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A variation of a classical Turán-type extremal problem

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Abstract

A variation of a classical Turán-type extremal problem (Erdős on Graphs: His Legacy of Unsolved Problems (1998) p. 36) is considered as follows: determine the smallest even integer $\sigma(K_{r,s}, n)$ such that every n-term graphic non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(K_{r,s}, n)$ has a realization G containing $K_{r,s}$ as a subgraph, where $K_{r,s}$ is a $r \times s$ complete bipartite graph. In this paper, we determine $\sigma(K_{r,s}, n)$ exactly for every fixed $s \geq r \geq 3$ when $n \geq n_0(r, s)$, where $m = [\frac{(r+s+1)^2}{4}]$ and

$$n_0(r,s) = \begin{cases} m + 3s^2 - 2s - 6, & \text{if } s \leq 2r \text{ and } s \text{ is even,} \\ m + 3s^2 + 2s - 8, & \text{if } s \leq 2r \text{ and } s \text{ is odd,} \\ m + 2s^2 + (2r-6)s + 4r - 8, & \text{if } s \geq 2r + 1. \end{cases}$$

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative integers with $d_i \le n-1$ for each i is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic non-increasing sequences in NS_n is denoted by GS_n. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For a given graph H, a sequence $\pi \in GS_n$ is said to be *potentially* (resp. *forcibly*) H*-graphic* if there exists a realization of π containing H as a subgraph (resp. each realization of π contains H as a subgraph).

It is well known (see [1] for example) that one of the classical extremal problems in extremal graph theory is to determine the smallest integer $t(H, n)$ such that every graph G on n vertices with edge number $e(G) \ge t(H, n)$ contains H as a subgraph. The number $t(H, n)$ is called the *Turán number* of H. The classical Turán theorem [1] determined the Turán number $t(K_r, n)$ for K_r, a complete graph on r vertices. For the Turán number $t(K_{r,s}, n)$, Kővári et al. [2] gave the general upper bound as follows: $t(K_{r,s}, n) \le cn^{2-\frac{1}{r}}$ for $2 \le r \le s$. In chapter 3 of [3], a conjecture has been made that $t(K_{r,r}, n) \ge cn^{2-\frac{1}{r}}$. Erdős et al. [4] proved $t(K_{2,2}, n) \sim \frac{1}{2}n^{\frac{3}{2}}$. Recently, Füredi [5] proved $t(K_{3,3}, n) \sim \frac{1}{2}n^{\frac{5}{3}}$.

In terms of graphic sequences, the number $2t(H, n)$ is the smallest even integer such that each sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge 2t(H, n)$ is forcibly H-graphic. Gould et al. [6] considered the following variation of the classical Turán number $t(H, n)$: determine the smallest even integer $\sigma(H, n)$ such that every sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge \sigma(H, n)$ is potentially H-graphic. If $H = K_{r+1}$, this problem was considered by Erdős et al. [7] where they showed that $\sigma(K_3, n) = 2n$ for $n \ge 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for sufficiently large n. Gould et al. [6] and Li and Song [8] proved that the conjecture holds for $r = 3$ and $n \ge 8$, respectively. Recently, Li et al. [9, 10] further proved that the conjecture is true for $r = 4$ and $n \ge 10$ and for $r \ge 5$ and $n \ge \binom{r}{2} + 3$. For $H = K_{r,s}$, Gould et al. [6] determined $\sigma(K_{2,2}, n)$ for $n \ge 4$. Recently, Yin and Li [11, 12] determined $\sigma(K_{3,3}, n)$ for $n \ge 6$ and $\sigma(K_{4,4}, n)$ for $n \ge 8$, and also determined $\sigma(K_{r,r}, n)$ for even $r (\ge 4)$ and $n \ge 4r^2 - r - 6$ and for odd $r (\ge 3)$ and $n \ge 4r^2 + 3r - 8$. The purpose of the paper is to determine $\sigma(K_{r,s}, n)$ for large n. The paper is organized as follows. The second section will give a sufficient condition for a sequence being potentially $K_{r,s}$-graphic (see Theorem 2.11). In the third section, we will determine $\sigma(K_{r,s}, n)$ exactly for $s \ge r \ge 3$ and $n \ge n_0(r, s)$ (see Theorems 3.1.2, 3.2.2 and 3.3.2), where $m = [\frac{(r+s+1)^2}{4}]$ and

$$n_0(r, s) = \begin{cases} m + 3s^2 - 2s - 6, & \text{if } s \le 2r \text{ and } s \text{ is even,} \\ m + 3s^2 + 2s - 8, & \text{if } s \le 2r \text{ and } s \text{ is odd,} \\ m + 2s^2 + (2r-6)s + 4r - 8, & \text{if } s \ge 2r + 1. \end{cases}$$

More specifically, when $s = r$, our main results become exactly the main results of [12].

2. Preliminaries

In order to prove our main results, we need the following notations and results.

For a non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $f'(\pi) = \max\{i : d_i \geq i\}$ and define an n-by-n matrix $\overline{A} = (a_{ij})$ as follows: if $d_i \geq i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i + 1 \text{ and } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and if $d_i < i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i, \\ 0 & \text{otherwise.} \end{cases}$$

$f'(\pi)$ and $\overline{A}$ are called the *trace* and the *left-most off-diagonal matrix* of π, respectively. The column sum vector of $\overline{A}$, denoted by $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, is called the *corrected conjugate vector* of π. Clearly, the row sum vector of $\overline{A}$ is π and $\sigma(\overline{\pi}) = \sigma(\pi)$.

Theorem 2.1 (See [13]). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f'(\pi)$.*

For a non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, let $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the rearrangement of $d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1}$. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the *residual sequence* of π. It is easy to see that if $\pi' \in GS_{n-1}$ then $\pi \in GS_n$, since a realization G of π can be obtained from a realization G' of π' by adding a new vertex of degree d_n to G' and joining it to the vertices whose degrees are reduced by one in going from π to π'. In fact more is true:

Theorem 2.2 (See [14]). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence. Then $\pi \in GS_n$ if and only if $\pi' \in GS_{n-1}$.*

Theorem 2.3 (See [10]). *Let $k \geq 5$. Then $\sigma(K_{k+1}, n) \leq 2n(k-2) + 8$ for $2k + 2 \leq n \leq \binom{k}{2} + 3$ and $\sigma(K_{k+1}, n) = (k-1)(2n-k) + 2$ for $n \geq \binom{k}{2} + 3$.*

Theorem 2.4 (See [11] or [12]). *Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_{r+s} \geq r + s - 1$ and $d_n \geq r$. Then π is potentially $K_{r,s}$-graphic.*

Theorem 2.5 (See [11] or [12]). *Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_r \geq r + s - 1$, $d_{r+s} \leq r + s - 2$ and $d_n \geq r$. If $n \geq (r+2)(s-1)$, then π is potentially $K_{r,s}$-graphic.*

Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n$. If π has a realization H with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that $d_i = d_H(v_i)$ for $1 \leq i \leq n$ and H contains $K_{r,s}$ as a subgraph, where $\{v_1, \ldots, v_r\}$ and $\{v_{r+1}, \ldots, v_{r+s}\}$ is the bipartite partition of the vertex set of $K_{r,s}$, then π is called *potentially $A_{r,s}$-graphic*. It is easy to see that a potentially $A_{r,s}$-graphic sequence must be potentially $K_{r,s}$-graphic, but the inverse is not true in general. On the potentially $A_{r,s}$-graphic sequences, we have the following

Proposition 2.6 (See [11]). *Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n$, where $d_1 \geq \cdots \geq d_r \geq s$, $d_{r+1} \geq \cdots \geq d_{r+s} \geq r$ and $d_{r+s+1} \geq \cdots \geq d_n \geq r$. Let*

$$\pi_1' = \begin{cases} (d_2, \ldots, d_r, d_{r+1} - 1, \ldots, d_{r+d_1} - 1, \\ \quad d_{r+d_1+1}, \ldots, d_n), & \text{if } d_1 \le n - r, \\ (d_2 - 1, \ldots, d_{d_1+r-n+1} - 1, d_{d_1+r-n+2}, \ldots, d_r, \\ \quad d_{r+1} - 1, \ldots, d_n - 1), & \text{if } d_1 > n - r, \end{cases}$$

and $\pi_1'' = (d_2^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}, \ldots, d_{r+s}^{(1)}, d_{r+s+1}^{(1)}, \ldots, d_n^{(1)})$*, where* $d_2^{(1)} \ge \cdots \ge d_r^{(1)}$ *is the rearrangement of the first* $r - 1$ *terms in* π_1', $d_{r+i}^{(1)} = d_{r+i} - 1$ *for* $1 \le i \le s$ *and* $d_{r+s+1}^{(1)} \ge \cdots \ge d_n^{(1)}$ *is the rearrangement of the final* $n - r - s$ *terms in* π_1'*. If* π_1'' *is potentially* $A_{r-1,s}$*-graphic, then* π *is potentially* $A_{r,s}$*-graphic.*

For the sequence π_1'', if $d_2^{(1)} \ge \cdots \ge d_r^{(1)} \ge s$, we can define similarly the sequence π_2'' as follows: let

$$\pi_2' = \begin{cases} (d_3^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)} - 1, \ldots, d_{r+d_2^{(1)}}^{(1)} - 1, \\ \quad d_{r+d_2^{(1)}+1}^{(1)}, \ldots, d_n^{(1)}), & \text{if } d_2^{(1)} \le n - r, \\ (d_3^{(1)} - 1, \ldots, d_{d_2^{(1)}+r-n+2}^{(1)} - 1, d_{d_2^{(1)}+r-n+3}^{(1)}, \ldots, d_r^{(1)}, \\ \quad d_{r+1}^{(1)} - 1, \ldots, d_n^{(1)} - 1), & \text{if } d_2^{(1)} > n - r, \end{cases}$$

and $\pi_2'' = (d_3^{(2)}, \ldots, d_r^{(2)}, d_{r+1}^{(2)}, \ldots, d_{r+s}^{(2)}, d_{r+s+1}^{(2)}, \ldots, d_n^{(2)})$, where $d_3^{(2)} \ge \cdots \ge d_r^{(2)}$ is the rearrangement of the first $r - 2$ terms in π_2', $d_{r+i}^{(2)} = d_{r+i}^{(1)} - 1$ for $1 \le i \le s$ and $d_{r+s+1}^{(2)} \ge \cdots \ge d_n^{(2)}$ is the rearrangement of the final $n - r - s$ terms in π_2'. For $k = 3, 4, \ldots, r$ in turn, if $d_k^{(k-1)} \ge \cdots \ge d_r^{(k-1)} \ge s$, the definitions of π_k' and π_k'' are similar.

Proposition 2.7 (See [11]). *Let* $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in NS_n$ *be a sequence in Proposition 2.6, and let* π_r'' *be defined as above. If* π_r'' *is graphic, then* π *is potentially* $A_{r,s}$*-graphic.*

For the defined sequence $\pi_r'' = (d_{r+1}^{(r)}, \ldots, d_{r+s}^{(r)}, d_{r+s+1}^{(r)}, \ldots, d_n^{(r)})$ in Proposition 2.7, if $d_{r+1}^{(r)} \ge \cdots \ge d_{r+s}^{(r)} \ge 1$ and $d_{r+s+1}^{(r)} \ge \cdots \ge d_n^{(r)} \ge 1$, we define

$$\pi_{r+1}' = (d_{r+2}^{(r)} - 1, \ldots, d_{r+d_{r+1}^{(r)}+1}^{(r)} - 1, d_{r+d_{r+1}^{(r)}+2}^{(r)}, \ldots, d_n^{(r)}),$$

and $\pi_{r+1}'' = (d_{r+2}^{(r+1)}, \ldots, d_{r+s}^{(r+1)}, d_{r+s+1}^{(r+1)}, \ldots, d_n^{(r+1)})$, where $d_{r+2}^{(r+1)} \ge \cdots \ge d_{r+s}^{(r+1)}$ is the rearrangement of the first $s - 1$ terms in π_{r+1}' and $d_{r+s+1}^{(r+1)} \ge \cdots \ge d_n^{(r+1)}$ is the rearrangement of the final $n - r - s$ terms in π_{r+1}'. It is easy to see that if π_{r+1}'' is graphic then π_r'' is graphic, since a realization H of π_r'' can be obtained from a realization H' of π_{r+1}'' by adding a new vertex of degree $d_{r+1}^{(r)}$ to H' and joining it to the vertices whose degrees are reduced by one in going from π_r'' to π_{r+1}''. For $k = 2, 3, \ldots, s$ in turn, if $d_{r+k}^{(r+k-1)} \ge \cdots \ge d_{r+s}^{(r+k-1)} \ge 1$ and $d_{r+s+1}^{(r+k-1)} \ge \cdots \ge d_n^{(r+k-1)} \ge 1$, the definitions of π_{r+k}' and π_{r+k}'' are similar.

Proposition 2.8. *Let $\pi''_r = (d^{(r)}_{r+1}, \ldots, d^{(r)}_{r+s}, d^{(r)}_{r+s+1}, \ldots, d^{(r)}_n)$ be a defined sequence as in Proposition 2.7, $1 \le k \le s$ and let π''_{r+k} be defined as above. If π''_{r+k} is graphic, then π''_r is also graphic.*

Proof. It follows from π''_{r+k} being graphic that π''_{r+j} is graphic for $j = k-1, k-2, \ldots, 0$ in turn. □

Proposition 2.9. *Let $(d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence. For $1 \le t \le n$, let $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n$ be the rearrangement of $d_1-1, d_2-1, \ldots, d_t-1, d_{t+1}, \ldots, d_n$.*

(1) *If $d_1 > d_n$ and $\rho_n = d_n - 1$, then $\rho_1 = d_1 - 1$;*
(2) *If $d_1 - d_n \le 1$, then $\rho_1 - \rho_n \le 1$.*

Proof. (1) Suppose $d_t > d_n$. Then $d_t - 1 \ge d_n$. Hence $\rho_n = d_n$, a contradiction. Thus $d_t = d_{t+1} = \cdots = d_n$. Moreover, $d_1 - 1 \ge \max\{d_2 - 1, \ldots, d_t - 1, d_{t+1}, \ldots, d_n\}$. Hence $\rho_1 = d_1 - 1$.
(2) is evident. □

Lemma 2.10 (See [11] or [12]). *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $m = \max\{d_1, d_2, \ldots, d_n\}$ and $\sigma(\pi)$ be even. And let $\pi^* = (d^*_1, d^*_2, \ldots, d^*_n)$ be the rearrangement sequence of π, where $m = d^*_1 \ge d^*_2 \ge \cdots \ge d^*_n$ is the rearrangement of $d_1, d_2, \ldots, d_n$. If there exists an integer $n_1 \le n$ such that $d^*_{n_1} \ge h \ge 1$ and $n_1 \ge \frac{1}{h}[\frac{(m+h+1)^2}{4}]$, then $\pi \in GS_n$.*

The following is a main result in this section.

Theorem 2.11. *Let $s \ge r \ge 3$, $n \ge \frac{(r+s)^2}{4} + \frac{r+s}{2}$ and $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, \ldots, d_n) \in GS_n$, where $d_r \le r + s - 2$ and $d_n \ge r$. If there exists an integer t, $1 \le t \le \min\{r-1, \lceil \frac{s}{2} \rceil - 1\}$ such that $d_{r+t} \ge r + s - 1 - t$ and $d_{r+s} \ge r + t$, then π is potentially $K_{r,s}$-graphic.*

Proof. Define a new sequence $(p_1, \ldots, p_{r-t-1}, p_{r-t}, \ldots, p_r, p_{r+1}, \ldots, p_{r+t}, p_{r+t+1}, \ldots, p_{r+s}, p_{r+s+1}, \ldots, p_n)$ as follows: $p_i = d_i$ if $1 \le i \le r-t-1$, $p_i = d_{i+t}$ if $r-t \le i \le r$, $p_i = d_{i-t-1}$ if $r+1 \le i \le r+t$ and $p_i = d_i$ if $r+t+1 \le i \le n$. For convenience, the new sequence is denoted by π. Clearly, we have

(1) $n-1 \ge p_1 \ge p_2 \ge \cdots \ge p_r \ge r+s-1-t$ and $p_{r-t} \le r+s-2$;
(2) $n-1 \ge p_{r+1} \ge p_{r+2} \ge \cdots \ge p_{r+s} \ge r+t$ and $p_{r+t+1} \le r+s-2$;
(3) $r+s-2 \ge p_{r+s+1} \ge p_{r+s+2} \ge \cdots \ge p_n \ge r$.

By Proposition 2.7, it is enough to prove that π''_r is graphic. It follows from the definition of π''_{r-t-1} that $\pi''_{r-t-1} = (p^{(r-t-1)}_{r-t}, \ldots, p^{(r-t-1)}_r, p^{(r-t-1)}_{r+1}, \ldots, p^{(r-t-1)}_{r+s}, p^{(r-t-1)}_{r+s+1}, \ldots, p^{(r-t-1)}_n)$ satisfies:

(4) $r+s-2 \ge p^{(r-t-1)}_{r-t} \ge \cdots \ge p^{(r-t-1)}_r \ge d_{r+t} - (r-t-1) \ge (r+s-1-t) - (r-t-1) \ge s$;
(5) $n-r+t \ge p^{(r-t-1)}_{r+1} \ge \cdots \ge p^{(r-t-1)}_{r+t} \ge p^{(r-t-1)}_{r+t+1} \ge \cdots \ge p^{(r-t-1)}_{r+s} \ge 2t+1$ and $p^{(r-t-1)}_{r+t+1} \le s+t-1$;
(6) $r+s-2 \ge p^{(r-t-1)}_{r+s+1} \ge \cdots \ge p^{(r-t-1)}_n \ge t+1$.

Hence, by $(p_{r-t}^{(r-t-1)}-s)+\cdots+(p_r^{(r-t-1)}-s) \le (r-2)(t+1) \le (r-2)\lceil \frac{s}{2} \rceil < \frac{rs+r}{2}-s < \frac{(r+s)^2}{4} - \frac{r+s}{2} \le n-(r+s)$, $\pi_r'' = (p_{r+1}^{(r)}, \ldots, p_{r+t}^{(r)}, p_{r+t+1}^{(r)}, \ldots, p_{r+s}^{(r)}, p_{r+s+1}^{(r)}, \ldots, p_n^{(r)})$ satisfies:

(7) $n-r-1 \ge p_{r+1}^{(r)} \ge \cdots \ge p_{r+t}^{(r)} \ge p_{r+t+1}^{(r)} \ge \cdots \ge p_{r+s}^{(r)} \ge t$ and $p_{r+t+1}^{(r)} \le s-2$;

(8) $r+s-2 \ge p_{r+s+1}^{(r)} \ge \cdots \ge p_n^{(r)} \ge t$ and there exists an integer $x_0 \ge n-(r+s)-(\frac{rs+r}{2}-s) \ge \frac{(r+s)^2}{4} - \frac{r+s}{2} - (\frac{rs+r}{2}-s)$ such that $p_{r+s+x_0}^{(r)} \ge t+1$.

Thus, $\pi_{r+t}'' = (p_{r+t+1}^{(r+t)}, \ldots, p_{r+s}^{(r+t)}, p_{r+s+1}^{(r+t)}, \ldots, p_n^{(r+t)})$ satisfies:

(9) $s-2 \ge p_{r+t+1}^{(r+t)} \ge \cdots \ge p_{r+s}^{(r+t)} \ge t-t=0$;

(10) $r+s-2 \ge p_{r+s+1}^{(r+t)} \ge \cdots \ge p_n^{(r+t)} \ge t-t=0$ and $p_{r+s+x_0}^{(r+t)} \ge t+1-t=1$.

By Proposition 2.8, we only need to prove that π_{r+t}'' is graphic. Let $\pi^* = (d_1^*, d_2^*, \ldots, d_{n-r-t}^*)$ be the rearrangement sequence of π_{r+t}'', where $d_1^* \ge d_2^* \ge \cdots \ge d_{n-r-t}^*$ is the rearrangement of $p_{r+t+1}^{(r+t)}, \ldots, p_{r+s}^{(r+t)}, p_{r+s+1}^{(r+t)}, \ldots, p_n^{(r+t)}$. We consider the following three cases:

Case 1. $p_n^{(r+t)} \ge 2$. Then $d_{n-r-s}^* \ge p_n^{(r+t)} \ge 2$ and $d_1^* = \max\{p_{r+t+1}^{(r+t)}, p_{r+s+1}^{(r+t)}\} \le r+s-2$. Since $\frac{1}{2}[\frac{(r+s-2+2+1)^2}{4}] \le \frac{(r+s)^2}{4} - \frac{r+s}{2} \le n-(r+s)$, by Lemma 2.10, π_{r+t}'' is graphic.

Case 2. $p_n^{(r+t)} = 1$. Then $d_{n-r-s}^* \ge p_n^{(r+t)} = 1$. If $p_{r+s+1}^{(k)} > p_n^{(k)}$ for each $k = 0, 1, \ldots, r+t$, where $p_{r+s+1}^{(0)} = p_{r+s+1}$ and $p_n^{(0)} = p_n$, then by Proposition 2.9(1), $p_{r+s+1}^{(0)} - p_{r+s+1}^{(r+t)} \ge p_n^{(0)} - p_n^{(r+t)}$, i.e., $p_{r+s+1}^{(r+t)} \le p_{r+s+1}^{(0)} - p_n^{(0)} + 1 \le (r+s-2)-r+1 = s-1$. If there exists an integer $j, 0 \le j \le r+t$ such that $p_{r+s+1}^{(j)} = p_n^{(j)}$, then by Proposition 2.9(2), $p_{r+s+1}^{(k)} - p_n^{(k)} \le 1$ for each $k = j, \ldots, r+t$, in particular, $p_{r+s+1}^{(r+t)} \le p_n^{(r+t)} + 1 = 2 \le s-1$. Hence $d_1^* = \max\{p_{r+t+1}^{(r+t)}, p_{r+s+1}^{(r+t)}\} \le s-1$. Since $[\frac{(s-1+1+1)^2}{4}] \le \frac{s^2}{4} + \frac{s}{2} + \frac{1}{4} \le \frac{(r+s)^2}{4} - \frac{r+s}{2} \le n-(r+s)$, by Lemma 2.10, π_{r+t}'' is graphic.

Case 3. $p_n^{(r+t)} = 0$. Then $d_{x_0}^* \ge p_{r+s+x_0}^{(r+t)} \ge 1$. If $p_{r+s+1}^{(k)} > p_n^{(k)}$ for each $k = 0, 1, \ldots, r+t$, similarly by Proposition 2.9(1), we have $p_{r+s+1}^{(r+t)} \le p_{r+s+1}^{(0)} - p_n^{(0)} \le (r+s-2)-r = s-2$. If there exists an integer $j, 0 \le j \le r+t$ such that $p_{r+s+1}^{(j)} = p_n^{(j)}$, then by Proposition 2.9(2), $p_{r+s+1}^{(r+t)} \le p_n^{(r+t)} + 1 = 1 \le s-2$. Thus $d_1^* = \max\{p_{r+t+1}^{(r+t)}, p_{r+s+1}^{(r+t)}\} \le s-2$. By $[\frac{(s-2+1+1)^2}{4}] = \frac{s^2}{4} \le \frac{(r+s)^2}{4} - \frac{r+s}{2} - (\frac{rs+r}{2}-s) \le x_0$ and Lemma 2.10, π_{r+t}'' is also graphic. □

3. Main results

For convenience, we also introduce the following notations: let $m = [\frac{(r+s+1)^2}{4}]$ and

$$f(r,s,n)=\left(2r+\frac{s}{2}-2\right)n-(r-1)r-\frac{sr}{2}+\frac{s}{4}\left(\frac{s}{2}+1\right),$$
$$g(r,s,n)=\left(2r+\frac{s}{2}-\frac{5}{2}\right)n-(r-1)r-\left(\frac{s}{2}-\frac{1}{2}\right)r+\frac{1}{8}(s^2+8s-1),$$
$$h(r,s,n)=(r+s-2)n-\frac{(r-1)(2s-r)}{2},$$

and let

$$A_1=\{(r,s,n)\mid s \text{ is even and } f(r,s,n) \text{ is also even}\},$$
$$A_2=\{(r,s,n)\mid s \text{ is even and } f(r,s,n) \text{ is odd}\},$$
$$B_1=\{(r,s,n)\mid s \text{ is odd and } g(r,s,n) \text{ is even}\},$$
$$B_2=\{(r,s,n)\mid s \text{ is odd and } g(r,s,n) \text{ is also odd}\},$$
$$C_1=\{(r,s,n)\mid h(r,s,n) \text{ is even}\},$$
$$C_2=\{(r,s,n)\mid h(r,s,n) \text{ is odd}\}.$$

3.1. $\sigma(K_{r,s},n)$ for $3\le r\le s\le 2r$, even s and $n\ge m+3s^2-2s-6$

Theorem 3.1.1. *Let $3\le r\le s\le 2r$, where s is even, and let $n\ge r+s$. Then*

$$\sigma(K_{r,s},n)\ge\begin{cases} f(r,s,n)+2, & \text{if } (r,s,n)\in A_1,\\ f(r,s,n)+1, & \text{if } (r,s,n)\in A_2.\end{cases}$$

Proof. Assume $(r,s,n)\in A_1$. Let $\pi=((n-1)^{r-1},r+s-2,r+s-3,\dots,r+\frac{s}{2},(r+\frac{s}{2}-1)^{n-r-\frac{s}{2}+2})$, where the symbol x^y stands for y consecutive terms x. Then $\sigma(\pi)=f(r,s,n)$ is even and $f'(\pi)=r+\frac{s}{2}-1$. It is easy to see from the left-most off-diagonal matrix $\overline{A}$ of π that $\overline{\pi}=(\overline{d_1},\overline{d_2},\dots,\overline{d_n})$ satisfies $n-1=\overline{d_1}=\overline{d_2}=\cdots=\overline{d_{f'(\pi)}}\ge\overline{d_{f'(\pi)+1}}\ge\cdots\ge\overline{d_n}$. Clearly, $d_1+d_2+\cdots+d_i\le\overline{d_1}+\overline{d_2}+\cdots+\overline{d_i}$ for $i=1,2,\dots,f'(\pi)$. Hence by Theorem 2.1, $\pi\in GS_n$. Let $\pi_1=(s-1,s-2,\dots,\frac{s}{2}+1,(\frac{s}{2})^{n-r-\frac{s}{2}+2})$. If π is potentially $K_{r,s}$-graphic, then there exist integers r_1 and s_1, $1\le r_1\le\frac{s}{2}$ and $s_1=s+1-r_1$ such that π_1 is potentially K_{r_1,s_1}-graphic, and hence there are at least r_1 terms in π_1 which are greater than or equal to $s+1-r_1$, a contradiction. So π is not potentially $K_{r,s}$-graphic. Thus $\sigma(K_{r,s},n)\ge\sigma(\pi)+2=f(r,s,n)+2$.

Now assume $(r,s,n)\in A_2$. Let $\pi=((n-1)^{r-1},r+s-2,r+s-3,\dots,r+\frac{s}{2},(r+\frac{s}{2}-1)^{n-r-\frac{s}{2}+1},r+\frac{s}{2}-2)$. Then $\sigma(\pi)=f(r,s,n)-1$ is even and $f'(\pi)=r+\frac{s}{2}-1$. By the left-most off-diagonal matrix $\overline{A}$ of $\pi,\overline{\pi}=(\overline{d_1},\overline{d_2},\dots,\overline{d_n})$ satisfies $\overline{d_1}=\overline{d_2}=\cdots=\overline{d_{f'(\pi)-1}}=n-1$ and $n-2=\overline{d_{f'(\pi)}}\ge\overline{d_{f'(\pi)+1}}\ge\cdots\ge\overline{d_n}$. It is easy to see that $d_1+d_2+\cdots+d_i\le\overline{d_1}+\overline{d_2}+\cdots+\overline{d_i}$ for $i=1,2,\dots,f'(\pi)$. By Theorem 2.1, $\pi\in GS_n$. Similarly, we can also prove that π is not potentially $K_{r,s}$-graphic. Thus $\sigma(K_{r,s},n)\ge\sigma(\pi)+2=f(f,s,n)+1$. □

Lemma 3.1.1. *Let $3\le r\le s\le 2r$, where s is even, and let $n=m$. Then*

$$\sigma(K_{r,s},n)\le f(r,s,n)+2+\left(\tfrac{3}{2}s^3-s^2-3s\right).$$

Proof. Clearly, $m \le \binom{r+s-1}{2} + 3$. Hence by Theorem 2.3,

$$\begin{aligned}\sigma(K_{r,s}, n) &\le \sigma(K_{r+s}, n) \le (2r+2s-6)n+8\\ &= f(r,s,n)+2+\left(\frac{3}{2}s-4\right)m+(r-1)r+\frac{sr}{2}-\frac{s}{4}\left(\frac{s}{2}+1\right)+6\\ &\le f(r,s,n)+2+\left(\frac{3}{2}s-4\right)(s^2+s)+(s-1)s+\frac{s^2}{2}-\frac{s}{4}\left(\frac{s}{2}+1\right)+6\\ &\le f(r,s,n)+2+\left(\tfrac{3}{2}s^3-s^2-3s\right). \quad \square\end{aligned}$$

Lemma 3.1.2. *Let $3 \le r \le s \le 2r$, where s is even, and let $n \ge m$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_n \ge r$. If $\sigma(\pi) \ge f(r,s,n)+2$, then π is potentially $K_{r,s}$-graphic.*

Proof. If $d_{r+s} \ge r+s-1$, then by Theorem 2.4, π is potentially $K_{r,s}$-graphic. If $d_{r+s} \le r+s-2$ and $d_r \ge r+s-1$, then by $n \ge m = [\frac{(r+s+1)^2}{4}] \ge (r+2)(s-1)$ and Theorem 2.5, π is potentially $K_{r,s}$-graphic. Now assume that $d_r \le r+s-2$. If $d_{r+t} \le r+s-2-t$ for any $t \in \{1, 2, \ldots, \frac{s}{2}-1\}$, then $\sigma(\pi) \le (r-1)(n-1)+(r+s-2)+(r+s-3)+\cdots+(r+\frac{s}{2})+(r+\frac{s}{2}-1)(n-r-\frac{s}{2}+2) = f(r,s,n) < \sigma(\pi)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, \frac{s}{2}-1\}$ such that $d_{r+t} \ge r+s-1-t$. If $d_{r+s} \le r+\frac{s}{2}-2$, then by $r \le s \le 2r$,

$$\begin{aligned}\sigma(\pi) &\le (r-1)(n-1)+s(r+s-2)+\left(r+\frac{s}{2}-2\right)(n-r-s+1)\\ &= \left(2r+\frac{s}{2}-2\right)n-n-r^2+\frac{s^2}{2}-\frac{rs}{2}+2r+\frac{s}{2}-1\\ &\le \left(2r+\frac{s}{2}-2\right)n-(r-1)r-\frac{rs}{2}+\frac{s^2}{2}+r+\frac{s}{2}-1-\left[\frac{(r+s+1)^2}{4}\right]\\ &\le \left(2r+\frac{s}{2}-2\right)n-(r-1)r-\frac{rs}{2}+\frac{s^2}{2}+r+\frac{s}{2}-1\\ &\quad -\frac{(r+s)^2+2(r+s)}{4}\\ &\le \left(2r+\frac{s}{2}-2\right)n-(r-1)r-\frac{rs}{2}+\frac{s^2}{2}+r+\frac{s}{2}-1\\ &\quad -\frac{\left(\frac{s}{2}+s\right)^2+2(r+r)}{4}\\ &= \left(2r+\frac{s}{2}-2\right)n-(r-1)r-\frac{rs}{2}-\frac{1}{16}s^2+\frac{s}{2}-1\\ &< f(r,s,n)+2 \le \sigma(\pi), \text{ a contradiction.}\end{aligned}$$

Hence $d_{r+s} \ge r+\frac{s}{2}-1 \ge r+t$. Now by Theorem 2.11, π is potentially $K_{r,s}$-graphic. $\square$

Lemma 3.1.3. *Let $3 \le r \le s \le 2r$, where s is even, and let $n = m+t$, where $0 \le t \le 3s^2-2s-6$. Then $\sigma(K_{r,s}, n) \le f(r,s,n)+2+(\frac{3}{2}s^3-s^2-3s)-\frac{st}{2}$.*

Proof. We use induction on t. It is known from Lemma 3.1.1 that the result holds for $t = 0$. Now assume that the result holds for $t-1$, $0 \le t-1 \le 3s^2-2s-7$. Let $n = m+t$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge f(r,s,n)+2+(\frac{3}{2}s^3-s^2-3s)-\frac{st}{2}$. We only need to prove that π is potentially $K_{r,s}$-graphic. Obviously, $\sigma(\pi) \ge f(r,s,n)+2$. If $d_n \ge r$, then by Lemma 3.1.2, π is potentially $K_{r,s}$-graphic. If $d_n \le r-1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi)-2d_n \ge f(r,s,n-1)+2+(\frac{3}{2}s^3-s^2-3s)-\frac{s(t-1)}{2}$. By the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. □

Lemma 3.1.4. *Let $3 \le r \le s \le 2r$, where s is even, and let $n \ge m+3s^2-2s-6$. Then $\sigma(K_{r,s}, n) \le f(r,s,n)+2$.*

Proof. It is enough to prove that (∗): if $\pi = (d_1, \ldots, d_n) \in GS_n$ and $\sigma(\pi) \ge f(r,s,n)+2$, then π is potentially $K_{r,s}$-graphic. Apply induction on n. By Lemma 3.1.3, (∗) holds for $n = m+3s^2-2s-6$. Now suppose that (∗) holds for $n-1 \ge m+3s^2-2s-6$. We will prove that (∗) holds for n. If $d_n \ge r$, then by Lemma 3.1.2, π is potentially $K_{r,s}$-graphic. If $d_n \le r-1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi)-2d_n \ge f(r,s,n-1)+2$. By the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. □

Theorem 3.1.2. *Let $3 \le r \le s \le 2r$, where s is even, and let $n \ge m+3s^2-2s-6$. Then*

$$\sigma(K_{r,s}, n) = \begin{cases} f(r,s,n)+2, & \text{if } (r,s,n) \in A_1, \\ f(r,s,n)+1, & \text{if } (r,s,n) \in A_2. \end{cases}$$

Proof. It follows from Theorem 3.1.1 and Lemma 3.1.4 that $\sigma(K_{r,s}, n) = f(r,s,n)+2$ for $(r,s,n) \in A_1$ and $f(r,s,n)+1 \le \sigma(K_{r,s}, n) \le f(r,s,n)+2$ for $(r,s,n) \in A_2$. Since $\sigma(K_{r,s}, n)$ is even, we have $\sigma(K_{r,s}, n) = f(r,s,n)+1$ for $(r,s,n) \in A_2$. □

3.2. $\sigma(K_{r,s}, n)$ for $3 \le r \le s \le 2r$, odd s and $n \ge m+3s^2+2s-8$

Theorem 3.2.1. *Let $3 \le r \le s \le 2r$, where s is odd, and let $n \ge r+s$. Then*

$$\sigma(K_{r,s}, n) \ge \begin{cases} g(r,s,n)+2, & \text{if } (r,s,n) \in B_1, \\ g(r,s,n)+1, & \text{if } (r,s,n) \in B_2. \end{cases}$$

Proof. Suppose $(r,s,n) \in B_1$. Let $\pi = ((n-1)^{r-1}, r+s-2, r+s-3, \ldots, r+\frac{s}{2}+\frac{1}{2}, (r+\frac{s}{2}-\frac{1}{2})^{\frac{s}{2}+\frac{3}{2}}, (r+\frac{s}{2}-\frac{3}{2})^{n-r-s+1})$. Then $\sigma(\pi) = g(r,s,n)$ is even and $f'(\pi) = r+\frac{s}{2}-\frac{1}{2}$. By the left-most off-diagonal matrix $\overline{A}$ of π, $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$ satisfies $\overline{d_1} = \overline{d_2} = \cdots = \overline{d_{f'(\pi)-1}} = n-1$ and $r+s-2 = \overline{d_{f'(\pi)}} \ge \overline{d_{f'(\pi)+1}} \ge \cdots \ge \overline{d_n}$. Clearly, $d_1+d_2+\cdots+d_i \le \overline{d_1}+\overline{d_2}+\cdots+\overline{d_i}$ for $i = 1, 2, \ldots, f'(\pi)$, and so by Theorem 2.1, $\pi \in GS_n$. Let $\pi_1 = (s-1, s-2, \ldots, \frac{s}{2}+\frac{3}{2}, (\frac{s}{2}+\frac{1}{2})^{\frac{s}{2}+\frac{3}{2}}, (\frac{s}{2}-\frac{1}{2})^{n-r-s+1})$. Assume that π is potentially $K_{r,s}$-graphic. Then there exist integers r_1 and s_1, $1 \le r_1 \le \frac{s}{2}+\frac{1}{2}$ and $s_1 = s+1-r_1$ such that π_1 is potentially K_{r_1,s_1}-graphic. If $r_1 < \frac{s}{2}+\frac{1}{2}$, then there are at least r_1 terms in π_1 which are greater than or equal to $s+1-r_1$, which is impossible. If $r_1 = \frac{s}{2}+\frac{1}{2}$, then there are at least $s+1$ terms in π_1 which are greater than or equal to $\frac{s}{2}+\frac{1}{2}$, which is also impossible. Hence π is not potentially $K_{r,s}$-graphic. Thus $\sigma(K_{r,s}, n) \ge \sigma(\pi)+2 = g(r,s,n)+2$.

Now suppose $(r, s, n) \in B_2$. Let $\pi = ((n-1)^{r-1}, r+s-2, r+s-3, \ldots, r+\frac{s}{2}+\frac{1}{2}, (r+\frac{s}{2}-\frac{1}{2})^{\frac{s}{2}+\frac{3}{2}}, (r+\frac{s}{2}-\frac{3}{2})^{n-r-s}, r+\frac{s}{2}-\frac{5}{2})$. Then $\sigma(\pi) = g(r, s, n) - 1$ is even and $f'(\pi) = r + \frac{s}{2} - \frac{1}{2}$. By using the similar method, we also can prove that π is graphic and not potentially $K_{r,s}$-graphic. Hence $\sigma(K_{r,s}, n) \geq \sigma(\pi) + 2 = g(r, s, n) + 1$. □

Lemma 3.2.1. *Let* $3 \leq r \leq s \leq 2r$*, where s is odd, and let* $n = m$*. Then*

$$\sigma(K_{r,s}, n) \leq g(r, s, n) + 2 + \left(\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4\right).$$

Proof. Since $m \leq \binom{r+s-1}{2} + 3$, by Theorem 2.3,

$$\begin{aligned}
\sigma(K_{r,s}, n) &\leq \sigma(K_{r+s}, n) \leq (2r+2s-6)n + 8 \\
&= g(r, s, n) + 2 + \left(\frac{3}{2}s - \frac{7}{2}\right)m + (r-1)r + \left(\frac{s}{2} - \frac{1}{2}\right)r \\
&\quad - \frac{1}{8}(s^2 + 8s - 1) + 6 \\
&\leq g(r, s, n) + 2 + \left(\frac{3}{2}s - \frac{7}{2}\right)(s^2 + s) + (s-1)s + \left(\frac{s}{2} - \frac{1}{2}\right)s \\
&\quad - \frac{1}{8}(s^2 + 8s - 1) + 6 \\
&= g(r, s, n) + 2 + \left(\frac{3}{2}s^3 - \frac{5}{8}s^2 - 6s + \frac{49}{8}\right) \\
&\leq g(r, s, n) + 2 + \left(\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4\right). \quad \square
\end{aligned}$$

Lemma 3.2.2. *Let* $3 \leq r \leq s \leq 2r$*, where s is odd, and let* $n \geq m$ *and* $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ *with* $d_n \geq r$*. If* $\sigma(\pi) \geq g(r, s, n) + 2$*, then* π *is potentially* $K_{r,s}$*-graphic.*

Proof. If $d_{r+s} \geq r + s - 1$, then by Theorem 2.4, π is potentially $K_{r,s}$-graphic. If $d_{r+s} \leq r + s - 1$ and $d_r \geq r + s - 1$, then by $n \geq m \geq (r+2)(s-1)$ and Theorem 2.5, π is potentially $K_{r,s}$-graphic. Now assume that $d_r \leq r + s - 2$. If $d_{r+t} \leq r + s - 2 - t$ for any $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{1}{2}\}$, then $\sigma(\pi) \leq (r-1)(n-1) + (r+s-2) + (r+s-3) + \cdots + (r+\frac{s}{2}-\frac{1}{2}) + (r+\frac{s}{2}-\frac{3}{2})(n-r-\frac{s}{2}+\frac{3}{2}) < g(r, s, n) + 2 \leq \sigma(\pi)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{1}{2}\}$ such that $d_{r+t} \geq r + s - 1 - t$. We consider the following two cases:

Case 1. There exists an integer $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{3}{2}\}$ such that $d_{r+t} \geq r + s - 1 - t$. If $d_{r+s} \leq r + \frac{s}{2} - \frac{5}{2}$, then by $r \leq s \leq 2r$,

$$\begin{aligned}
\sigma(\pi) &\leq (r-1)(n-1) + s(r+s-2) + \left(r + \frac{s}{2} - \frac{5}{2}\right)(n - r - s + 1) \\
&\leq \left(2r + \frac{s}{2} - \frac{5}{2}\right)n - (r-1)r - \left(\frac{s}{2} - \frac{1}{2}\right)r + \frac{s^2}{2} + s + r - \frac{3}{2}
\end{aligned}$$

$$
\begin{aligned}
&- \left\lceil \frac{(r+s+1)^2}{4} \right\rceil \\
&\le \left(2r + \frac{s}{2} - \frac{5}{2}\right)n - (r-1)r - \left(\frac{s}{2} - \frac{1}{2}\right)r + \frac{s^2}{2} + s + r - \frac{3}{2} \\
&\quad - \frac{(r+s)^2 + 2(r+s)}{4} \\
&\le \left(2r + \frac{s}{2} - \frac{5}{2}\right)n - (r-1)r - \left(\frac{s}{2} - \frac{1}{2}\right)r + \frac{s^2}{2} + s + r - \frac{3}{2} \\
&\quad - \frac{\left(\frac{s}{2} + s\right)^2 + 2(r+r)}{4} \\
&= \left(2r + \frac{s}{2} - \frac{5}{2}\right)n - (r-1)r - \left(\frac{s}{2} - \frac{1}{2}\right)r - \frac{1}{16}s^2 + s - \frac{3}{2} \\
&< g(r,s,n) + 2 \le \sigma(\pi), \text{ a contradiction.}
\end{aligned}
$$

Hence $d_{r+s} \ge r + \frac{s}{2} - \frac{3}{2} \ge r + t$. By Theorem 2.11, π is potentially $K_{r,s}$-graphic.

Case 2. $d_{r+t} \le r + s - 2 - t$ for any $t \in \{1, 2, \ldots, \frac{s}{2} - \frac{3}{2}\}$ and $d_{r+t} \ge r + s - 1 - t$ for $t = \frac{s}{2} - \frac{1}{2}$. If $d_{r+s} \le r + \frac{s}{2} - \frac{3}{2}$, then $\sigma(\pi) \le (r-1)(n-1) + (r+s-2) + (r+s-3) + \cdots + (r + \frac{s}{2} + \frac{1}{2}) + (r + \frac{s}{2} - \frac{1}{2})(\frac{s}{2} + \frac{3}{2}) + (r + \frac{s}{2} - \frac{3}{2})(n - r - s + 1) < g(r,s,n) + 2 \le \sigma(\pi)$, which is impossible. Hence $d_{r+s} \ge r + \frac{s}{2} - \frac{1}{2}$. Thus by Theorem 2.11, π is also potentially $K_{r,s}$-graphic. □

Lemma 3.2.3. *Let* $3 \le r \le s \le 2r$, *where* s *is odd, and let* $n = m + t$, *where* $0 \le t \le 3s^2 + 2s - 8$. *Then*

$$
\sigma(K_{r,s}, n) \le g(r,s,n) + 2 + \left(\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4\right) - \frac{(s-1)t}{2}.
$$

Proof. Use induction on t. If follows from Lemma 3.2.1 that the result holds for $t = 0$. Now suppose that the result holds for $t - 1$, $0 \le t - 1 \le 3s^2 + 2s - 9$. We will prove that the result holds for t. Let $n = m + t$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge g(r,s,n) + 2 + (\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4) - \frac{(s-1)t}{2}$. Then $\sigma(\pi) \ge g(r,s,n) + 2$. If $d_n \ge r$, then by Lemma 3.2.2, π is potentially $K_{r,s}$-graphic. If $d_n \le r - 1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \ge g(r,s,n-1) + 2 + (\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4) - \frac{(s-1)(t-1)}{2}$. By the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. Thus $\sigma(K_{r,s}, n) \le g(r,s,n) + 2 + (\frac{3}{2}s^3 - \frac{s^2}{2} - 5s + 4) - \frac{(s-1)t}{2}$. □

Lemma 3.2.4. *Let* $3 \le r \le s \le 2r$, *where* s *is odd, and let* $n \ge m + 3s^2 + 2s - 8$. *Then* $\sigma(K_{r,s}, n) \le g(r,s,n) + 2$.

Proof. We only need to prove that (∗): if $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge g(r,s,n) + 2$, then π is potentially $K_{r,s}$-graphic. Apply induction on n. By Lemma 3.2.3, (∗) holds for $n = m + 3s^2 + 2s - 8$. Now suppose that (∗) holds for $n - 1 \ge m + 3s^2 + 2s - 8$. We will prove that (∗) holds for n. If $d_n \ge r$, then by Lemma 3.2.2, π is potentially

$K_{r,s}$-graphic. If $d_n \leq r-1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \geq g(r,s,n-1)+2$. Thus by the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. □

Theorem 3.2.2. *Let* $3 \leq r \leq s \leq 2r$, *where* s *is odd, and let* $n \geq m + 3s^2 + 2s - 8$. *Then*

$$\sigma(K_{r,s}, n) = \begin{cases} g(r,s,n)+2, & \text{if } (r,s,n) \in B_1, \\ g(r,s,n)+1, & \text{if } (r,s,n) \in B_2. \end{cases}$$

Proof. The result follows from Theorem 3.2.1, Lemma 3.2.4 and $\sigma(K_{r,s}, n)$ being even. □

3.3. $\sigma(K_{r,s}, n)$ *for* $r \geq 3$, $s \geq 2r+1$ *and* $n \geq m + 2s^2 + (2r-6)s + 4r - 8$

Theorem 3.3.1. *Let* $r \geq 3$, $s \geq 2r+1$ *and* $n \geq r+s$. *Then*

$$\sigma(K_{r,s}, n) \geq \begin{cases} h(r,s,n)+2, & \text{if } (r,s,n) \in C_1, \\ h(r,s,n)+1, & \text{if } (r,s,n) \in C_2. \end{cases}$$

Proof. Assume $(r,s,n) \in C_1$. Let $\pi = ((n-1)^{r-1}, r+s-2, r+s-3, \ldots, s, (s-1)^{n-2r+2})$. Then $\sigma(\pi) = h(r,s,n)$ is even and $f'(\pi) = s-1$. By the left-most off-diagonal matrix $\overline{A}$ of π, $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$ satisfies $n-1 = \overline{d_1} = \overline{d_2} = \cdots = \overline{d_{f'(\pi)}} \geq \overline{d_{f'(\pi)+1}} \geq \cdots \geq \overline{d_n}$, and so $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for $i = 1, 2, \ldots, f'(\pi)$. Thus by Theorem 2.1, $\pi \in GS_n$. Let $\pi_1 = (s-1, s-2, \ldots, s-r+1, (s-r)^{n-2r+2})$. If π is potentially $K_{r,s}$-graphic, then there exist integers r_1 and s_1, $1 \leq r_1 \leq r$ and $s_1 = s+1-r_1$ such that π_1 is potentially K_{r_1,s_1}-graphic. Hence there are at least r_1 terms in π_1 which are greater than or equal to $s+1-r_1$, which is impossible. So π is not potentially $K_{r,s}$-graphic. Thus $\sigma(K_{r,s}, n) \geq \sigma(\pi) + 2 = h(r,s,n)+2$.

Now assume $(r,s,n) \in C_2$. Let $\pi = ((n-1)^{r-1}, r+s-2, r+s-3, \ldots, s, (s-1)^{n-2r+1}, s-2)$. Then $\sigma(\pi) = h(r,s,n)-1$ is even and $f'(\pi) = s-1$. Similarly, we also can prove that π is graphic and not potentially $K_{r,s}$-graphic. Thus $\sigma(K_{r,s}, n) \geq \sigma(\pi)+2 = h(r,s,n)+1$. □

Lemma 3.3.1. *Let* $r \geq 3$, $s \geq 2r+1$ *and* $n = m$. *Then*

$$\sigma(K_{r,s}, n) \leq h(r,s,n) + 2 + \frac{2s^3 + (2r-6)s^2 + 4rs - 8s}{2}.$$

Proof. Since $m \leq \binom{r+s-1}{2} + 3$, by Theorem 2.3,

$$\begin{aligned} \sigma(K_{r,s}, n) &\leq \sigma(K_{r+s}, n) \leq (2r+2s-6)n + 8 \\ &= h(r,s,n) + 2 + (r+s-4)m + 6 + \frac{(r-1)(2s-r)}{2} \\ &\leq h(r,s,n) + 2 + (r+s-4)(s^2+s) + 6 + \frac{(r-1)(2s-r)}{2} \\ &= h(r,s,n) + 2 + \frac{2s^3 + (2r-6)s^2 + 4rs - 10s - r^2 + r + 12}{2} \\ &\leq h(r,s,n) + 2 + \frac{2s^3 + (2r-6)s^2 + 4rs - 8s}{2}. \quad \square \end{aligned}$$

Lemma 3.3.2. *Let $r \ge 3, s \ge 2r+1$ and $n \ge m$, and let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_n \ge r$. If $\sigma(\pi) \ge h(r, s, n) + 2$, then π is potentially $K_{r,s}$-graphic.*

Proof. If $d_{r+s} \ge r+s-1$, then by Theorem 2.4, π is potentially $K_{r,s}$-graphic. If $d_{r+s} \le r+s-2$ and $d_r \ge r+s-1$, then by $n \ge m \ge (r+2)(s-1)$ and Theorem 2.5, π is potentially $K_{r,s}$-graphic. Now assume that $d_r \le r+s-2$. If $d_{r+t} \le r+s-2-t$ for any $t \in \{1, 2, \ldots, r-1\}$, then $\sigma(\pi) \le (r-1)(n-1) + (r+s-2) + (r+s-3) + \cdots + s + (s-1)(n-2r+2) = h(r, s, n) < \sigma(\pi)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, r-1\}$ such that $d_{r+t} \ge r+s-1-t$. If $d_{r+s} \le 2r-2$, then by $s \ge 2r+1$ and $n \ge [\frac{(r+s+1)^2}{4}] \ge \frac{(r+s)^2+2(r+s)}{4}$,

$$\begin{aligned}
\sigma(\pi) &\le (r-1)(n-1) + s(r+s-2) + (2r-2)(n-r-s+1) \\
&= (r+s-2)n - (s-2r+1)(n-r-s+1) \\
&\le (r+s-2)n - (2r+1-2r+1)\left(\frac{(r+s)^2+2(r+s)}{4} - r - s + 1\right) \\
&= (r+s-2)n - \frac{(r+s)^2 - 2(r+s) + 4}{2} \\
&\le h(r, s, n) < \sigma(\pi), \text{ a contradiction.}
\end{aligned}$$

Hence $d_{r+s} \ge 2r-1 \ge r+t$. Thus by Theorem 2.11, π is potentially $K_{r,s}$-graphic. □

Lemma 3.3.3. *Let $r \ge 3, s \ge 2r+1$ and $n = m+t$, where $0 \le t \le 2s^2 + (2r-6)s + 4r - 8$. Then $\sigma(K_{r,s}, n) \le h(r, s, n) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{st}{2}$.*

Proof. We use induction on t. It follows from Lemma 3.3.1 that the result holds for $t = 0$. Now assume that the result holds for $t-1, 0 \le t-1 \le 2s^2 + (2r-6)s + 4r - 9$. Let $n = m+t$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \ge h(r, s, n) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{st}{2}$. Then $\sigma(\pi) \ge h(r, s, n) + 2$. If $d_n \ge r$, then by Lemma 3.3.2, π is potentially $K_{r,s}$-graphic. If $d_n \le r-1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \ge h(r, s, n-1) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{st}{2} + s - r \ge h(r, s, n-1) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{st}{2} + \frac{s}{2} = h(r, s, n-1) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{s(t-1)}{2}$. Hence by the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. Thus $\sigma(K_{r,s}, n) \le h(r, s, n) + 2 + \frac{2s^3+(2r-6)s^2+4rs-8s}{2} - \frac{st}{2}$. □

Lemma 3.3.4. *Let $r \ge 3, s \ge 2r+1$ and $n \ge m + 2s^2 + (2r-6)s + 4r - 8$. Then $\sigma(K_{r,s}, n) \le h(r, s, n) + 2$.*

Proof. It is enough to prove that $(*)$: if $\pi = (d_1, \ldots, d_n) \in GS_n$ and $\sigma(\pi) \ge h(r, s, n)+2$, then π is potentially $K_{r,s}$-graphic. Apply induction on n. By Lemma 3.3.3, $(*)$ holds for $n = m + 2s^2 + (2r-6)s + 4r - 8$. Now suppose that $(*)$ holds for $n-1 \ge m + 2s^2 + (2r-6)s + 4r - 8$. We will prove that $(*)$ holds for n. If $d_n \ge r$, then by Lemma 3.3.2, π is potentially $K_{r,s}$-graphic. If $d_n \le r-1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \ge h(r, s, n-1) + 2$. By the induction hypothesis, π' is potentially $K_{r,s}$-graphic, and hence so is π. □

Theorem 3.3.2. *Let* $r \geq 3$, $s \geq 2r+1$ *and* $n \geq m + 2s^2 + (2r-6)s + 4r - 8$. *Then*

$$\sigma(K_{r,s}, n) = \begin{cases} h(r,s,n)+2, & \textit{if } (r,s,n) \in C_1, \\ h(r,s,n)+1, & \textit{if } (r,s,n) \in C_2. \end{cases}$$

Proof. The result follows from Theorem 3.3.1, Lemma 3.3.4 and $\sigma(K_{r,s}, n)$ being even. □

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