue our investigation of simultaneous interpolation by considering Hermite interpolation at the endpoints -1 and 1, in addition to usual Lagrange interpolation in the interior of the interval. Specifically, we may let $x_1, ..., x_n$ be nodes such that $-1 < x_1 < ... < x_n < 1$, and we set $x_0 = -1$ and $x_{n+1} = 1$. We will assume that f is in $C^q[-1, 1]$ for some fixed integer $q > 0$. The derivatives $f^{(0)}, ..., f^{(r-1)}$ are then interpolated at the points 1 and -1, where $2r = q$, if q is even, and $2r = q + 1$ if q is odd. An interpolation operator H_m is then constructed which approximates f with a polynomial $H_m f$ of degree at most m, where $m = n - 1 + 2r$. Specifically,

$$H_m f(x) = \sum_{k=0}^{r-1} f^{(k)}(x_0)\, r_{0,k}(x) + \sum_{k=0}^{r-1} f^{(k)}(x_{n+1})\, r_{n+1,k}(x) + \sum_{j=1}^{n} f(x_j)\,(1-x_j^2)^{-r}\,(1-x^2)^r\, l_j(x), \tag{6}$$

in which the l_j are the fundamental polynomials defined in (2), and, for $k = 0, ..., r-1$, the polynomials $r_{0,k}$ and $r_{n+1,k}$ are respectively defined by

$$\begin{aligned} r_{0,k}(x_j) = r_{n+1,k}(x_j) = 0 \qquad & \text{for } j = 1, ..., n \quad \text{and} \quad k = 0, ..., r-1 \\ r_{0,k}^{(i)}(-1) = r_{n+1,k}^{(i)}(1) = \delta_{ki} \qquad & \text{for } i, k = 0, ..., r-1 \\ r_{0,k}^{(i)}(1) = r_{n+1,k}^{(i)}(-1) = 0 \qquad & \text{for } i = 0, ..., r-k-1, \quad k = 0, ..., r-1. \end{aligned} \tag{7}$$

An explicit formula for the polynomials $r_{0,k}$ and $r_{n+1,k}$ may be given in the form

$$r_{0,k} = (1-x)^r\, W_n(x) \sum_{i=0}^{r-k-1} c_i^{(0,k)}\, (1+x)^{k+1}$$

$$r_{n+1,k} = (1+x)^r\, W_n(x) \sum_{i=0}^{r-k-1} c_i^{(n+1,k)}\, (1-x)^{k+1}$$

in which $W_n(x)$ is given in (1), and the coefficients may be computed explicitly from the conditions listed in (7). However, these explicit formulas will not be needed here.

For the mixed Lagrange–Hermite interpolation just described, our results will be stated in Theorem 2, where a form of *weighted* interpolation will also assume a position of importance. For $l_1, ..., l_n$ as already defined, we will let

$$L_n^* f(x) = \sum_{j=1}^{n} f(x_j) \left(\frac{1-x^2}{1-x_j^2}\right)^{1/2} l_j(x) \tag{8}$$

and we note that

$$\|L_n^*\| = \left\| \sum_{j=1}^{n} \left(\frac{1-x^2}{1-x_j^2}\right)^{1/2} |l_j(x)| \right\|. \tag{9}$$

The range of L_n^* is the space of weighted polynomials of degree $n-1$ or less, with weight $(1-x^2)^{1/2}$. The method of proof used in Kilgore [6] suffices to show that this is a Bernstein–Erdös optimal interpolation space. We will show here (Corollary) that $\|L_n^*\| = O(\log n)$ on the nodes situated at the zeroes of the Tchebycheff polynomials.

THEOREM 2. *For q a fixed positive integer, let f be in $C^q[-1, 1]$. Let $\{H_m\}$ be any sequence of modified Lagrange–Hermite interpolation operators as described, $\{L_n\}$ the associated sequence of Lagrange interpolation operators, and $\{L_n^*\}$ the associated sequence of weighted interpolation operators. Then, for all x in $[-1, 1]$ and for $i = 0, ..., q$:*

(a) *For q even and $m = n-1+q$,*

$$|f^{(i)}(x) - H_m^{(i)} f(x)| = O(n^{i-q})\, \omega(f^{(q)}; 1/n)\, \|L_n\|.$$

Uniform convergence of $H_m^{(i)} f$ to $f^{(i)}$ occurs for $i = 0, ..., q$ provided that $n^{i-q}\, \omega(f^{(q)}; 1/n)\, \|L_n\| \to 0$.

(b) *For q odd and $m = n+q$,*

$$|f^{(i)}(x) - H_m^{(i)} f(x)| = O(n^{i+1-q})\, \omega(f^{(q)}; 1/n)\, \|L_n\|.$$

Uniform convergence of $H_m^{(i)} f$ to $f^{(i)}$ occurs for $i = 0, ..., q$ provided that $n^{i+1-q}\, \omega(f^{(q)}; 1/n)\, \|L_n\| \to 0$.

(c) *For q odd and $m = n+q$, a sharper result than* (b) *is*

$$|f^{(i)}(x) - H_m^{(i)} f(x)| = O(n^{i-q})\, \omega(f^{(q)}; 1/n)(\|L_n\| + \|L_n^*\|).$$

Uniform convergence of $H_m^{(i)} f$ to $f^{(i)}$ occurs for $i = 0, ..., q$ provided that $\omega(f^{(q)}; 1/n)(\|L_n\| + \|L_n^*\|) \to 0$.

For estimating the error in simultaneous approximation by Lagrange interpolation, an apparent discrepancy exists between Theorem 1 (global result) and Szabados [9] (particular result). Here, in contrast, the particular result is that of Muneer [7, Theorem 1.2], which we list below as a corollary of Theorem 2. In addition there are many other choices of systems of nodes which give $\|L_n\| = O(\log n)$ besides the ones used by Muneer (cf. Remark 1), and none which gives an essentially slower rate of growth. Our theorem is more flexible in its potential for application, but Muneer gives rates of convergence which apparently cannot be improved.

COROLLARY [7, Theorem 2.1] *If the nodes of interpolation for L_n in Theorem 2 are based at the zeroes of the Tchebycheff polynomial $T_n(x) = \cos(n \arccos x)$, then $\|L_n\| = O(\log n)$ and $\|L_n^*\| = O(\log n)$, and these values may be used in Theorem 2.*

EXISTING RESULTS

Our proofs will be based on the Markov–Bernstein inequality and on a theorem of Gopengauz [4] which states the existence of certain polynomials of approximation with rather useful properties. We list these results.

MARKOV INEQUALITY. *Let p_n be a polynomial of degree n or less. Then* $\|p_n^{(i)}\| = O(1)\, n^{2i}\, \|p_n\|$, *the norm being the usual sup norm, taken on the interval* $[-1, 1]$.

BERNSTEIN INEQUALITY. (a) Let T_n be a trigonometric polynomial of degree n or less. Then $\|T_n'\| \leqslant n\, \|T_n\|$.

(b) For p_n a polynomial of degree n or less, and for any x in $(-1, 1)$, $|p_n^{(i)}(x)| = O(1)\, n^i (1-x^2)^{-i/2}\, \|p_n\|$.

THEOREM OF GOPENGAUZ. *Let q be a fixed non-negative integer, and let f be in* $C^q[-1, 1]$. *Then, for every* $m \geqslant 4q+5$, *there exists a polynomial* G_m *of degree at most m such that, for* $i = 0, 1, ..., q$ *and for x in* $[-1, 1]$,

$$|f^{(i)}(x) - G_m^{(i)}(x)| = O(1)\, m^{i-q}\, (1-x^2)^{(q-i)/2}\, \omega(f^{(q)}; 1/m).$$

SOME CONSEQUENCES OF THE MARKOV–BERNSTEIN INEQUALITIES

The Markov and Bernstein inequalities in conjunction imply the following simple and useful inequalities, of which [7, Lemma 3.1] is a particular case for $k = 2r$, for successive derivatives of a polynomial with multiple zeroes at 1 and -1. We list this result as Lemma 1.

LEMMA 1. *Consider for fixed non-negative r a polynomial of the form* $(1-x^2)^r g_n(x)$, *where* g_n *is a polynomial of degree n or less. Then, for* $k = 0, ..., 2r$ *and with* $O(1)$ *depending only on r, and for* $|x| \leqslant 1$,

$$|[(1-x^2)^r g_n(x)]^{(k)}| = O(1)\, n^k\, \|g_n\|. \tag{10}$$

Proof. We first write the derivative on the left in expanded form

$$[(1-x^2)^r g_n(x)]^{(k)} = \sum_{i=0}^{k} \binom{k}{i} [(1-x^2)^r]^{(k-i)}\, g_n^{(i)}(x),$$

where $\binom{k}{i}$ is the binomial coefficient. For convenience, we will write $|[(1-x^2)^r]^{(k-i)}\, g_n^{(i)}(x)| = A_i(x)$.

Since $k \leqslant 2r$, and r is fixed, it will be sufficient to show for each i in the sum that $A_i(x) = O(n^k)\, \|g_n\|$. To see this, we consider three possibilities:

(a) $r-k+i\leqslant 0$. This implies that $i\leqslant k/2$, and the Markov inequality implies that $A_i(x)=O(n^{2i})\,\|g_n\|$, which is sufficient because $2i\leqslant k$.

(b) $r-k+i\geqslant i/2$. In this case, it can be said by use of the Bernstein inequality that

$$A_i(x)\leqslant C_i|(1-x^2)^{i/2}\,g_n^{(i)}(x)|=O(n^i)\,\|g_n\|,$$

which suffices because $i\leqslant k$.

(c) $0<r-k+i<i/2$. One begins by noting that, in this case, $i/2<k-r$, whence $2i<k$. Therefore

$$A_i(x)\leqslant C_i|(1-x^2)^{r-k+i}\,g_n^{(i)}(x)|=O(1)\,n^{2r-2k+2i}\,\|g_n^{(i-2r+2k-2i)}\|,$$

using the Bernstein inequality. Now, using the Markov inequality,

$$\begin{aligned} n^{2r-2k+2i}\,\|g_n^{(i-2r+2k-2i)}\| &= O(1)(n^{2r-2k+2i+2i-4r+4k-4i})\,\|g_n\| \\ &= O(1)(n^{2(k-r)})\,\|g_n\|=O(n^k)\,\|g_n\|, \end{aligned}$$

this last following because $k<2r$, and our proof is completed.

Remark on Lemma 1. It is immediate from (10) that, under the same hypotheses, one has for $k=1,...,2r-1$ and for $|x|\leqslant 1$ that

$$|[(1-x^2)^r\,g_n(x)]^{(k)}|=O(1)\,n^{k-1}\|(1-x^2)\,g_n'(x)-2rxg_n(x)\|. \quad (11)$$

One simply differentiates once and then applies (10).

Also based on the Bernstein inequality is the following result.

LEMMA 2. *Assume that g_n is a polynomial of degree $n-1$ or less on $[-1,1]$. Then for $|X|\leqslant 1$*

$$|(1-x^2)\,g_n'(x)|\leqslant \|g_n\|+n\|(1-x^2)^{1/2}\,g_n(x)\|. \quad (12)$$

Proof. We write, using the substitution $x=\cos t$, the expression $(1-x^2)^{1/2}\,g_n(x)$ in trigonometric form, and invoke (a) of the Bernstein inequality, obtaining

$$\|-(\sin^2 t)\,g_n'(\cos t)+(\cos t)\,g_n(\cos t)\|\leqslant n\|(\sin t)\,g_n(\cos t)\|$$

and, using the triangle inequality and back-substitution, we obtain

$$\|(1-x^2)\,g_n'(x)\|-\|xg_n(x)\|\leqslant n\|(1-x^2)^{1/2}\,g_n(x)\|,$$

From this our result follows.

A REMARK ON THE GOPENGAUZ THEOREM

The following difference inequalities are an immediate consequence of the theorem of Gopengauz and will be used in proving Theorem 2, where, when q is odd, one encounters $i = 1$. For $i = 0$, the estimate has previously been used by Muneer and Szabados.

COROLLARY OF THE GOPENGAUZ THEOREM. *Let q be a positive integer, and let f be in $C^q[-1, 1]$. For an arbitrary $m > 4q + 5$, let G_m be the Gopengauz polynomial of f. Then for $i = 0, ..., q$, and for $|x| \leqslant 1$, G_m satisfies*

$$\left\| \frac{f(x) - G_m(x)}{(1-x^2)^{(q+i)/2}} \right\| = O(m^{i-q})\, \omega(f^{(q)}; 1/m). \tag{13}$$

Proof. Assume that x is such as to cause the indicated norm to be attained. Our conclusion certainly holds if x is between the values of, for example, $-2^{-1/2}$ and $2^{-1/2}$. There is also no problem if x is at 1 or -1, in view of the fact that $f^{(i)}(x) = G_m^{(i)}(x)$ for $x = 1$ or $x = -1$ and $i = 0, ..., q-1$, as an immediate consequence of the theorem of Gopengauz. Assume, therefore, without loss of generality, that $2^{-1/2} < x < 1$. Then repeated use of Cauchy's lemma (used in the standard proof of l'Hospital's rule) demonstrates the existence of y satisfying $x < y < 1$, such that

$$\left| \frac{f(x) - G_m(x)}{(1-x^2)^{(q-i)/2}} \right| = O(1) \left| \frac{f^{(i)}(y) - G_m^{(i)}(y)}{(1-y^2)^{(q-i)/2}} \right| = O(m^{i-q})\, \omega(f^{(q)}; 1/m).$$

PROOF OF THEOREM 1.

For $n \geqslant 4q + 5$, q a fixed integer, let G_n be the polynomial of approximation to f guaranteed by the theorem of Gopengauz, observing that, by the properties of L_n as a linear projection operator, $L_n^{(i)} G_n = G_n^{(i)}$ and $L_n^{(i)} f - L_n^{(i)} G_n = L_n^{(i)}(f - G_n)$ for $i = 0, ..., q$ are algebraic identities. Thus, for $i = 0, ..., q$ and for arbitrary x in $[-1, 1]$, we have

$$|f^{(i)}(x) - L_n^{(i)} f(x)| \leqslant |f^{(i)}(x) - G_n^{(i)}(x)| + |L_n^{(i)}(f - G_n)(x)|.$$

Analysing separately the two quantities on the right, we note, using the theorem of Gopengauz and the Markov inequality, that the second of the two satisfies

$$\begin{aligned}|L_n^{(i)}(f-G_n)(x)| &\leqslant \|f-G_n\| \sum_{j=1}^{n} |l_j^{(i)}(x)| \\ &= O(1)(n^{-q})\, \omega(f^{(q)}; 1/n)(n^{2i}) \left\| \sum_{j=1}^{n} |l_j(x)| \right\| \\ &= O(n^{2i-q})\, \omega(f^{(q)}; 1/n)\, \|L_n\|,\end{aligned}$$

in the estimation of which the inequality (5) has played an implicit part. By dilution of the theorem of Gopengauz, we have as well

$$|f^{(i)}(x) - G_n^{(i)}(x)| = O(n^{i-q})\, \omega(f^{(q)}; 1/n)\, \|L_n\|,$$

and part (a) of Theorem 1 is completed; the statement concerning convergence clearly follows as well.

Proof of Theorem 2

We may begin in a fashion similar to that used in the proof of Theorem 1. Let G_m be the Gopengauz polynomial of degree at most m, where, recall, $m = n - 1 + q$ if q is even and, $m = n + q$ if q is odd. We then employ the triangle inequality, obtaining

$$|f^{(i)}(x) - H_m^{(i)} f(x)| \leqslant |f^{(i)}(x) - G_m^{(i)}(x)| + |H_m^{(i)}(f - G_m)(x)|. \tag{14}$$

Since the first term on the right clearly satisfies the conclusions of our theorem, we will confine our attentions to the second. Writing that term explicitly, we have

$$|H_m^{(i)}(f-G_m)(x)| = \left| \sum_{j=1}^{n} (f(x_j) - G_m(x_j)) \left[\frac{(1-x^2)^r}{(1-x_j^2)^r} l_j(x) \right]^{(i)} \right|, \tag{15}$$

where $r = q/2$ if q is even, and $r = (q+1)/2$ if q is odd. We have, after regrouping in the expression on the right,

$$|H_m^{(i)}(f-G_m)(x)| = \left| \sum_{j=1}^{n} \left[\frac{f(x_j) - G_m(x_j)}{(1-x_j^2)^r} \right] [(1-x^2)^r\, l_j(x)]^{(i)} \right|. \tag{16}$$

And now we may make the estimate

$$|H_m^{(i)}(f-G_m)(x)| \leqslant \left\| \frac{f(x) - G_m(x)}{(1-x^2)^r} \right\| \cdot \left\| \sum_{j=1}^{n} |[(1-x^2)^r\, l_j(x)]^{(i)}| \right\|. \tag{17}$$

If q is even, the right side of this inequality is

$$\left\| \frac{f(x) - G_m(x)}{(1-x^2)^{q/2}} \right\| \cdot \left\| \sum_{j=1}^{n} |[(1-x^2)^{q/2} l_j(x)]^{(i)}| \right\|, \tag{18}$$

and we may use the difference-quotient estimate (13) (with $i=0$) and the Markov–Bernstein inequality in the form (10) followed by the use of (5) on the respective components of (18) to reach

$$[O(n^{-q})\,\omega(f^{(q)};1/n)]\cdot[O(n^i)\,\|L_n\|]=O(n^{i-q})\,\omega(f^{(q)};1/n)\,\|L_n\|.$$

This completes the proof of Theorem 2, part (a).

If q is odd, then the right side of (15) is

$$\left|\sum_{j=1}^{n}\left[\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{(q+1)/2}}\right][(1-x^2)^{(q+1)/2}\,l_j(x)]^{(i)}\right|. \tag{19}$$

Now we may use the difference-quotient estimate (13) (for $i=1$) and the Markov–Bernstein inequality in the form (11) on the respective components of this expression, following with (5), and we obtain

$$[O(n^{-q+1})\,\omega(f^{(q)};1/n)]\cdot[O(n^i)\|L_n\|]=O(n^{i+1-q})\,\omega(f^{(q)};1/n)\,\|L_n\|.$$

This concludes the proof of Theorem 2, part (b).

For Theorem 2, *part* (c), we should return to (19), where we obtain, by means of (11) and (5),

$$|H_m^{(i)}(f-G_m)(x)|\leqslant O(n^{i-1})\left\|\sum_{j=1}^{n}\left|\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{q/2}}\right|\cdot\left|\frac{(1-x^2)\,l_j'(x)}{(1-x_j^2)^{1/2}}\right|\right\|$$
$$+O(n^{i-1})\left\|\sum_{j=1}^{n}\left|\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{(q+1)/2}}\right|\cdot|l_j(x)|\right\|. \tag{20}$$

Now, analysing the first of the sums on the right of (20), we have, using (12), (13), and (5),

$$O(n^{i-1})\left\|\sum_{j=1}^{n}\left|\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{q/2}}\right|\cdot\left|\frac{(1-x^2)\,l_j'(x)}{(1-x_j^2)^{1/2}}\right|\right\|$$
$$\leqslant O(n^{i-1})\left[\left\|\sum_{j=1}^{n}\left|\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{(q+1)/2}}\right|\,|l_j(x)|\right\|\right.$$
$$\left.+n\left\|\sum_{j=1}^{n}\left|\frac{f(x_j)-G_m(x_j)}{(1-x_j^2)^{q/2}}\right|\cdot\left|\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}}\,l_j(x)\right|\right\|\right]$$
$$\leqslant O(n^{i-1})\left\|\frac{f(x)-G_m(x)}{(1-x^2)^{(q+1)/2}}\right\|\cdot\left\|\sum_{j=1}^{n}|l_j(x)|\right\|$$
$$+O(n^i)\left\|\frac{f(x)-G_m(x)}{(1-x^2)^{q/2}}\right\|\cdot\left\|\sum_{j=1}^{n}\left|\frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}}\,l_j(x)\right|\right\|$$
$$=O(n)^{i-1}\left(\frac{1}{n}\right)^{q-1}\omega\left(f^{(q)};\frac{1}{n}\right)\|L_n\|+O(n^i)\left(\frac{1}{n}\right)^{q}\omega\left(f^{(q)};\frac{1}{n}\right)\|L_n^*\|.$$

By a similar argument, also using (13), the second of the sums on the right of (20) is dominated by

$$O(n^{i-1}) \left\| \frac{f(x)-G_m(x)}{(1-x^2)^{(q+1)/2}} \right\| \|L_n\| = O(n^{i-1}) \cdot \left(\frac{1}{n}\right)^{q-1} \omega\left(f^{(q)}; \frac{1}{n}\right) \|L_n\|.$$

This concludes the proof of Theorem 2, part (c).

Proof of the Corollary. The norm $\|L_n\|$ is well known to satisfy $O(\log n)$ on the system of nodes generated by the Tchebycheff polynomials for $n = 1, 2, \ldots$ by $T_n(x) := \cos(n \arccos x)$. It is necessary only to show that the same asymptotic estimate is valid for $\|L_n^*\|$. For convenience, we begin by reviewing some of the basic, well-known facts which can be stated about the Tchebycheff polynomials and about this weighted interpolation. First of all, it is advantageous in this case to number the nodes for each n in reverse order, beginning at the right of the interval $[-1, 1]$ instead of the left, and then we have the explicit formulation that for $j = 1, \ldots, n$

$$x_j = \cos\left(\frac{2j-1}{2n}\right)\pi.$$

We then note that the function W_n of (1) takes on the form $T_n(x)$, and thus for each n and for $j = 1, 2, \ldots, n$

$$|T_n'(x_j)| = n/(1-x_j)^{1/2}.$$

The functions $l_1, \ldots, l_n$ are defined for each n as in (2), and the following estimate is also known (Fejér [3]) to hold independently of n,

$$\sum_{j=1}^{n} (l_j(x))^2 \leqslant 2,$$

from which we may conclude in particular, setting $x_0 := 1$ and $x_{n+1} := -1$ for convenience in what follows, that for any x in $[-1, 1]$, for any n, and for any $j = 1, \ldots, n-1$, $|l_j(x)| + |l_{j+1}(x)| = O(1)$. A last observation is that, if x is any fixed number in $(-1, 1]$, there is for each n an integer k (which depends on n) such that x lies in $[x_{k+1}, x_k]$ (we will assume for the sake of unicity that if x is an endpoint of such an interval, it will be the point x_k). For this k, the following estimates follow from fundamental trigonometric identities:

$$|x_k - x_j| \sim (n^{-2})\,|k-j| \cdot |k+j-1| \qquad \text{for} \quad j<k$$

$$|x_k - x_j| \sim (n^{-2})\,|j-k-1| \cdot |j+k| \qquad \text{for} \quad j>k+1.$$

We now show that, for any fixed but arbitrary x in $[-1, 1]$,

$$\sum_{j=1}^{n} \left(\frac{1-x^2}{1-x_j^2}\right)^{1/2} |l_j(x)| = O(\log n).$$

To this end, we note that the expression is zero if $x = 1$ or $x = -1$, and so we may assume that x lies in the *open* interval. Moreover, we note that the expression is an even function of x, and so we may assume with no loss of generality that x lies in $(-1, 0]$, in which case we clearly have

$$\limsup n/k = 2.$$

We now combine all of the preceding remarks in the following estimates:

$$\begin{aligned}\sum_{j=1}^{n} \left(\frac{1-x^2}{1-x_j^2}\right)^{1/2} |l_j(x)| &= O(1) + \sum_{\substack{j\neq k\\ j\neq k-1}} \frac{(1-x^2)^{1/2}}{(1-x_j^2)^{1/2}} |l_j(x)| \\ &= O(1) + \sum_{\substack{j\neq k\\ j\neq k-1}} \frac{(1-x^2)^{1/2}\,|T_n(x)|}{n\,|x-x_j|}.\end{aligned}$$

In turn, we may now state that

$$\begin{aligned}\sum_{\substack{j\neq k\\ j\neq k+1}} \frac{(1-x^2)^{1/2}\,|T_n(x)|}{n\,|x-x_j|} &\leqslant \frac{1}{n}\left[\sum_{j>k+1} \frac{1}{|x_k-x_j|} + \sum_{j<k} \frac{1}{|x_k-x_j|}\right] \\ &= O(1)\cdot\frac{1}{n}\sum_{j<k} \frac{n^2}{|k-j|\cdot|k+j-1|} + O(1)\cdot\frac{1}{n}\sum_{j>k-1} \frac{n^2}{|j-k-1|\cdot|j+k|} \\ &= O(1)\,\frac{n}{k}\left[\sum_{j=1}^{k-1} \frac{1}{|k-j|} + \sum_{j=k+2}^{n} \frac{1}{|j-k-1|}\right] \\ &= O(1)\sum_{j=1}^{n} \frac{1}{j} = O(\log n).\end{aligned}$$

This concludes the proof of the Corollary to Theorem 2.

References

1. K. Balázs, On the convergence of the derivatives of the Lagrange interpolation polynomials, *Acta Math. Hungar.*, in press.
2. L. Brutman, On the Lebesgue function for polynomial approximation, *SIAM J. Numer. Anal.* **15** (1978), 694–704.
3. L. Fejér, "Gesammelte Arbeiten," Akad. Kiadó, Budapest, 1970.

4. I. E. GOPENGAUZ, On a theorem of A. F. Timan on approximation of functions by polynomials on a finite interval, *Mat. Zametki* **1** (1967), 163–172 [in Russian]; *Math. Notes* **1** (1967), 110–116 [English translation].
5. G. GRÜNWALD AND P. TURÁN, Uber Interpolation, *Ann. Scuola Norm. Sup. Pisa* **7** (1938), 137–146.
6. T. KILGORE, Optimal interpolation with polynomials having fixed roots, *J. Approx. Theory* **49** (1987), 378–389.
7. Y. E. MUNEER, On Lagrange and Hermite interpolation, I, *Acta Math. Hungar.* **49** (1987), 293–305.
8. P. RUNCK AND P. VÉRTESI, Some good point systems for derivatives of Lagrange interpolatory operators, unpublished manuscript.
9. J. SZABADOS, On the convergence of the derivatives of projection operators, *Analysis* **7** (1987), 349–357.
10. G. SZEGÖ, "Orthogonal Polynomials," Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1978.
11. P. VÉRTESI, On the optimal Lebesgue constants for polynomial interpolation, *Acta Math. Hungar.* **47** (1986), 165–178.

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