A new technique for the characterization of graphs with a maximum number of spanning trees

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Abstract

Let \( \Gamma(n,e) \) denote the class of all simple graphs on \( n \) nodes and \( e \) edges. The number of spanning trees of a graph \( G \) is denoted by \( t(G) \). A graph \( G_0 \in \Gamma(n,e) \) is said to be \( t \)-optimal if \( t(G_0) \geq t(G) \) for all \( G \in \Gamma(n,e) \). The problem of characterizing \( t \)-optimal graphs for arbitrary \( n \) and \( e \) is still open, although characterizations of \( t \)-optimal graphs for specific pairs \((n,e)\) are known. We introduce a new technique for the characterization of \( t \)-optimal graphs, based on an upper bound for the number of spanning trees of a graph \( G \) in terms of the degree sequence and the number of induced paths of length two of the complement of \( G \). The technique yields the following new results:

(1) Complete, almost-regular multipartite graphs are \( t \)-optimal.
(2) A complete characterization of \( t \)-optimal graphs in \( \Gamma(n,e) \) for \( n(n-1)/2 - 3n/2 \leq e \leq n(n-1)/2 - n \) is obtained for \( n \geq n_0 \), where \( n_0 \) can be explicitly determined.

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1. Introduction

An open extremal problem, with applications to the synthesis of reliable networks, is the characterization of those graphs having a maximum number of spanning trees for a given number of nodes, \( n \), and edges, \( e \). Such graphs are said to be \( t \)-optimal.

Characterizations of \( t \)-optimal graphs for specific pairs \((n,e)\) have appeared in the literature [2,4,6–10,12]. In [11] we introduced a lower bound for the trace of the \( k \)th power of the Laplacian matrix of a graph in terms of its degree sequence. Using this inequality we developed an upper bound for the number of spanning trees of a graph...
in terms of the degree sequence of its complement that is sharp for, and only for, complete multipartite graphs (asymptotic bounds for the number of spanning trees of regular graphs have recently been obtained in [5]).

Here we will develop a powerful refinement of the upper bounding technique for the number of spanning trees established in [11]. The improved bound yields a new technique to characterize many hitherto unknown types of \( t \)-optimal graphs. The rest of the paper is organized as follows:

In Section 2, we introduce notation and basic results that will be used throughout the paper. In Section 3, we establish a lower bound for the trace of the \( k \)th power of the Laplacian matrix of a graph in terms of its degree sequence. In Section 4, we use the results in the previous section to establish an upper bound for \( C(G,x) \), the characteristic polynomial of the Laplacian matrix of \( G \), in terms of the degree sequence and the number of induced paths of length two of \( G \). This yields an upper bound for the number of spanning trees of a graph that is sharp for, and only for, complete multipartite graphs. Section 5 is concerned with the characterization of degree sequences that maximize the upper bound derived in Section 4.

In the remaining sections we use the previous results to characterize hitherto unknown types of \( t \)-optimal graphs. Thus, in Section 6, we show that a complete, almost-regular multipartite graph \( G \) is the unique \( t \)-optimal graph among all simple graphs on \( n(G) \) vertices and \( e(G) \) edges. Previously, this was known to be true for complete, regular multipartite graphs [4]. In Section 7, we outline a general technique to characterize \( t \)-optimal graphs. The technique is based on two key theorems which, taken together, state that asymptotically, if \( G \) is a graph with the same number of vertices and edges as \( mK_{a+1} \cup G_0 \cup hK_a \) (\( G_0 \) being a graph with \( a - 1 \leq \delta(G_0) \) and \( A(G_0) \leq a \), where \( \delta(G_0) \) and \( A(G_0) \) denote the minimum and maximum degree of \( G_0 \) then \( \tilde{G} \), the complement of \( G \), is \( t \)-optimal only if \( G \) is almost-regular and has a minimum number of induced paths of length two.

In the next two sections, we carry out the program presented in Section 7, to provide a complete characterization of \( t \)-optimal graphs on \( n \) nodes and \( e \) edges for \( n(n - 1)/2 - 3n/2 \leq e \leq n(n - 1)/2 - n \) and \( n \geq n_0 \). We also determine \( n_0 \) (previously, \( t \)-optimal graphs had been completely characterized when \( n(n - 1)/2 - n \leq e \leq n(n - 1)/2 \) [8,10]).

The characterization involves two major steps:

1. Characterize almost-regular graphs with the minimum number of induced paths of length two in \( \Gamma(n,e) \) when \( n \leq e \leq 3n/2 \). This is done in Section 8.

2. Determine \( n_0 \) such that, for \( n \geq n_0 \), the complements of the almost-regular graphs with the minimum number of induced paths of length two, characterized in Section 8, are \( t \)-optimal. This is the object of Section 9.

In Section 10, we conclude with some open problems and conjectures. In order to make this paper self-contained, we will repeat some of the proofs that have appeared elsewhere [11].
2. Preliminaries

For any terms not defined here, see [3]. We deal exclusively with simple graphs. The number of nodes (resp. edges) of a graph \( G \) is denoted by \( n(G) \) (resp. \( e(G) \)). Let \( \Gamma(n,e) \) stand for the collection of all non-isomorphic simple graphs on \( n \) nodes and \( e \) edges. For a graph \( G \), we let \( \Gamma(G) = \Gamma(n(G), e(G)) \). The Laplacian (or admittance) matrix of a graph \( G \) on vertex set \( \{v_1, v_2, \ldots, v_n\} \) is the \( n \times n \) matrix \( (h_{ij}) \) with \( h_{ii} \) being the degree of vertex \( v_i \), and, for \( i \neq j \), \( h_{ij} = -1 \) if \( \{v_i, v_j\} \) is an edge of \( G \). Otherwise, we denote the Laplacian matrix of \( G \) by \( \text{Lapl}(G) \). Let \( C(G, x) \) stand for the characteristic polynomial of \( H = \text{Lapl}(G) \), i.e., \( \det(xI_n - H) \). The complement of a graph \( G \) is denoted by \( \tilde{G} \). \( \tilde{\Gamma}(n,e) \) is the collection of all graphs \( G \) such that \( \tilde{G} \in \Gamma(n,e) \). The number of spanning trees of a graph \( G \) is denoted by \( \tau(G) \), and \( v(G) \) stands for the number of subgraphs of \( G \) induced by three vertices and having exactly two edges (such subgraphs resemble the letter “v” when suitably drawn, hence the notation; they can also be characterized as induced paths of length two).

A graph is said to be almost-regular if the degrees of any two of its vertices differ by no more than one. The class of all non-isomorphic almost-regular graphs on \( n \) nodes and \( e \) edges is denoted by \( \Gamma_A(n,e) \).

A graph \( G \) on \( n \) nodes and \( e \) edges is said to be t-optimal iff \( t(G) \geq t(G') \) for all \( G' \in \Gamma(n,e) \). A graph \( G \in \Gamma(n,e) \) is said to be \( \tau \)-max in \( \Gamma_A(n,e) \), or almost-regular-\( \tau \)-max (resp. \( v \)-min in \( \Gamma_A(n,e) \), or almost-regular-v-min), if \( G \) is almost-regular and \( \tau(G) \geq \tau(G') \) (resp. \( v(G) \leq v(G') \)) for all \( G' \in \Gamma_A(n,e) \).

The trace (sum of the diagonal elements) of a matrix \( M \) is denoted by \( \text{tr}(M) \). The diagonal matrix with diagonal entries \( a_1, a_2, \ldots, a_n \) is denoted by \( \text{diag}(a_1, a_2, \ldots, a_n) \).

We use the following notation for a sequence \( s \) of \( n \) numbers \( s_1, s_2, \ldots, s_n \): \( s = [s_i]_{1 \leq i \leq n} \) or simply \( s = [s_i] \). Repetitions of a value in a sequence are indicated by superscripting that value with the number of times it appears, enclosed in parentheses; for example, \( [a^{(2)}, b, c^{(4)}] \) is a 7-element sequence where \( a \) appears two times, \( b \) appears one time, and \( c \) appears four times. It is understood that the order in which the elements of a sequence appear is immaterial.

The following theorem is proved in [1, Proposition 6.6]:

**Theorem 1.** If \( G \) is a graph on \( n \) nodes, \( t(G) = n^{-2}C(\tilde{G}, n) \).

An immediate consequence of Theorem 1 is the following:

**Corollary 2.** Let \( G \) be a graph on \( n \) nodes. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of \( \text{Lapl}(\tilde{G}) \), then \( t(G) = n^{n-2} \prod_{i=1}^{n}(1 - \lambda_i/n) \).

We state without proof the following lemma, concerning two elementary identities satisfied by the eigenvalues of the Laplacian matrix of a graph:
Lemma 3. Let $G$ be a graph on $n$ nodes with degree sequence $d_1, d_2, \ldots, d_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of $H = \text{Lapl}(G)$. Then

$$\text{tr}(H) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i,$$

$$\text{tr}(H^2) = \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i(1 + d_i).$$

The following lemma establishes a relationship between $\tau(G)$, $v(G)$, and $[d_i]_1 \leq i \leq n$ (the degree sequence of $G$).

Lemma 4. $\sum_{i=1}^{n} \left(\frac{d_i}{2}\right) = 3\tau(G) + v(G)$.

Proof. $\sum_{i=1}^{n} \left(\frac{d_i}{2}\right)$ counts the number of unordered pairs of incident edges. Each induced path of length two contributes one such pair, whereas each triangle contributes three.

3. A lower bound for the trace of $\text{Lapl}(G)^k$

We need the following two lemmas:

Lemma 5. Let $a_1, a_2, \ldots, a_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers. The function $f(x) = (a_1\lambda_1^{x+1} + a_2\lambda_2^{x+1} + \cdots + a_n\lambda_n^{x+1})/(a_1\lambda_1^x + a_2\lambda_2^x + \cdots + a_n\lambda_n^x)$ is either constant or strictly increasing for all real $x$.

Proof. The function $f(x)$ is clearly constant if all $\lambda_i$’s are equal. Otherwise, assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Suppose $x > y$. It suffices to show that

$$\left(\sum_{i=1}^{n} a_i\lambda_i^{x+1}\right)\left(\sum_{i=1}^{n} a_i\lambda_i^y\right) - \left(\sum_{i=1}^{n} a_i\lambda_i^{y+1}\right)\left(\sum_{i=1}^{n} a_i\lambda_i^x\right) > 0.$$

The left hand side can be rewritten as follows:

$$\sum_{1 \leq i < j \leq n} a_ia_j\lambda_i^{x+1}\lambda_j^y((\lambda_i/\lambda_j)^y - (\lambda_i/\lambda_j)^x)(\lambda_i - \lambda_j).$$

Since there exist $i, j$ with $\lambda_i > \lambda_j$, this last expression is positive.

Lemma 6. Let $G$ be a graph with Laplacian matrix $H$. Then $\text{tr}(H^3) = \sum_{i=1}^{n} d_i(1 + d_i)^2 + 2\tau(G)$. 

Proof. If $G$ has adjacency matrix $A$ and degree sequence $[d_1,d_2,\ldots,d_n]$, then $H = D - A$, where $D = \text{diag}(d_1,d_2,\ldots,d_n)$. We have
\[
\text{tr}(H^3) = \text{tr}(D^3) - 3\text{tr}(D^2A) + 3\text{tr}(DA^2) - \text{tr}(A^3).
\]
(1)
The result follows by rewriting the terms on the right hand side of (1) according to the following identities: $\text{tr}(D^3) = \sum_{i=1}^n d_i^3$, $\text{tr}(D^2A) = 0$, $\text{tr}(DA^2) = \sum_{i=1}^n d_i^2$, $\text{tr}(A^3) = 6\tau(G)$, and by Lemma 4, $\sum_{i=1}^n \left( \frac{d_i}{2} \right) = 3\tau(G) + v(G)$. □

Theorem 7. If $G$ is a simple graph with Laplacian matrix $H = (h_{ij})$ and degree sequence $(d_1,d_2,\ldots,d_n)$, then $\text{tr}(H^k) \geq \sum_{i=1}^n d_i (1 + d_i)^{k-1}$ for every positive integer $k$. For $k \geq 3$, equality occurs if and only if $G$ is a disjoint union of cliques.

Proof. We actually prove the following stronger result:
\[
h_{ii}^{(k)} \geq d_i(1 + d_i)^{k-1}
\]
(2)
for all vertices $i$, where $h_{ii}^{(k)}$ is the $i$th diagonal entry of the matrix $H^k$. Since $H$ is real symmetric, it can be diagonalized by means of an orthonormal matrix $M = (m_{ij})$. Thus $H^k = M\lambda^k M^T$, with $\lambda = \text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, where $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of $H$. Therefore,
\[
h_{ii}^{(k)} = m_{i1}^2 \lambda_1^k + m_{i2}^2 \lambda_2^k + \cdots + m_{in}^2 \lambda_n^k.
\]
If $h_{ii}(=d_i) = 0$, then, since $h_{ii} = m_{i1}^2 \lambda_1 + m_{i2}^2 \lambda_2 + \cdots + m_{in}^2 \lambda_n$, we conclude that $m_{ij}^2 \lambda_j = 0$ for all $j$, $1 \leq j \leq n$. Hence $h_{ii}^{(k)} = 0$ for all $k \geq 1$, and (2) is trivially true. Otherwise there is at least one $j$ such that $m_{ij}^2 \lambda_j > 0$, thus the function
\[
f_i(x) = \frac{m_{i1}^2 \lambda_1^{x+1} + m_{i2}^2 \lambda_2^{x+1} + \cdots + m_{in}^2 \lambda_n^{x+1}}{m_{i1}^2 \lambda_1^x + m_{i2}^2 \lambda_2^x + \cdots + m_{in}^2 \lambda_n^x}
\]
is well defined. Discarding the terms which are zero in the above expression, and taking into account that, since $H$ is positive semidefinite, $\lambda_i \geq 0$ for all $i$, we see that Lemma 5 applies to $f_i(x)$. Therefore $f_i(x)$ is a non-decreasing function for all real $x$. Thus, for $k \geq 1$, we have
\[
f_i(k) = \frac{h_{ii}^{(k+1)}}{h_{ii}^{(k)}} \geq f_i(1) = \frac{h_{ii}^{(2)}}{h_{ii}^{(1)}} = \frac{d_i^2 + d_i}{d_i} = 1 + d_i.
\]
Therefore, we conclude
\[
\frac{h_{ii}^{(k+1)}}{h_{ii}^{(k)}} \geq 1 + d_i
\]
for all $k \geq 1$. Hence,
\[
h_{ii}^{(k)} = \frac{h_{ii}^{(k)}}{h_{ii}^{(k-1)}} \frac{h_{ii}^{(k-1)}}{h_{ii}^{(k-2)}} \cdots \frac{h_{ii}^{(2)}}{h_{ii}^{(1)}} h_{ii}^{(1)} \geq (1 + d_i)^{k-1} d_i.
\]
(3)
Suppose that, for some \( k \geq 3 \), \( tr(H^k) = \sum_{i=1}^{n} d_i (1 + d_i)^k - 1 \). This is equivalent to \( h^{(k)}_i = d_i (1 + d_i)^k - 1 \) for all \( i \). From (3) and Lemma 5, this can occur if and only if all the functions \( f_i \) corresponding to non-isolated nodes \( i \) are constant, which, again by Lemma 5, is equivalent to \( h^{(2)}_i = h^{(1)}_i = 1 + d_i \) for every non-isolated \( i \); since \( h^{(2)}_i = d_i (1 + d_i) \), and since \( h^{(3)}_i \geq d_i (1 + d_i)^2 \) for all \( i \), this is equivalent to \( tr(H^3) = \sum_{i=1}^{n} d_i (1 + d_i)^2 \). By Lemma 6 this is possible if and only if \( v(G) = 0 \). It can be easily seen that this is equivalent to saying that \( G \) is a union of cliques.

4. An upper bound for \( C(G, x) \)

**Theorem 8.** Let \( G \) be a graph on \( n \) vertices with degree sequence \([d_i]_1 \leq i \leq n\). Assume \( \tilde{G} \) is connected. Then \( C(G, x) \leq x^n e^{-2\nu(G)/(3x^3)} \prod_{i=1}^{n} \left( 1 - \frac{1 + d_i}{x} \right)^{d_i/(1 + d_i)} \) for \( x \geq n \). The inequality is strict unless \( G \) is a disjoint union of cliques.

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) denote the eigenvalues of \( H = \text{Lap}(G) \). Then \( tr(H^k) = \sum_{i=1}^{n} \lambda_i^k \). Since \( \tilde{G} \) is connected, we have \( \max[\lambda_i] < n \) (see [1, pp. 29 (4e), p. 40, Proposition 6.6]) and \( \max[d_i] < n - 1 \). The validity of the power series manipulations that follow is justified by these two facts. Using Lemma 6 and Theorem 7 we get

\[
\sum_{k=1}^{\infty} \frac{\sum_{i=1}^{n} \lambda_i^k}{kx^k} = \sum_{k=1}^{\infty} \frac{tr(H^k)}{kx^k} \geq \frac{2\nu(G)}{3x^3} + \sum_{k=1}^{\infty} \frac{\sum_{i=1}^{n} d_i (1 + d_i)^k}{kx^k}. \tag{4}
\]

Interchanging summations, we obtain

\[
\sum_{k=1}^{\infty} \frac{\sum_{i=1}^{n} \lambda_i^k}{kx^k} \geq \frac{2\nu(G)}{3x^3} + \sum_{i=1}^{n} d_i \frac{\sum_{k=1}^{\infty} (1 + d_i)^k}{kx^k}.
\]

Using the power series expansion \(-\log(1 - x) = \sum_{k=1}^{\infty} x^k/k\), which is valid for \(|x| < 1\), results in

\[
\sum_{i=1}^{n} \log \left( 1 - \frac{\lambda_i}{x} \right) \leq -\frac{2\nu(G)}{3x^3} + \sum_{i=1}^{n} \frac{d_i}{1 + d_i} \log \left( 1 - \frac{1 + d_i}{x} \right),
\]

which, by exponentiation, is equivalent to

\[
\prod_{i=1}^{n} \left( 1 - \frac{\lambda_i}{x} \right) \leq e^{-2\nu(G)/(3x^3)} \prod_{i=1}^{n} \left( 1 - \frac{1 + d_i}{x} \right)^{d_i/(1 + d_i)}. \tag{5}
\]

The theorem follows by observing that the left hand side of (5) is just \( C(G, x)/x^n \), and that, by Theorem 7, inequality (4) is strict unless \( G \) is a union of cliques.
Corollary 9. Let \( G \) be a graph on \( n \) vertices with degree sequence \([d_i]\) for \( 1 \leq i \leq n\). Assume \( \tilde{G} \) is connected. Then \( t(\tilde{G}) \leq n^{n-2} e^{-(G)/(3n^3)} \prod_{i=1}^{n} (1 - (1 + d_i)/n)^{d_i/(1 + d_i)} \). The inequality is strict unless \( G \) is a union of cliques.

Proof. Follows from Theorem 8 when \( x = n \) and the fact that \( t(\tilde{G}) = C(G,n)/n^2 \) by Theorem 1. \( \square \)

5. Almost-regular sequences and the relation \( \prec \) among non-negative integer sequences

A sequence \( d = [d_i]_1 \leq i \leq n \) is said to be almost-regular if \( |d_i - d_j| \leq 1 \) for all \( i, j \). Given two non-negative integer sequences \( d = [d_i], \; d' = [d'_i] \), we write \( d \prec d' \) if, up to a permutation, the sequence \( d' \) can be obtained from \( d \) by the following operation:

(\( \mathcal{O} \)) Pick two indices \( i, j \) such that \( d_j - d_i \geq 2 \) (if they exist). Replace \( d_i \) by \( d_i + 1 \) and \( d_j \) by \( d_j - 1 \). In other words, \( d' \) is obtained from \( d \) by bringing closer together (by one unit each) two terms in \( d \) that differ by two or more.

Let \( \prec^+ \) denote the transitive closure of \( \prec \). That is, \( d \prec^+ d' \) if \( d \prec d_2 \prec \cdots \prec d_m = d' \) with \( m > 1 \). Likewise, let \( \prec^* \) denote the reflexive and transitive closure of \( \prec \). That is, \( d \prec^* d' \) if \( d = d' \) (up to a permutation) or \( d \prec^+ d' \). We have the following lemma:

Lemma 10. For any non-negative integer sequence \( d \), there exists a unique (up to a permutation) \( d' \) such that \( d' \) is almost-regular and \( d \prec^* d' \).

Proof. For a non-negative integer sequence \( d \), define \( S_2(d) = \sum_{i=1}^{n} d_i^2 \) (the norm-square of \( d \)). Clearly \( d \prec d' \) implies \( S_2(d) > S_2(d') \). Since \( S_2(d) \) is integer and non-negative, after finitely many applications of operation (\( \mathcal{O} \)) we obtain sequences \( d_1, d_2, \ldots, d_m \) such that \( d \prec d_1 \prec d_2 \prec \cdots \prec d_m \) and such that (\( \mathcal{O} \)) cannot be applied to \( d_m \). This can occur only if \( d_m \) is almost-regular. It is easily verified that \( d_m \) is uniquely determined (up to a permutation). Indeed, if \( s = \sum_{i=1}^{n} d_i \) and if \( s = nq + r \) where \( q \) and \( r \) are the quotient and remainder of \( s \) divided by \( n \) then \( d_m \) is, up to a permutation, the sequence consisting of \( (n - r) \) entries equal to \( q \) and \( r \) entries equal to \( q + 1 \). \( \square \)

The following definitions will be used in the sequel:

Definition 11. Let \( d = [d_1, d_2, \ldots, d_n] \). The function \( f(d, x) \) is defined as follows:

\[
f(d, x) = \prod_{i=1}^{n} \left( 1 - \frac{1 + d_i}{x} \right)^{d_i/(1 + d_i)}.
\]

Definition 12. Let \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \). The function \( g(\lambda, x) \) is defined thus

\[
g(\lambda, x) = \prod_{i=1}^{n} \left( 1 - \frac{\lambda_i}{x} \right).
\]
The following is a key lemma:

**Lemma 13.** If \( d \prec d' \) then \( f(d,x) < f(d',x) \) for \( x > 1 + \Delta \), where \( \Delta = \max[d_i] \).

**Proof.** It suffices to show \( d \prec d' \) implies that \( f(d,x) < f(d',x) \) for \( x > 1 + \Delta \). Taking log and expanding into a power series (which is valid if \( x > 1 + \Delta \)), we obtain

\[
\log f(d,x) = -\sum_{k=1}^{\infty} \frac{\phi_k(d)}{kx^k},
\]

where

\[
\phi_k(d) = \sum_{i=1}^{n} d_i(1 + d_i)^{k-1}.
\]

We have reduced the proof to showing that, for all \( k \geq 2 \), \( \phi_k(d) > \phi_k(d') \) if \( d \prec d' \). In turn, we can write

\[
\phi_k(d) = \sum_{j=0}^{k-1} \binom{k-1}{j} S_{j+1}(d),
\]

where

\[
S_j(d) = \sum_{i=1}^{n} d_i^j.
\]

Thus we just need to show that, for all \( j \geq 2 \), \( S_j(d) > S_j(d') \) when \( d \prec d' \). Let \( d = [d_i] \).

Since \( d \prec d' \), there are indexes \( u, v \) such that \( d_v - d_u \geq 2 \) and \( d' \) is obtained from \( d \) by replacing \( d_u \) with \( d_u + 1 \) and \( d_v \) with \( d_v - 1 \). Thus \( S_j(d) - S_j(d') = (d_v^j - (d_v - 1)^j) - ((d_v + 1)^j - d_u^j) \). Using \( a^j - b^j = (a - b) \sum_{i=0}^{j-1} a^{j-1-i} b^i \) we obtain \( S_j(d) - S_j(d') = \sum_{i=0}^{j-1} (d_v^{j-1-i} - d_u^j - (d_v + 1)^{j-1-i} d_u^i) \). Because of \( d_v - d_u \geq 2 \), each term in the last sum is positive. Therefore \( S_j(d) - S_j(d') > 0 \). □

6. Complete, almost-regular multipartite graphs are \( t \)-optimal

Cheng [4] showed that complete regular multipartite graphs are \( t \)-optimal. We now show that, more generally, complete almost-regular multipartite graphs are \( t \)-optimal.

**Theorem 14.** Let \( G \) be a disjoint union of cliques whose orders differ by at most one. Then \( t(\tilde{G}) > t(\tilde{G}') \) for every \( G' \in \Gamma(G) - \{G\} \).

**Proof.** Suppose \( G' \in \Gamma(G) - \{G\} \). We can assume \( \tilde{G}' \) is connected. From Corollary 9, we have, using Definition 11:

\[
t(\tilde{G}') \leq n^{n-2} e^{-2t(\tilde{G}')/(3n)} f(d',n),
\]

(6)
where \( d' = [d'_1, d'_2, \ldots, d'_n] \) is the degree sequence of \( G' \). By Lemma 10, there is a unique (up to a permutation) almost-regular sequence \( d = [d_1, d_2, \ldots, d_n] \) such that \( d' \prec^* d \). By Lemma 13, \( f(d', n) \leq f(d, n) \). Thus

\[
  t(G') \leq n^n e^{-2t(G')/(3n)} f(d', n) \leq n^n e^{-2t(G)}/(3n) f(d, n) = t(G).
\]

(7)

Thus \( t(G') \leq t(G) \). In addition, if \( t(G') = t(G) \), then all the inequalities in (7) are equalities. Thus, \( v(G') = 0 \), and therefore \( G' \) must be a union of cliques (by Corollary 9). Also, we must have \( f(d', n) = f(d, n) \). By Lemma 13, this can happen only if \( d' = d \) (up to a permutation), i.e., if \( d' \) is almost-regular. Thus, \( G' \) is an almost-regular union of cliques, and therefore \( G' = G \). □

7. A technique for the characterization of \( t \)-optimal graphs

Corollary 9 yields a general technique to prove the \( t \)-optimality of many types of graphs. The technique can be summarized as follows:

1. Let \( G_0 \) be an almost-regular-\( v \)-min graph with \( a - 1 \leq \delta(G_0) \) and \( \Delta(G_0) \leq a \).
2. Consider the family of graphs \( G(m, h) = mK_{a+1} \cup G_0 \cup hK_a \) \((m, h \geq 0)\). Assume \( G(m, h) \) is almost-regular-\( v \)-min for all \( m, h \geq 0 \).
3. Show that for \( n(G(m, h)) = m(a + 1) + n(G_0) + ha \) sufficiently large, if \( G \in \Gamma(G(m, h)) \) is \( t \)-optimal, then \( G \) must be almost-regular-\( v \)-min.
4. Characterize almost-regular-\( v \)-min graphs in \( \Gamma(G(m, h)) \).
5. If there are several almost-regular-\( v \)-min graphs in \( \Gamma(G(m, h)) \), compare the characteristic polynomials of their Laplacian matrices to decide which ones have a \( t \)-optimal complement.

Step 3 above is justified by the following two theorems:

**Theorem 15.** Let \( G_0 \) be an almost-regular graph with \( a - 1 \leq \delta(G_0) \) and \( \Delta(G_0) \leq a \). Let \( G(m, h) = mK_{a+1} \cup G_0 \cup hK_a \) \((m, h \geq 0)\). There exists \( n_0 = n_0(G_0) \) such that for \( n(G(m, h)) = m(a + 1) + n(G_0) + ha \geq n_0 \), every \( t \)-optimal graph in \( \Gamma(G(m, h)) \) is almost-regular.

**Proof.** We show that for \( n(G(m, h)) = m(a + 1) + n(G_0) + ha \) sufficiently large, a non-almost-regular graph in \( \Gamma(G(m, h)) \) has fewer spanning trees than \( G(m, h) \).

Suppose \( G_0 \) has \( p = r + s \) nodes with \( r \) nodes of degree \( a - 1 \) and \( s \) nodes of degree \( a \). We allow \( r = 0 \) or \( s = 0 \). Assume the degree sequence \( d \) of \( G \in \Gamma(G(m, h)) \) is not almost-regular. By Lemma 10 there are sequences \( d_0, d_1, \ldots, d_l \) with \( l \geq 1 \) such that \( d = d_0 \prec d_1 \prec \cdots \prec d_{l-1} \prec d_l \) and \( d_l \) is almost-regular; \( d_l \) must be the degree sequence of \( G(m, h) \), namely \( [(a - 1)^{(r+ha)}, a^{(s+m(a+1))}] \). Since \( d_{l-1} \prec d_l \), \( d_{l-1} \) must be
one of the following sequences:

\[ s_1 = [a - 2, (a - 1)^{(r + ha - 2)}, d^{s + m(a + 1)}], \]
\[ s_2 = [(a - 1)^{(r + ha + 1)}, d^{s + m(a + 1) - 2}, a + 1], \]
\[ s_3 = [a - 2, (a - 1)^{(r + ha - 1)}, d^{s + m(a + 1) - 1}, a + 1]. \]

It is also clear that \( s_3 \prec s_1 \) and \( s_3 \prec s_2 \). We conclude that \( d \prec^* s_1 \) or \( d \prec^* s_2 \).

Let \( [\lambda_1, \lambda_2, \ldots, \lambda_p] \) be the eigenvalue sequence of \( \text{Lapl}(G_0) \). Since the eigenvalue sequence of \( \text{Lapl}(K_n) \) is \([0, n^{n-2}]\), the eigenvalue sequence of \( \text{Lapl}(G(m, n)) \) is \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_p, 0^{(m + h)}, d^{l(a - 1)}], (a + 1)^{(ma)} \). Let the functions \( f \) and \( g \) be defined as in Section 5.

**Claim.** There is an \( x_0 = x_0(r, s, a, \lambda_1, \ldots, \lambda_p) \) such that for \( x \geq x_0 \), \( f(s_i, x) < g(\lambda, x) \) for \( i = 1, 2 \).

This claim, together with Corollary 9, Definition 11, Lemma 13, and Theorem 1 yields, with \( n = p + ha + m(a + 1) \),

\[ t(\tilde{G}) \leq n^{n-2} f(d, n) \leq n^{n-2} f(s_i, n) < n^{n-2} g(\lambda, n) = t(\text{Lapl}(m, n)) \]

and for \( n \geq x_0 \), that is, for \( p + ha + m(a + 1) \geq x_0 \). This is what we wanted to show.

The proof will be complete once the above claim has been justified. To this end, we define \( u_i(x) = g(\lambda, x)/f(s_i, x) \) for \( i = 1, 2 \). We have

\[ u_1(x) = \left(1 - \frac{a - 1}{x}\right)^{-(a - 2)/(a - 1)} \left(1 - \frac{a}{x}\right)^{-(a - 1)/(r - 2)/a} \times \left(1 - \frac{a + 1}{x}\right)^{-a(s + 1)/(a + 1)} v(x), \]
\[ u_2(x) = \left(1 - \frac{a}{x}\right)^{-(a - 1)(r + 1)/a} \left(1 - \frac{a + 1}{x}\right)^{-a(s - 2)/(a + 1)} \times \left(1 - \frac{a + 2}{x}\right)^{-(a + 1)/(a + 2)} v(x), \]

where \( v(x) = \prod_{i=1}^{p} (1 - \lambda_i/x) \). The claim is equivalent to \( u_1(x) > 1 \) and \( u_2(x) > 1 \) for \( x \) sufficiently large. To show \( u_1(x) > 1 \), we take derivative of \( \log u_1(x) \) to obtain

\[ x^2(\log u_1(x))' = -\frac{a - 2}{1 - (a - 1)/x} - \frac{(a - 1)(r - 2)}{1 - a/x} - \frac{a(s + 1)}{1 - (a + 1)/x} + \sum_{i=1}^{p} \frac{\lambda_i}{1 - \lambda_i/x}. \]
Expanding all the terms on the right hand side in powers of $1/x$ (which can be done for large enough $x$), we obtain

$$(\log u_1(x))' = \sum_{j=0}^{\infty} \frac{b_j}{x^{j+2}},$$

where

$$b_j = -(a - 2)(a - 1)' - (a - 1)(r - 2)a' - a(s + 1)(a + 1)' + \sum_{i=0}^{p} \lambda_i^{j+1}.$$ 

Applying Lemma 3 to $G_0$, we obtain $\sum_{i=1}^{p} \lambda_i = r(a - 1) + sa$ and $\sum_{i=1}^{p} \lambda_i^2 = r(a - 1)a + sa(a + 1)$. A simple calculation then shows that $b_0 = 0$ and $b_1 = -2$. Thus

$$(\log u_1(x))' = -\frac{2}{x^3} + O \left( \frac{1}{x^4} \right).$$

Therefore $(\log u_1(x))' < 0$ for $x$ sufficiently large, and thus, $u_1(x)$ is strictly decreasing for $x$ sufficiently large. Since $\lim_{x \to \infty} u_1(x) = 1$, it follows that, for $x$ sufficiently large, $u_1(x) > 1$. By the same reasoning one shows that $u_2(x) > 1$ for $x$ large. Notice that how large $x$ must be depends only on $r, s, a, \lambda_1, \ldots, \lambda_p$, which in turn, depend only on $G_0$. Thus, there is $x_0 = x_0(G_0)$ such that for $x \geq x_0$, $g(\lambda, x) > f(s, x)$ for $i = 1, 2$, as claimed.  

**Theorem 16.** Let $G_0$ be an almost-regular graph with $a - 1 \leq \delta(G_0)$ and $\Delta(G_0) \leq a$. Let $G(m, h)$ be defined as in Theorem 15. Suppose $G(m, h)$ is almost-regular-$v$-min for all $m, h \geq 0$. Then there exists $n_0 = n_0(G_0)$ such that, for $n(G(m, h)) \geq n_0$, if the complement of $G \in \Gamma_d(G(m, h))$ is $t$-optimal in $\Gamma_d(G(m, h))$, then $v(G) = v(G(m, h))$.

**Proof.** Let $v_0 = v(G_0) = v(G(m, h))$. Assume $G(m, h)$ is almost-regular-$v$-min for all $m, h \geq 0$. We show that for $n(G(m, h))$ sufficiently large, $G(m, h)$ has more spanning trees than any other graph $\tilde{G}$ such that $G \in \Gamma_d(G(m, h))$ and $v(G) > v_0$. The proof technique is similar to the one used to prove Theorem 15.

Suppose $G_0$ has $p = r + s$ nodes, $r$ nodes of degree $a - 1$, and $s$ nodes of degree $a$ ($r = 0$ or $s = 0$ are allowed). Then $d = [(a - 1)(r + ha), d^{(s+m(a+1))}]$ is the degree sequence common to all graphs in $\Gamma_d(G(m, h))$. Clearly, $G(m, h)$ has eigenvalue sequence $\lambda = [g(m+h), d^{(s+1)}, \ldots, \lambda_p]$, where $[\lambda_1, \ldots, \lambda_p]$ is the eigenvalue sequence of $G_0$. Let $\hat{f}(d, v_0, x) = e^{-2(v_0+3)/3x^3} f(d, x)$ where $f(d, x)$ is given by Definition 11, and $g(\lambda, x)$ is given by Definition 12. We have

$$\hat{f}(d, v_0, x) = e^{-2(v_0+3)/3x^3} \left( 1 - \frac{a}{x} \right)^{(r+ha)(a-1)/a} \left( 1 - \frac{a + 1}{x} \right)^{(s+m(a+1))a/(a+1)}, \quad (8)$$

$$g(\lambda, x) = \left( 1 - \frac{a}{x} \right)^{h(a-1)} \left( 1 - \frac{a + 1}{x} \right)^{ma} \prod_{i=1}^{p} \left( 1 - \frac{\lambda_i}{x} \right). \quad (9)$$
We claim that there is an \( x_0 = x_0(G_0) \) such that \( f(d, v_0, x) < g(\lambda, x) \) for \( x \geq x_0 \). Let us define
\[
    w(v_0, x) = \frac{g(\lambda, x)}{f(d, v_0, x)}.
\]
We have
\[
    w(v_0, x) = e^{2(v_0+3)/(3x^3)} \left( 1 - \frac{a}{x} \right)^{-(a-1)/a} \left( 1 - \frac{a + 1}{x} \right)^{-sa/(a+1)} \prod_{i=1}^{p} \left( 1 - \frac{\lambda_i}{x} \right)
\]
and
\[
    \left( \log w(v_0, x) \right)' = \frac{-2(v_0 + 3)}{x^4} - \frac{r(a-1)}{x^2} \frac{1}{1 - a/x} \frac{1}{x^2} \frac{1}{1 - (a+1)/x} + \sum_{i=1}^{p} \frac{\lambda_i}{x^2} \frac{1}{1 - \lambda_i/x}.
\]
Expanding all the terms on the right hand side in powers of \( 1/x \), which is valid for \( x \) sufficiently large, we get
\[
    \left( \log w(v_0, x) \right)' = \sum_{j=0}^{\infty} b_j \frac{1}{x^{j+2}},
\]
where
\[
    b_j = -r(a-1)a^j - sa(a+1)^j + \sum_{i=1}^{p} \lambda_i^{j+1} \quad \text{for } j \neq 2
\]
and
\[
    b_2 = -r(a-1)a^2 - sa(a+1)^2 + \sum_{i=1}^{p} \lambda_i^3 - 2(v_0 + 3). \]
Taking into account \( \sum_{i=1}^{p} \lambda_i = r(a-1) + sa \), \( \sum_{i=1}^{p} \lambda_i^2 = r(a-1)a + sa(a+1) \), and \( \sum_{i=1}^{p} \lambda_i^3 = r(a-1)a^2 + sa(a+1)^2 + 2v_0 \) (from Lemmas 3 and 6), we obtain \( b_0 = b_1 = 0 \) and \( b_2 = -6 \). Thus
\[
    \left( \log w(v_0, x) \right)' = \frac{-6}{x^4} + O \left( \frac{1}{x^3} \right).
\]
Therefore, \( w(v_0, x) \) is a decreasing function of \( x \) for \( x \) sufficiently large, and since \( \lim_{x \to \infty} w(v_0, x) = 1 \), we conclude that there is an \( x_0 = x_0(G_0) \) such that, for \( x \geq x_0 \), \( w(v_0, x) > 1 \), or \( g(\lambda, x) > f(d, v_0, x) \), as claimed.

Now let \( G \in \mathcal{T}_\lambda(G(m, h)) \) be almost-regular and such that \( v(G) > v_0 = v(G(m, h)) = v(G_0) \). Then \( v(G) \geq v_0 = v_0 + 3 \) from Lemma 4. Therefore, by Corollary 9,
\[
    \tau(G) \leq n^{n-2} e^{-2(g(G)/(3n^3)} f(d, n) \leq n^{n-2} \hat{f}(d, v_0, n) < n^{n-2} g(\lambda, n) = \tau(G(m, h))
\]
will hold for \( n = n(G(m, h)) \geq x_0 \).

Theorems 15 and 16 deal with a two-parametric family of graphs \( G(m, h) = mK_{a+1} \cup G_0 \cup hK_2 \). Analogous results hold for the one-parametric family \( G(m) = mK_{a+1} \cup G_0 \), \( m \geq 0 \). The proofs are identical except that we set \( h = 0 \) throughout. Thus we have:
Theorem 17. Let $G_0$ be an almost-regular graph with $a - 1 \leq \delta(G_0)$ and $A(G_0) \leq a$. Let $G(m) = mK_{a+1} \cup G_0$ ($m \geq 0$). There exists $n_0 = n_0(G_0)$ such that for $n(G(m)) = m(a + 1) + n(G_0) \geq n_0$, every $t$-optimal graph in $\Gamma(G(m))$ is almost-regular.

Theorem 18. Let $G_0$ be an almost-regular graph with $a - 1 \leq \delta(G_0)$ and $A(G_0) \leq a$. Let $G(m)$ be defined as in Theorem 17. Suppose $G(m)$ is almost-regular-$v$-min for all $m \geq 0$. Then there exists $n_0 = n_0(G_0)$ such that, for $n(G(m)) \geq n_0$, if the complement of $G \in \Gamma_A(G(m))$ is $t$-optimal in $\Gamma_A(G(m))$, then $v(G) = v(G(m))$.

The significance of Theorems 15–18 lies in the fact that they reduce the problem of characterizing $t$-optimal graphs in classes of the form $\Gamma(G(m,h))$ or $\Gamma(G(m))$ to that of characterizing almost-regular-$v$-min graphs in classes of the form $\Gamma(G(m,h))$ with $G(m,h) = mK_{a+1} \cup G_0 \cup hK_a$ or $\Gamma(G(m))$ with $G(m) = mK_{a+1} \cup G_0$ satisfying the hypotheses of the theorems. Taken together, the theorems assert the existence of $n_0 = n_0(G_0)$ such that, for $n(G(m,h)) \geq n_0$ (resp. $n(G(m)) \geq n_0$) a $t$-optimal graph in $\Gamma(G(m))$ (resp. $\Gamma(G(m)))$ must be almost-regular, and its complement must be $v$-min among the almost-regular graphs in $\Gamma(G(m,h))$ (resp. $\Gamma(G(m)))$. It is clear from the proof of Theorem 15 (where $u_1(x)$ and $u_2(x)$ are defined) and from the proof of Theorem 16 (where $w(v(x))$ is defined) that it is sufficient to take $n_0$ such that, for $x \geq n_0$, the following inequalities hold:

\begin{align*}
  &u_1(x) > 1, \quad (12) \\
  &u_2(x) > 1, \quad (13) \\
  &w(v_0,x) > 1. \quad (14)
\end{align*}

Thus it would seem that, in order to compute $n_0$, it should be necessary to solve inequalities (12)–(14) separately. In fact, for $x$ large enough, (14) implies (12) and (13), as the next lemma shows.

Lemma 19. For $x > \max(4 + v_0, a + 1)$, $u_1(x) > w(v_0,x)$. For $x > \max(4 + v_0, a + 2)$, $u_2(x) > w(v_0,x)$.

Proof. Let

\[ k(a,x) = e^{-2(x+3)/(3x)} \left( 1 - \frac{a - 1}{x} \right)^{-(a-2)/(a-1)} \]

\[ \times \left( 1 - \frac{a}{x} \right)^{2(a-1)/a} \left( 1 - \frac{a + 1}{x} \right)^{-a/(a+1)}. \]

A simple computation shows that $u_1(x)/w(v_0,x) = k(a,x)$ and $u_2(x)/w(v_0,x) = k(a+1,x)$. We show $k(a,x) > 1$ for $x > \max(4 + v_0, a + 1)$. Indeed, taking log, and differentiating,
one obtains

\[
(\log k(a,x))'
= -2x^4-(4+v_0)x^3+3a(v_0+3)x^2-(3a^2-1)(v+3)x+a(a^2-1)(v_0+3)
\]
\[
x^4(x-a)(x-a+1)(x-a-1).
\]

Thus \(k(a,x) > 1\) for \(x > \max(4+v_0,a+1)\), since \(k(a,x)\) is strictly decreasing in that range and \(k(a,x) \to 1\) when \(x \to \infty\).

From Theorems 15, 16, and Lemma 19, we obtain the following:

**Corollary 20.** Let \(G_0\) be an almost-regular graph with \(a-1 \leq \delta(G_0)\) and \(\Delta(G_0) \leq a\). Suppose \(G(m,h) = mK_{a+1} \cup G_0 \cup hK_a\) is almost-regular-v-min for all \(m,h \geq 0\). Let \(v_0 = v(G_0)\), and let \(n_0\) be such that \(n_0 > \max(4+v_0,a+2)\) and \(w(v_0,x) > 1\) for \(x \geq n_0\). For \(n(G(m,h)) \geq n_0\), if \(\tilde{G} \in \Gamma(G(m,h))\) is t-optimal, then \(G\) is almost-regular-v-min.

An analogous corollary follows from Theorems 17, 18, and Lemma 19:

**Corollary 21.** Let \(G_0\) be an almost-regular graph with \(a-1 \leq \delta(G_0)\) and \(\Delta(G_0) \leq a\). Suppose \(G(m) = mK_{a+1} \cup G_0\) is almost-regular-v-min for all \(m \geq 0\). Let \(v_0 = v(G_0)\), and let \(n_0\) be such that \(n_0 > \max(4+v_0,a+2)\) and \(w(v_0,x) > 1\) for \(x \geq n_0\). For \(n(G(m)) \geq n_0\), if \(\tilde{G} \in \Gamma(G(m))\) is t-optimal, then \(G\) is almost-regular-v-min.

In the next two sections we use the results in this section to characterize t-optimal graphs in \(\Gamma(n,e)\) for \(n \leq e \leq 3n/2\) and \(n\) sufficiently large. In Section 8 we show that there is a finite family of graphs \(G_j\) with \(2 \leq \delta(G_j)\) and \(\Delta(G_j) \leq 3\) such that for \(n \leq e \leq 3n/2\) the class \(\Gamma(n,e)\) coincides with one of the classes \(\Gamma(mK_4 \cup G_j \cup hK_3)\). We then characterize almost-regular-v-min graphs in \(\Gamma(mK_4 \cup G_j \cup hK_3)\), motivated by Theorems 15–18, which state that the complements of t-optimal graphs in \(\Gamma(mK_4 \cup G_j \cup hK_3)\) must be almost-regular-v-min when \(n \geq nj\). In Section 9 we determine the value \(nj\) for each of the classes \(\Gamma(mK_4 \cup G_j \cup hK_3)\) using Corollaries 20 and 21. We also resolve ties among competing almost-regular-v-min graphs in the same class. Letting \(n_0 = \max[nj]\), we will have a complete characterization of t-optimal graphs in \(\Gamma(n,e)\) for \(n \leq e \leq 3n/2\) and \(n \geq n_0\).

**8. Characterization of almost-regular-v-min graphs for \(n \leq e \leq 3n/2\)**

By Theorems 15–18, for large \(n(G(m,h))\) or large \(n(G(m))\), a necessary condition for \(\tilde{G} \in \Gamma(G(m,h))\) or \(\tilde{G} \in \Gamma(G(m))\) to be t-optimal is that \(G\) be almost-regular-v-min. We now proceed to characterize almost-regular-v-min graphs in \(\Gamma(n,e)\) for \(n \leq e \leq 3n/2\). Lemma 23 shows that for some \(m,h \geq 0\), \(\Gamma(n,e)\) must equal \(\Gamma(G_j(m,h))\) or \(\Gamma(G_j(m))\), where \(G_j(m,h)\) and \(G_j(m)\) are given by the following definition:
Definition 22. (1) \( G_0(m, h) = mK_4 \cup hK_3 \),
(2) \( G_1(m, h) = mK_4 \cup F_1 \cup hK_3 \),
(3) \( G_2(m, h) = mK_4 \cup F_2 \cup hK_3 \),
(4) \( G_3(m, h) = mK_4 \cup C_4 \cup hK_3 \),
(5) \( G_4(m, h) = mK_4 \cup S \cup hK_3 \),
(6) \( G_5(m, h) = mK_4 \cup C_5 \cup hK_3 \),
(7) \( G_6(m) = mK_4 \cup Q_1 \),
(8) \( G_7(m) = mK_4 \cup Q_2 \),
(9) \( G_8(m) = mK_4 \cup W_1 \),
(10) \( G_9(m) = mK_4 \cup W_2 \),
(11) \( G_{10}(m) = mK_4 \cup H_2 \),
where the graphs \( K_3, C_4, S, K_4, C_5, Q_1, Q_2, F_1, W_1, W_2, F_2, H_2 \) are shown in Fig. 1.

Lemma 23. If \( n \leq e \leq 3n/2 \) then \( \Gamma(n, e) = \Gamma(G_j(m, h)) \) or \( \Gamma(n, e) = \Gamma(G_j(m)) \) for some \( j, m, h \), where \( G_j(m, h) \) and \( G_j(m) \) are given by Definition 22.
Proof. Given \( n \) and \( e \) with \( n \leq e \leq 3n/2 \), pick an almost-regular degree sequence \( d \) corresponding to an almost-regular graph on \( n \) nodes and \( e \) edges. Assume \( d \) has \( 2i \) entries equal to three and \( r = n - 2i \) entries equal to two where \( 0 \leq i \leq n/2 \). We have two cases:

Case 1: \( i \) is even. If \( r \geq 3 \), we can realize the degree sequence \( d \) as follows: First, since \( i \) is even, we can use \( m = i/2 \) \( K_4 \)'s to account for all three degrees. Now let \( h \) and \( r' \) denote the quotient and remainder of \( r - 3 \) divided by 3. We have \( r = 3h + r' + 3 \), with \( h \geq 0 \) and \( r' + 3 = 3, 4, \) or 5. Thus, we can use \( hK_3 \)'s plus another \( K_3 \), or \( C_4 \), or \( C_5 \) to account for all two degrees. This proves that \( \Gamma(n,e) = \Gamma(G_0(m,h+1)) \) or \( \Gamma(G_3(m,h)) \) or \( \Gamma(G_5(m,h)) \).

If, on the other hand, \( r < 3 \), to realize \( d \) we use \( m = (i-2)/2 \) \( K_4 \)'s to account for all but four 3-degrees. Depending on whether \( r = 0, 1, \) or 2 we use an additional \( K_4 \), \( Q_2 \), or \( W_1 \) to account for the four 3-degrees and the \( r \) 2-degrees. Thus \( \Gamma(n,e) = \Gamma(G_0(m+1,0)) \) or \( \Gamma(G_3(m)) \) or \( \Gamma(G_5(m)) \).

Case 2: \( i \) is odd. If \( r \geq 4 \), \( d \) is realized as follows: Start with \( m = (i-1)/2 \) \( K_4 \)'s to account for all but two 3-degrees. Let \( h \) and \( r' \) be the quotient and remainder of \( r - 4 \) divided by 3. Since \( r = 4h + r' + 4 \) with \( h \geq 0 \) and \( r' + 4 = 4, 5, \) or 6, we can add \( h \) \( K_3 \)'s plus \( F_1 \), or \( S \cup K_3 \) or \( F_2 \) to take care of all the 2-degrees plus the two 3-degrees that are left. Thus \( \Gamma(n,e) = \Gamma(G_1(m,h)) \) or \( \Gamma(G_4(m,h+1)) \) or \( \Gamma(G_5(m,h)) \).

When \( 2 \leq r < 4 \), we realize \( d \) using \( m = (i-1)/2 \) \( K_4 \)'s, leaving two 3-degrees unaccounted for. Then, depending on whether \( r \) is 2 or 3, we add \( S \) or \( Q_1 \) to account for the 2-degrees. Thus \( \Gamma(n,e) = \Gamma(G_3(m,0)) \) or \( \Gamma(G_6(m)) \).

Finally, when \( r < 2 \), we must have \( i \geq 3 \) (assuming \( n \geq 6 \)). In this case we let \( m = (i-3)/2 \). Then, depending on whether \( r = 0 \) or 1, we realize \( d \) with \( mK_4 \cup H_2 \) or \( mK_4 \cup W_2 \). Therefore \( \Gamma(n,e) = \Gamma(G_{10}(m)) \) or \( \Gamma(G_9(m)) \). □

Lemma 23 reduces the characterization of almost-regular-\( v \)-min graphs for \( n \leq e \leq 3n/2 \) to the characterization of almost-regular-\( v \)-min in \( \Gamma(G_j(m,h)) \) for \( 0 \leq j \leq 5 \) and in \( \Gamma(G_j(m)) \) for \( 6 \leq j \leq 10 \), which we now undertake.

First we dispose of the case \( j = 0 \); since \( v(G_0(m,h)) = 0 \), an almost-regular-\( v \)-min graph \( G' \) in \( \Gamma(G_0(m,h)) \) must have \( v(G') = 0 \), which implies \( G' \) must be an almost-regular disjoint union of cliques. Thus \( G' \) must equal \( G_0(m,h) \).

For the remaining cases, we will provide a characterization of almost-regular-\( \tau \)-max graphs. By Lemma 4, this is equivalent to characterizing almost-regular-\( v \)-min graphs.

The following lemma will be used repeatedly:

**Lemma 24.** Let \( G \) be a graph without cliques of order 4, such that for each \( v \in V(G) \), \( 2 \leq \text{degree}(v) \leq 3 \). We have

(a) Any two triangles of \( G \) that share a vertex also share an edge. (We say that any such triangles are adjacent.)

(b) Let \( p \) denote the number of pairs of adjacent triangles in \( G \), and let \( k \) denote the number of degree-3 nodes of \( G \). Then \( 3\tau(G) - n(G) \leq 2p \leq k \).
Proof. Part (a): Immediate.

Part (b): Each pair of adjacent triangles contributes four vertices to $G$, and the remaining triangles contribute three vertices each. Therefore $n(G) \geq 4p + 3(\tau(G) - 2p)$. Hence $2p \geq 3(\tau(G) - n(G))$. Also, each pair of adjacent triangles contributes at least two degree-3 points to $G$. Therefore $2p \leq k$. \hfill \qed

Lemma 25. $G \in \Gamma_A(G_1(m,h))$ is almost-regular-$\tau$-max iff $G = G_1(m,h)$.

Proof. By induction on $m$.

Basis: Let $m = 0$, and suppose that $G$ is $\tau$-max in $\Gamma_A(G_1(0,h))$. Then $\tau(G) \geq \tau(G_1(0,h)) = h + 2$. Also, $n(G) = 3h + 6$. Since $G$ has exactly two degree-3 nodes, it follows from Lemma 24(b) that $0 \leq 2p \leq 2$. Thus $p = 0$ or $1$. If $p = 1$, $G = S \cup hK_3 \cup G'$ where $n(G') = 2$ and $e(G') = 2$, which is impossible, since $G'$ must be a simple graph. Therefore we must have $p = 0$; since $\tau(G) \geq h + 2$, this implies that $(h + 2)K_3$ is a spanning subgraph of $G$ with one edge less that $G$. Therefore $G = G_1(0,h)$.

Inductive step: Suppose that $m \geq 1$ and that the statement is true for $m - 1$. Let $G \in \Gamma_A(G_1(m,h))$ be almost-regular-$\tau$-max. First, we show that $G$ must contain a $K_4$. Suppose this is not the case. Since $G$ is assumed to be $\tau$-max in $\Gamma_A(G_1(m,h))$, we must have $\tau(G) \geq \tau(G_1(m,h)) = 4m + h + 2$. Let $p$ be the number of pairs of adjacent triangles in $G$. Since $n(G) = 4m + 3h + 6$, and since $G$ has $4m + 2$ nodes of degree 3, we have $8m \leq 2p \leq 4m + 2$ by Lemma 24(b). This contradicts $m \geq 1$. Thus $G$ contains a $K_4$, and we must have $G = K_4 \cup G'$ where $G'$ is $\tau$-max in $\Gamma_A(G_1(m,h))$. By the inductive hypothesis, $G' = G_1(m - 1,h)$, thus $G = G_1(m,h)$. \hfill \qed

Lemma 26. Let $G \in \Gamma_A(G_2(m,h))$.

1. If $m = 0$, $G$ is almost-regular-$\tau$-max iff $G = G_2(0,h)$ or $G = Q_4 \cup (h + 1)K_3$ or $G = S \cup C_4 \cup hK_3$.

2. If $m \geq 1$, $G$ is almost-regular-$\tau$-max iff $G = G_2(m,h)$ or $G = mK_4 \cup Q_4 \cup (h + 1)K_3$ or $G = (m - 1)K_4 \cup 3S \cup hK_3$ or $G = mK_4 \cup S \cup C_4 \cup hK_3$.

Proof. First, we make the general observation that if $G \in \Gamma_A(G_2(m,h))$ is almost-regular-$\tau$-max and $G$ does not contain a $K_4$, then the following inequalities hold by Lemma 24(b):

\[2p \geq 3(\tau(G) - n(G)) \geq 3(4m + h + 2) - (4m + 3h + 8) = 8m - 2, \quad (15)\]

\[2p \leq 4m + 2. \quad (16)\]

Thus

\[4m + 2 \geq 2p \geq 8m - 2. \quad (17)\]

First we consider the case $m = 0$. If $G \in \Gamma_A(G_2(m,h))$ is almost-regular-$\tau$-max then $G$ is $K_4$-free (since it contains only two degree-3 nodes). Thus, by inequality (17), we must have $p = 0$ or $1$. If $p = 0$, $G$ contains a subgraph $G' = (h + 2)K_3$. It can be
Lemma 27. Let \( G \in \Gamma_d(G_2(m,h)) \) is almost-
regular-\( \tau \)-max. If \( G \) is \( K_4 \)-free then \( p = 3 \) by (17). It is easily seen that this forces \( G = 3S \cup hK_3 \). Otherwise \( G \) contains a \( K_4 \), and \( G \in \Gamma_d(G_2(0,h)) \) must be almost-regular-\( \tau \)-max. Therefore \( G' = F_2 \cup hK_3 \) or \( G' = Q_1 \cup (h + 1)K_3 \) or \( G' = S \cup C_4 \cup hK_3 \). Therefore \( G \) is of the stated form. 

(2) Consider next the case \( m = 1 \), and suppose that \( G \in \Gamma_d(G_2(m,h)) \) is almost-
regular-\( \tau \)-max. If \( G \) is \( K_4 \)-free then \( p = 3 \) by (17). It is easily seen that this forces \( G = 3S \cup hK_3 \). Otherwise \( G \) contains a \( K_4 \), and \( G \in \Gamma_d(G_2(0,h)) \) must be almost-regular-\( \tau \)-max. Therefore \( G' = F_2 \cup hK_3 \) or \( G' = Q_1 \cup (h + 1)K_3 \) or \( G' = S \cup C_4 \cup hK_3 \). Therefore \( G \) is of the stated form. 

(3) When \( m \geq 2 \), (17) cannot occur. Therefore, a almost-regular-\( \tau \)-max \( G \in \Gamma_d(G_2(m,h)) \) must contain a \( K_4 \). Thus \( G = K_4 \cup G' \) with \( G' \in \Gamma_d(G_2(m-1,h)) \). By induction, \( G' = (m-2)K_4 \cup 3S \cup hK_3 \) or \( G' = (m-1)K_4 \cup S \cup hK_3 \) or \( G' = (m-1)K_4 \cup Q_1 \cup (h + 1)K_3 \) or \( G' = (m-1)K_4 \cup S \cup C_4 \cup hK_3 \). Thus \( G \) is of the required form. \( \square \)

The following lemmas are proven in a similar fashion. We omit the proofs.

Lemma 28. \( G \in \Gamma_d(G_8(m)) \) is almost-regular-\( \tau \)-max if and only if \( G = G_8(m) \) or \( G = mK_4 \cup H_1 \).

Lemma 29. For \( 4 \leq j \leq 10 \), \( j \neq 8 \), \( G \in \Gamma_d(G_j(m,h)) \) is almost-regular-\( \tau \)-max if and only if \( G = G_j(m,h) \).

9. Characterization of \( t \)-optimal graphs in \( \Gamma(n,e) \) when \( n \leq e \leq 3n/2 \)

(Throughout this section \( G_j(m,h) \) and \( G_j(m) \) are given by Definition 22.)

We now determine for each \( j \) an integer \( n_j \) such that if \( \tilde{G} \in \Gamma(\tilde{G}_j(m,h)) \) or \( \tilde{G} \in \Gamma(\tilde{G}_j(m)) \) is \( t \)-optimal then \( G \) is almost-regular-\( t \)-optimal whenever \( n(G_j(m,h)) \geq n_j \) or \( n(G_j(m)) \geq n_j \). By Corollaries 20 and 21, it is sufficient to find \( n_j \geq \max(v_0 + 4, a + 2) \) such that \( (\log w(v_0, x))' < 0 \) for \( x \geq n_j \), where \( v_0 \) is the number of induced paths of length two corresponding to an almost-regular-\( v \)-min graph in \( \Gamma(G_j(m,h)) \) or in \( \Gamma(G_j(m)) \). In the cases under consideration, we have \( a = 3 \).

In the proof of Theorem 16, the function \( (\log w(v_0, x))' \) corresponding to a graph \( mK_{n+1} \cup G_0 \cup hK_a \) was expressed in terms of the eigenvalues of Lapl\( G_0 \); it can also
be expressed as
\[ (\log w(v_0, x))' = -\frac{2(v_0 + 3)}{x^4} - \frac{r(a - 1)}{x^2 - ax} - \frac{sa}{x^2 - (a + 1)x} - \frac{p}{x} + \frac{C'(G_0, x)}{C(G_0, x)}, \]
where \( p = n(G_0) \), \( C(G_0, x) \) is the characteristic polynomial of \( \text{Lapl}(G_0) \), and \( C'(G_0, x) \) is its derivative with respect to \( x \). Thus, we can avoid the explicit computation of the eigenvalues.

We now list \( (\log w(v_0, x))' \) for each \( G_j(m, h) \) (1 \( \leq j \leq 5 \)) and for each \( G_j(m, h) \) (6 \( \leq j \leq 10 \)), together with an integer \( n_j \) such that \( n_j \geq \max(v_0 + 4, a + 2) \) and \( (\log w(v_0, x))' < 0 \):

1. **Case** \( G_1(m, h) = mK_4 \cup F_1 \cup hK_3 \) (\( v_0 = 4 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^4 - 74x^3 + 343x^2 - 518x + 168}{(x - 4)(x - 3)(x^2 - 5x + 2)x^4} < 0 \text{ for } x \geq n_1 = 25. \]

2. **Case** \( G_2(m, h) = mK_4 \cup F_2 \cup hK_3 \) (\( v_0 = 6 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^4 - 19x^3 + 32x^2 - 82x + 4}{(x - 4)(x - 3)(x^2 - 8x + 19x^2 - 14x + 2)x^4} < 0 \text{ for } x \geq n_2 = 33. \]

3. **Case** \( G_3(m, h) = mK_4 \cup C_4 \cup hK_3 \) (\( v_0 = 4 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^3 - 63x^2 + 182x - 168}{(x - 3)(x^2 - 6x + 8)x^4} < 0 \text{ for } x \geq n_3 = 21. \]

4. **Case** \( G_4(m, h) = mK_4 \cup S \cup hK_3 \) (\( v_0 = 2 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^3 - 45x^2 + 130x - 120}{(x - 4)(x - 3)(x - 2)x^4} < 0 \text{ for } x \geq n_4 = 15. \]

5. **Case** \( G_5(m, h) = mK_4 \cup C_5 \cup hK_3 \) (\( v_0 = 5 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^3 - 64x^2 + 160x - 120}{(x - 3)(x^2 - 5x + 5)x^4} < 0 \text{ for } x \geq n_5 = 22. \]

6. **Case** \( G_6(m) = mK_4 \cup Q_1 \) (\( v_0 = 6 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^6 - 115x^5 + 1153x^4 - 5154x^3 + 11673x^2 - 13185x + 5940}{(x - 3)(x - 4)(x^2 - 5x + 5)(x^2 - 7x + 11)x^4} < 0 \text{ for } x \geq n_6 = 39. \]

7. **Case** \( G_7(m) = mK_4 \cup Q_2 \) (\( v_0 = 7 \)):
\[ (\log w(v_0, x))' = -2 \frac{3x^4 - 120x^3 + 710x^2 - 1540x + 1200}{(x - 4)(x - 3)(x^2 - 7x + 10)x^4} < 0 \text{ for } x \geq n_7 = 40. \]
Thus each one almost-regular-$t$-optimal complement.

In Lemmas 25–29 we have characterized almost-regular-$v$-min graphs in the classes $\Gamma(G_j(m, h))$ and $\Gamma(G_j(m))$. In all these classes, with the exception of $\Gamma(G_2(m, h))$, $\Gamma(G_3(m, h))$, and $\Gamma(G_8(m))$, $G_j(m, h)$ and $G_j(m)$ are the unique almost-regular-$v$-min graphs, whose complements will therefore be the unique $t$-optimal graphs in $\Gamma(G_j(m, h))$ and in $\Gamma(G_j(m))$ when the number of nodes is at least $n_j$.

In each of the classes $\Gamma(G_2(m, h))$, $\Gamma(G_3(m, h))$, and $\Gamma(G_8(m))$ there is more than one almost-regular-$v$-min graph. By Theorem 1, it suffices to compare the characteristic polynomials of the Laplacian matrices of the $v$-min graphs in a given class to decide which ones have a $t$-optimal complement.

In $\Gamma(G_2(m, h))$ there are four $v$-min graphs for $m \geq 1$: $G_2(m, h) = mK_4 \cup F_2 \cup hK_3$, $G_2''(m, h) = mK_4 \cup Q_1 \cup (h + 1)K_3$, $G_2''(m, h) = mK_4 \cup S \cup C_4 \cup hK_3$, and $G_2''(m, h) = (m - 1)K_4 \cup 3S \cup hK_3$. For $m = 0$ there are only three $v$-min graphs, namely, $G_2(0, h)$, $G_2'(0, h)$, $G_2''(0, h)$. We have already shown that $t(G_2(m, h)) < t(G)$ for any $G \in \Gamma(G_2(m, h))$ that is not almost-regular-$v$-min if $n \geq n_2 = 33$. We now prove the $t$-optimality of $G_2(m, h)$ in $\Gamma(G_2(m, h))$ by showing that $C(G_2(m, h), x) > C(G_2'(m, h), x)$, $C(G_2''(m, h), x)$ for $x$ large enough. For $m \geq 1$, $C(G_2''(m, h), x) = C(G_2''(m, h), x)$. Thus $C(G_2(m, h), x)$ only needs to be compared with $C(G_2'(m, h), x)$ and $C(G_2''(m, h), x)$, and the characteristic polynomial of the Laplacian matrix of a disjoint union of graphs is the product of the characteristic polynomials of the Laplacian matrices of the components, it suffices to compare $C(F_2, x)$ against $C(Q_1 \cup K_3, x)$, and $C(S \cup C_4, x)$. A computation shows:

$$C(F_2, x) - C(Q_1 \cup K_3, x) = 3x^4 - 26x^3 + 75x^2 - 72x,$$

$$C(F_2, x) - C(S \cup C_4, x) = x^4 - 14x^3 + 58x^2 - 72x.$$
In $\Gamma(G_3(m,h))$ there are two $v$-min graphs for $m \geq 1$: $G_3(m,h) = mK_3 \cup C_4 \cup hK_3$ and $G'_3(m,h) = (m-1)K_4 \cup 2S \cup hK_3$. A simple calculation shows that $C(G_3(m,h),x) = C(G'_3(m,h),x)$. Therefore $t(G_3(m,h)) = t(G'_3(m,h))$, by Theorem 1. Thus for $m \geq 1$, $G_3(m,h)$ and $G'_3(m,h)$ are the only $t$-optimal graphs in $\Gamma(G_3(m,h))$ for $n(G_3(m,h)) \geq n_3 = 21$. When $m = 0$, only $G_3(m,h)$ is $t$-optimal.

Finally, in $\Gamma(G_5(m))$ there are two $v$-min graphs, namely, $G_5(m) = mK_4 \cup W_1$ and $G'_5(m) = mK_4 \cup H_1$, and $C(W_1,x) - C(H_1,x) = 3x^2 - 12x > 0$ for $x > 4$. This proves that $G_5(m)$ is the unique $t$-optimal graph in $\Gamma(G_5(m))$ for $n(G_5(m)) \geq n_5 = 41$.

Thus, for $n_0 = \max[n_j]_{1 \leq j \leq 10} = 60$, we have obtained a complete characterization of $t$-optimal graphs in $\Gamma(n,e)$ for $n_0 \leq n \leq e \leq 3n/2$.

We close this section with a theorem summarizing the characterization of $t$-optimal graphs in $\Gamma(n,e)$ for $n_0 = 60 \leq n \leq e \leq 3n/2$ in terms of the parameters $n$ and $e$, rather than in terms of the classes $\Gamma_j(G(m,h))$ and $\Gamma_j(G(m))$, via the proof of Lemma 23:

**Theorem 30.** Let $n \leq e \leq 3n/2$ with $n \geq 60$. Let $i = e - n$, $r = 3n - 2e$.

1. If $i$ is even, $r \geq 3$, then letting $m = i/2$, $h = \lfloor (r-3)/3 \rfloor$, and $r' = r - 3 \mod 3$, the graphs in $\Gamma(n,e)$ with $t$-optimal complement are:
   (a) $G_0(m,h+1)$ when $r' = 0$;
   (b) $G_3(m,h)$ when $r' = 1$;
   (c) $G_5(m,h)$ when $r' = 2$.

2. If $i$ is even, $r < 3$, then letting $m = (i-2)/2$, the graphs in $\Gamma(n,e)$ with $t$-optimal complement are:
   (a) $G_0(m+1,0)$ when $r = 0$;
   (b) $G_7(m)$ when $r = 1$;
   (c) $G_9(m)$ when $r = 2$.

3. If $i$ is odd, $r \geq 4$, then letting $m = (i-1)/2$, $h = \lfloor (r-4)/3 \rfloor$, and $r' = r - 4 \mod 3$, the graphs in $\Gamma(n,e)$ with $t$-optimal complement are:
   (a) $G_1(m,h)$ when $r' = 0$;
   (b) $G_4(m,h+1)$ and, for $m \geq 1$, $(m-1)K_4 \cup 2S \cup hK_3$, when $r' = 1$;
   (c) $G_2(m,h)$ when $r' = 2$.

4. If $i$ is odd, $2 \leq r < 4$, then letting $m = (i-1)/2$, the graphs in $\Gamma(n,e)$ with $t$-optimal complement are:
   (a) $G_4(m,0)$ when $r = 2$;
   (b) $G_6(m)$ when $r = 3$.

5. Finally, if $i$ is odd, $r < 2$, then letting $m = (i-3)/2$, the graphs in $\Gamma(n,e)$ with $t$-optimal complement are:
   (a) $G_{10}(m)$ when $r = 0$;
   (b) $G_8(m)$ when $r = 1$.

**Proof.** If $n \leq e \leq 3n/2$ then an almost-regular graph on $n$ nodes and $e$ edges has degree sequence $[2^r,3^{2i}]$ where $i = e - n$ and $r = 3e - 2n$. The theorem follows from the proof.
of Lemma 23 and the characterization of \( t \)-optimal graphs in the classes \( \overline{T_j(G(m,h))} \) and \( \overline{T(G_j(m))} \).

\[ \square \]

10. Conclusion

Theorems 15–18 shed some light on the characterization of \( t \)-optimal graphs in classes of the form \( \Gamma(mK_{a+1} \cup G_0 \cup hK_a) \), with \( a-1 \leq \delta(G_0) \) and \( \Delta(G_0) \leq a \), when the number of nodes is sufficiently large: the complements of \( t \)-optimal graphs in those classes must be almost-regular-\( v \)-min. As a consequence, the characterization of \( t \)-optimal graphs in the classes in question is reduced to the characterization of almost-regular-\( v \)-min graphs, or equivalently (by Lemma 4), to the characterization of almost-regular-\( \tau \)-max graphs. In \( \Gamma(n,e) \) with \( n \leq e \leq 3n/2 \), we were able to characterize almost-regular-\( \tau \)-max graphs with relative ease due to the fact that for \( G \in \Gamma_d(n,e) \) the subgraph of \( G \) consisting of all the vertices and edges lying in triangles of \( G \) has a particularly simple structure: it is a disjoint union of \( K_3 \)'s and \( K_4' \)'s, where \( K_4' \) is obtained from \( K_4 \) by the removal of a single edge.

We end with some conjectures regarding the structure of almost-regular-\( v \)-min graphs and \( t \)-optimal graphs. We believe the following to be true:

**Conjecture 1.** Let \( G_0 \) be a regular graph with degree of regularity \( a \), such that \( G_0 \) is regular-\( v \)-min (that is, \( v \)-min among all regular graphs in \( \Gamma(G_0) \)). Then \( G_0 \cup mK_{a+1} \) is regular-\( v \)-min for all \( m \geq 0 \). Conversely, if \( G' \in \Gamma(G_0 \cup mK_{a+1}) \) is regular-\( v \)-min, then there exists \( G'_0 \in \Gamma(G_0) \) such that \( G' = G'_0 \cup mK_{a+1} \).

Let us define an ordering among the graphs in \( \Gamma(n,e) \) as follows: if \( H = \text{Lapl}(G) \), let \( \gamma(G) = [\text{tr}(H^2), \text{tr}(H^3), \ldots, \text{tr}(H^n)] \). For \( G, G' \in \Gamma(n,e) \), we say that \( G <_{\text{lex}} G' \) when \( \gamma(G) <_{\text{lex}} \gamma(G') \), where the latter is the lexicographic order among numerical sequences. If Conjecture 1 is true then it is not difficult to see that for a regular graph \( G_0 \) whose degree of regularity is \( a \), and \( m \) sufficiently large, \( \bar{G} \in \Gamma(G_0 \cup mK_{a+1}) \) is \( t \)-optimal if and only if \( \bar{G} = G'_0 \cup mK_{a+1} \), where \( G'_0 \in \Gamma(G_0) \) is minimum according to the order \( <_{\text{lex}} \).

More generally, we conjecture the following:

**Conjecture 2.** If \( G \) is a graph whose complement is \( t \)-optimal, then \( G \) is minimum in the order \( <_{\text{lex}} \).

References


