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Relative formality theorem and quantisation of coisotropic submanifolds

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Abstract

We prove a relative version of Kontsevich's formality theorem. This theorem involves a manifold M and a submanifold C and reduces to Kontsevich's theorem if $C = M$. It states that the DGLA of multivector fields on an infinitesimal neighbourhood of C is L_∞-quasiisomorphic to the DGLA of multidifferential operators acting on sections of the exterior algebra of the conormal bundle. Applications to the deformation quantisation of coisotropic submanifolds are given. The proof uses a duality transformation to reduce the theorem to a version of Kontsevich's theorem for supermanifolds, which we also discuss. In physical language, the result states that there is a duality between the Poisson sigma model on a manifold with a D-brane and the Poisson sigma model on a supermanifold without branes (or, more properly, with a brane which extends over the whole supermanifold).

Keywords: Deformation quantisation; Formality; Coisotropic submanifolds; Strong homotopy algebras

1. Introduction

In [10] Kontsevich gave a solution to the problem of deformation quantisation of the algebra of functions on an arbitrary Poisson manifold. This solution is based on his formality theo-

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rem, stating that the differential graded Lie algebra (DGLA) of multidifferential operators is L_∞-quasiisomorphic to its cohomology, the DGLA of multivector fields. We consider here a version of the formality theorem for a pair (M, C) of manifolds $C \subset M$, which reduces to the original formality theorem if $C = M$. The algebra of functions on M is replaced here by the graded commutative algebra A of sections of the exterior algebra of the normal bundle NC. The (suitably completed) Hochschild complex of A, with Hochschild differential and Gerstenhaber bracket contains the sub-DGLA $\hat{\mathcal{D}}(A)$ of "multidifferential operators" on A, namely cochains built out of products of compositions of derivations of A. The statement is that this DGLA is L_∞-quasiisomorphic to its cohomology, which is identified with the DGLA $\mathcal{T}(M, C)$ of multivector fields on a formal neighbourhood of C with Schouten–Nijenhuis bracket and zero differential, see Theorem 4.8. The proof is based on a "Fourier transform" Theorem 4.1 which states that the Gerstenhaber algebra $\mathcal{T}(M, C)$ is isomorphic to the Gerstenhaber algebra of multivector fields on the supermanifold $N^*[1]C$, the conormal bundle with shifted parity of the fibres. In terms of supermanifolds, this isomorphism is obtained from an isomorphism of odd-symplectic graded supermanifolds $T^*[1]NC \to T^*[1]N^*[1]C$, a variant of what Roytenberg calls "Legendre transform" [15]. At this point the result follows from a version of Kontsevich's formality theorem for supermanifolds, see Theorem 4.6. The proof of the latter theorem is parallel to Kontsevich's [10], except for signs, which are already non-trivial in the original setting of ordinary manifolds. For this reason we work out all signs in Appendix A, and develop a formalism in which these signs appear in an essentially transparent way.

In the application to deformation quantisation we take C to be a coisotropic submanifold of M, which means that the vanishing ideal $I(C)$ of C is a Poisson subalgebra of $C^\infty(M)$. To these data one associates a Poisson algebra, the algebra of functions on the reduced phase space $C^\infty(\underline{C}) = N(I(C))/I(C)$, the quotient of the normaliser of the Lie algebra $I(C)$ by $I(C)$. Even if the reduced phase space $\underline{C}$, which is by definition the space of leaves of the characteristic foliation of C, is singular, the Poisson algebra $C^\infty(\underline{C})$ of "smooth functions" on it makes sense, and one can ask the question of quantising this algebra in the sense of deformation quantisation. It seems that this is not always possible because of anomalies, but it may be argued that the question is not correct and that one should not try to quantise $C^\infty(\underline{C})$, which can be trivial even for interesting C, but rather some kind of resolution of it. In fact there is a natural complex whose cohomology in degree zero is $C^\infty(\underline{C})$: the conormal bundle N^*C of a coisotropic manifold is naturally a Lie algebroid, and $C^\infty(\underline{C})$ is its zeroth Lie algebroid cohomology algebra. Thus one replaces $C^\infty(\underline{C})$ by the Lie algebroid cochain complex $\Gamma(C, \wedge NC)$. This differential graded algebra is however not a Poisson algebra; it turns out, as essentially noticed by Oh and Park [11], that the Poisson structure on M induces a P_∞-structure on $\Gamma(C, \wedge NC)$, namely an L_∞-structure whose structure maps are multiderivations, see Theorem 2.2. Algebraically, the P_∞-brackets are obtained from the Poisson structure on M as higher derived brackets in the sense of Voronov [18]. This L_∞-structure induces the Poisson bracket on cohomology. At this point the L_∞-machinery can be applied: the L_∞-structure can be understood as a solution of the Maurer–Cartan equation in $\mathcal{T}(M, C)$ and is mapped by the L_∞-quasiisomorphism to a solution of the Maurer–Cartan equation in $\hat{\mathcal{D}}(A)$. The latter is a deformation of the product in A as an A_∞-algebra, see Theorem 3.2. At this point one may want to pass to cohomology to quantise the original Poisson algebra $C^\infty(\underline{C})$, or more general the whole Lie algebroid cohomology. It is here that one meets the anomalies in general. Namely the A_∞-algebra obtained by this construction is not flat in general, namely its 0th product map μ_0 may not vanish (we use the not-quite-standard but natural definition of A_∞-algebra which allows for non-zero product $\mu_0 \in A$ and call an A_∞-algebra flat if $\mu_0 = 0$). In this case the first product μ_1 is not a differential and the cohomology is not defined. Removing μ_0 (which is at least quadratic in the deformation parameter) is a cohomological problem with an obstruction in the second Lie algebroid cohomology group. In some cases the obstruction vanishes even if the cohomology does not, see the second remark in 3.2. If the obstruction vanishes, one gets an associative algebra, which is however not always a flat deformation of $C^\infty(\underline{C})$. This time the obstruction is in the first cohomology group, see Corollary 3.3.

Some of the results presented here were announced in [5]. There the interpretation of these results in terms of topological quantum field theory is given: the L_∞-quasiisomorphism is constructed using a topological sigma model on the disk with the boundary condition that the boundary is sent to C. An alternative approach to this class of problems, based on Tamarkin's formality theorem, was proposed recently in [4]. See also the very recent preprint [12], in which a use of the formality for supermanifolds similar to ours is presented in a physics context and shown to be applicable to weak Poisson manifolds.

From the point of view of topological quantum field theory adopted in [5] this paper concerns the case of a single D-brane. The more general case of several D-branes will be studied elsewhere. It corresponds to the theory of (bi)modules over the deformed algebras.

Conventions. We work in the category of graded vector spaces (or free modules over a commutative ring) $V=\bigoplus_{j\in\mathbb{Z}}V^j$, and denote by $|a|$ the degree of a homogeneous element $a\in V^{|a|}$. We denote by $V[n]$ the graded vector space $\bigoplus_j V[n]^j$ with $V[n]^j=V^{n+j}$. The space of homomorphisms $f:V\to W$ of degree j (i.e., such that $f(V^i)\subset W^{j+i}$) is denoted by $\mathrm{Hom}^j(V,W)$. The Koszul sign rule holds. A derivation f of degree $|f|$ of a graded algebra A is a linear endomorphisms of degree $|f|$ obeying $f(ab)=f(a)b+(-1)^{|a||f|}af(b)$ for all $a,b\in A$. See A.1 for more details.

2. Coisotropic submanifolds of Poisson manifolds

2.1. Coisotropic submanifolds

Let (M,π) be a Poisson manifold, with Poisson bivector field $\pi\in\Gamma(M,\wedge^2TM)$ and Poisson bracket $\{f,g\}=\langle\pi,df\otimes dg\rangle$. Let $\pi^\sharp:T^*M\to TM$ be the bundle map induced by π on each cotangent space: $\langle\pi^\sharp(\alpha),\beta\rangle=\langle\pi,\alpha\otimes\beta\rangle$. A submanifold $C\subset M$ is called *coisotropic* [20] if $\pi^\sharp|_C$ maps the conormal bundle $N^*C=\mathrm{Ann}(TC)\subset T^*_CM$ to the tangent bundle TC. Equivalently, C is coisotropic if and only if the ideal $I(C)$ of the algebra $C^\infty(M)$ consisting of functions vanishing on C is closed under the Poisson bracket. Examples include M itself, Lagrangian submanifolds of symplectic manifolds, graphs of Poisson maps, zeros of equivariant moment maps and mechanical systems with first class constraints.

Coisotropic submanifolds come with interesting geometric and algebraic structures, which we turn to describe.

2.2. Characteristic foliation and reduced phase space

If C is coisotropic, the distribution $\pi^\sharp(N^*C)\subset TC$ of tangent subspaces is involutive since it is spanned by hamiltonian vector fields $X_h=\pi^\sharp\,dh$ with $h\in I(C)$, which commute on C by the coisotropy condition. The corresponding foliation is the *characteristic foliation* of C. The leaves of the characteristic foliation have points related by hamiltonian flows with hamiltonian functions in $I(C)$. The *reduced phase space* $\underline{C}$ of C is the space of leaves of the characteristic foliation. It can be a wild space so that it is better to consider the algebra of functions on it, which is, by definition, the algebra of functions on C that are invariant under the hamiltonian flows of $I(C)$:

$$C^\infty(\underline{C}) = \{ f \in C^\infty(C) \mid X_h(f) = 0 \ \forall h \in I(C) \}.$$

If $\underline{C}$ is a manifold, then the Poisson bivector field descends to $\underline{C}$ and gives it a structure of Poisson manifold. In general $C^\infty(\underline{C})$ is a Poisson algebra, namely a commutative algebra with a Lie bracket which is a derivation in each of its arguments. The Lie algebra structure is induced from the Lie algebra structure on $C^\infty(M)$ as is clear from the representation

$$C^\infty(\underline{C}) = N(I(C))/I(C),$$

as the quotient by $I(C)$ of the normaliser

$$N(I(C)) = \{ f \in C^\infty(M) \mid \{I(C), f\} \subset I(C) \}$$

of the Lie subalgebra $I(C)$.

2.3. *The Lie algebroid of a coisotropic submanifold*

The space of sections of the conormal bundle has a natural Lie algebra structure. The Lie bracket is uniquely defined by the conditions

$$[df, dg] = d\{f, g\}, \quad f, g \in I(C),$$
$$[f\alpha, \beta] = f[\alpha, \beta] - \pi^\sharp(\beta)(f)\alpha, \quad \alpha, \beta \in \Gamma(C, N^*C), \ f \in C^\infty(C).$$

By construction, $\pi^\sharp$ induces a Lie algebra homomorphism $\Gamma(C, N^*C) \to \Gamma(C, TC)$ from this Lie algebra to the Lie algebra of vector fields. In other words, N^*C is a Lie algebroid over C.

2.4. *The cochain complex of a coisotropic submanifold*

As for every Lie algebroid, the Lie algebroid of a coisotropic submanifold comes with a cochain complex, see [13]:

$$\cdots \to \Gamma(C, \wedge^j NC) \to \Gamma(C, \wedge^{j+1} NC) \to \cdots.$$

The differential δ on $\Gamma(C, \wedge^0 NC) = C^\infty(C)$ is $\delta f = \pi^\sharp d\tilde{f} \bmod TC$, for any extension $\tilde{f}$ of f to M: the class of $\pi^\sharp d\tilde{f}$ in $NC = T_C M/TC$ is independent of the choice of extension because of the coisotropy condition. The differential on $\Gamma(C, \wedge^1 NC)$ is the dual map to the Lie bracket $\Gamma(C, \wedge^2 N^*C) \to \Gamma(C, N^*C)$. The differential on general cochains is determined by the rule

$$\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \delta\beta, \quad \alpha, \beta \in \Gamma(C, \wedge NC).$$

The cohomology of this complex is the cohomology $H_\pi(N^*C)$ of the Lie algebroid N^*C. It is a graded commutative algebra. In degree 0 we have

$$H^0_\pi(N^*C) = C^\infty(\underline{C}).$$

The first cohomology group describes infinitesimal deformations of the imbedding of C as a coisotropic submanifold up to deformations induced by hamiltonian flows.

2.5. The P_∞-structure on the cochain complex

A natural question is whether the Poisson bracket on $H^0_\pi(N^*C) = C^\infty(\underline{C})$ comes from a structure on the cochain complex. We want to show that this structure is a flat P_∞-structure (defined up to homotopy), namely a graded commutative algebra structure with a compatible flat L_∞-structure. In particular, this flat P_∞-structure induces a Poisson bracket on the whole cohomology algebra $H_\pi(N^*M)$. The definitions are as follows. A *P_∞-algebra* (P for Poisson) is a graded commutative algebra A over a field of characteristic zero with a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$ of degree $2-n$, $n = 0, 1, 2, \ldots$, with the following properties (properties (i) and (iii) characterise L_∞-algebras) that are to hold for arbitrary $a_1, \ldots, a_n \in A$:

(i) $\lambda_n(\ldots, a_i, a_{i+1}, \ldots) = -(-1)^{|a_i|\cdot|a_{i+1}|}\lambda_n(\ldots, a_{i+1}, a_i, \ldots)$.
(ii) $a \mapsto \lambda_n(a_1, \ldots, a_{n-1}, a)$ is a derivation of degree $2 - n - \sum_{i=1}^{n-1} |a_i|$.
(iii) For all $n \geqslant 0$, the map

$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{q=0}^{n} \frac{(-1)^{q(n-q)}}{q!(n-q)!} \lambda_{n-q+1}\bigl(\lambda_q(a_1, \ldots, a_q), a_{q+1}, \ldots, a_n\bigr)$$

vanishes on the image of the alternation map $\mathrm{Alt}_n = \frac{1}{n!}\sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\sigma$.

A *flat* P_∞-algebra is a P_∞-algebra such that $\lambda_0 = 0$. Then λ_1 is a differential, λ_2 a chain map obeying the Jacobi identity up to exact terms. So the λ_1-cohomology of a flat P_∞-algebra is a graded Poisson algebra.

Remark. This notion of flat P_∞-structure is not completely standard: in the spirit of homotopical algebra one might prefer a notion in which also the commutativity of the product and the Leibniz rule hold only up to homotopy. Here only the bracket of a Poisson algebra is replaced by a sequence of higher brackets controlling the violation of the Jacobi identity.

The construction of a flat P_∞-structure on $A = \Gamma(C, \wedge NC)$ for a coisotropic submanifold $C \subset M$ depends on the choice of an identification of a tubular neighbourhood of C with the normal bundle of C, more precisely an embedding ι of NC into M sending the zero section identically to C and such that, for $x \in C$, the restriction of $\iota_* : T_x(NC) \to T_xM$ composed with the canonical projection is the identity $N_xC \to T_xM/T_xC$. As such an embedding is unique up to homotopy, the construction gives a flat P_∞-structure up to homotopy. We give the construction in the more general setting of a general submanifold of M, yielding a non-necessarily flat P_∞-structure. As the construction only involves a neighbourhood of the submanifold, we may as well assume that M is the total space of a vector bundle, which is then canonically the normal bundle of its zero section C.

Proposition 2.1. *Let C be a submanifold of a Poisson manifold M, not necessarily coisotropic. Assume that M is the total space of a vector bundle $p : E \to C$ and let $C \subset M$ be the zero section of E. Then there is a unique P_∞-structure on $\Gamma(C, \wedge NC) \simeq \Gamma(C, \wedge E)$ such that for $v_1, \ldots, v_n \in \Gamma(C, E)$, $f, g \in C^\infty(C) = \Gamma(C, \wedge^0 E)$, and $u, w \in \Gamma(C, E^*)$*

$$\lambda_n(v_1,\ldots,v_{n-2},f,g)=(-1)^{n-2}v_1\cdots v_{n-2}\{p^*f,p^*g\}|_C,$$
$$\big\langle\lambda_n(v_1,\ldots,v_{n-1},f),u\big\rangle=(-1)^{n-1}v_1\cdots v_{n-1}\{p^*f,u\}|_C,$$
$$\big\langle\lambda_n(v_1,\ldots,v_n),u\otimes w\big\rangle=(-1)^n v_1\cdots v_n\{u,w\}|_C.$$

On the right-hand side of these equations, u, w are regarded as functions on M linear on the fibres, and v_i as vertical vector fields on M.

The uniqueness part is clear: since λ_n is a multiderivation, it is sufficient to define it on $\Gamma(C,\wedge^j E)$ with $j=0,1$ as the algebra is generated by these spaces. Since λ_n is of degree $2-n$, λ_n vanishes on elements of degree 0 or 1 except in the three cases listed in the proposition. The fact that the λ_n extend to a P_∞-structure can be checked directly, but we will deduce it from a more general result below (see Proposition 4.4).

Theorem 2.2. *Assume that in Proposition* 2.1, *C is coisotropic. Then $\lambda_0=0$ so that $\Gamma(C,\wedge NC)$ is a flat P_∞-algebra, λ_1 is the differential of* 2.4 *and λ_2 induces the Poisson bracket on $H^0_\pi(C)=C^\infty(\underline{C})$ of* 2.2.

Proof. Let $u,v\in\Gamma(C,N^*C)$ considered as fibre-linear functions on NC. We have $\langle\lambda_0,u\otimes v\rangle=\{u,v\}|_C=0$ since u,v belong to the ideal $I(C)$, which is a Lie subalgebra if C is coisotropic. By definition, if $f\in C^\infty(C)$, $\langle\lambda_1(f),u\rangle=\{p^*f,u\}|_C$, so $\lambda_1(f)$ is the class in $\Gamma(C,NC)$ of $\pi^\sharp d\tilde f$ with $\tilde f=p^*f$. If $w\in\Gamma(C,NC)$, $\langle\lambda_1(w),u\otimes v\rangle=w\{u,v\}|_C=\langle w,[u,v]\rangle$. Therefore λ_1 is indeed the differential of 2.4. As for λ_2, recall that the Poisson bracket of $f,g\in H^0_\pi(N^*C)=\operatorname{Ker}(\lambda_1:C^\infty(C)\to\Gamma(C,E))$ is defined as $\{\tilde f,\tilde g\}$ for any extension $\tilde f,\tilde g$ of f,g to M. This is precisely the definition of λ_2, with $\tilde f=p^*f$. □

Using this result, we obtain a Poisson bracket induced by λ_2 on the cohomology $H_\pi(N^*C)$. A priori this bracket depends on the choice of embedding of NC into M. However we see by a standard homotopy argument that this is not the case:

Proposition 2.3. *Let $C\subset M$ be coisotropic. Then the Lie bracket induced by λ_2 on the cohomology $H_\pi(N^*C)$ is independent of the choice of embedding of NC into a tubular neighbourhood of C.*

2.6. Higher derived brackets and relative multivector fields

The maps λ_j on $\Gamma(C,\wedge NC)$ are a special case of higher derived brackets, see [18]. Let $\mathfrak a$ be an abelian graded Lie subalgebra of a graded Lie algebra $\mathfrak g$, with a projection $P:\mathfrak g\to\mathfrak a$, satisfying $P[a,b]=P[Pa,b]+P[a,Pb]$. Suppose we have an element $\pi\in\mathfrak g$ of degree 1 obeying $[\pi,\pi]=0$. Then the higher derived brackets

$$\{a_1,\ldots,a_n\}=P\big[\cdots[\pi,a_1],a_2\big],\ldots\big],a_n\big]$$

are graded *symmetric* multilinear functions of degree 1, obeying the Jacobi identities

$$\sum_k\sum_{\sigma\in S_n}\frac{(-1)^\epsilon}{k!(n-k)!}\big\{\{a_{\sigma(1)},\ldots,a_{\sigma(k)}\},\ldots,a_{\sigma(n)}\big\},$$

with the natural sign: $\epsilon = \sum_{i<j,\sigma(i)>\sigma(j)} \deg(a_i)\deg(a_j)$. The brackets

$$\lambda_n(a_1, \dots, a_n) = (-1)^{\sum_i (i-1)\deg(a_i)}\{a_1, \dots, a_n\}$$

are then *skew-symmetric* and of degree $2-n$ with respect to the shifted degree $|a| = \deg(a) + 1$ and they obey the L_∞-Jacobi identities (iii) above.

In our case, $\mathfrak{g} = \mathcal{T}(M, C)$ is the Lie algebra of *relative multivector fields* on the submanifold $C \subset M$ with the Schouten–Nijenhuis bracket. It is the inverse limit $\varprojlim \mathcal{T}(M)/I(C)^n\mathcal{T}(M)$ where $\mathcal{T}(M) = \bigoplus_{j=-1}^{\infty} \mathcal{T}^j(M)$ is the graded Lie algebra of multivector fields and $I(C)$ is the ideal in $C^\infty(M)$ of functions vanishing on C. The Lie subalgebra $\mathfrak{a} = \Gamma(C, \wedge NC)$ consists of sums of products of vector fields tangent to the fibres of $M = E \to C$ and constant along each fibre.

3. Quantisation of coisotropic submanifolds

3.1. A_∞-algebras and flat A_∞-algebras

An A_∞-algebra [16] over a commutative ring R is a free graded left R-module $A = \bigoplus_{j\in\mathbb{Z}} A^j$ with R-linear maps $\mu_n : A^{\otimes n} \to A[2-n]$ of degree 0 ($n = 0, 1, \dots$) obeying the associativity relations

$$\sum_{q=0}^{n} (-1)^{q(n-q)} \sum_{j=0}^{p-1} (-1)^{(q-1)j + \sum_{i=1}^{j} |a_i| \cdot q} \\ \times \mu_{n-q+1}\big(a_1, \dots, a_j, \mu_q(a_{j+1}, \dots, a_{j+q}), a_{j+q+1}, \dots, a_n\big) = 0.$$

A *flat* A_∞-algebra is an A_∞-algebra with $\mu_0 = 0$. In this case μ_1 is a differential and μ_2 induces an associative product on its cohomology. Associative algebras can be regarded as A_∞-algebras with product μ_2 and all other $\mu_i = 0$.

Let $F(V) = \bigoplus_{j\geqslant 0} V^{\otimes j}$ be the tensor coalgebra over R generated by a free R-module V and denote by $p_j : F(V) \to V^{\otimes j}$ the projection onto the jth summand. Let $A[1]$ be the graded R-module with homogeneous components $A[1]^j = A^{j+1}$ and let $s : A[1] \to A$ be the tautological map of degree 1. Then an A_∞-structure on A is the same as a coderivation Q of degree 1 of $F(A[1])$ obeying $[Q, Q] = 0$, see A.3 in Appendix A. The coderivation Q and the products μ_n are related by

$$\mu_n \circ (s \otimes \dots \otimes s) = s \circ p_1 \circ Q|_{A[1]^{\otimes n}}.$$

The "strange" signs in the associativity relations come from the Koszul rule, if we take into account that s has degree 1.

The following result is a graded, A_∞-version of the classical result relating first order associative deformations of algebras of smooth functions to Poisson brackets.

Proposition 3.1. *Let $A_0 = \bigoplus_i \Gamma(M, \wedge^i E)$ be the graded commutative algebra of sections of the exterior algebra of a vector bundle $E \to M$. Let $(\mu_n)_{n=0}^{\infty}$ be an A_∞-algebra structure on $A = A_0[\![\epsilon]\!]$ over $\mathbb{R}[\![\epsilon]\!]$ which reduces modulo ϵ to the algebra structure on $A_0 = A/\epsilon A$ and such that the structure maps μ_n are multidifferential operators. Let $\lambda_n = \frac{1}{\epsilon}\mu_n \circ \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\sigma \bmod \epsilon A$. Then $(\lambda_n)_{n=0}^{\infty}$ is a P_∞-structure on A_0.*

Proof. The products have the form $\mu_n = \mu_n^0 + \epsilon\mu_n^1 + \cdots$ with $\mu_n^0 = 0$ except for $n = 2$. The associativity relations can be expressed as $[\mu, \mu] = 0$ in terms of the Gerstenhaber bracket (see (6) below), so that to lower order in ϵ we have

$$[\mu^0, \mu^0] = 0, \qquad [\mu^0, \mu^1] = 0, \qquad [\mu^1, \mu^1] + 2[\mu^0, \mu^2] = 0.$$

The first equation is just the associativity of the product in A_0. The second equation states that μ_p^1 is a Hochschild cocycle for each p: $b\mu_p^1 = 0$. By the HKR theorem (see Lemma A.2), we have $\mu_p^1 = \tilde{\mu}_p^1 + b\varphi_p$ for some alternating multiderivation $\tilde{\mu}^1$. A straightforward direct calculation shows that, for any p-cochain φ_p,

$$b\varphi_p \circ \mathrm{Alt}_{p+1} = 0,$$

owing to the commutativity of the product. Thus $\lambda = \tilde{\mu}^1$ and therefore λ is an alternating multiderivation, i.e., obeys (i), (ii). Finally, the third equation, restricted to skew-symmetric tensors in $A_0^{\otimes n}$ becomes $[\lambda, \lambda] = 0$ as the term $[\mu^0, \mu^2] = b\mu^2$ does not contribute, again because of the commutativity of μ^0. This proves property (iii) of P_∞-structures. □

3.2. Quantisation

Our problem is to quantise the Poisson algebra $H_\pi(N^*C)$ (or at least the subalgebra $H_\pi^0(N^*C) = C^\infty(\underline{C})$) for coisotropic C, namely to find a *star-product*, i.e., an associative $\mathbb{R}[\![\epsilon]\!]$-bilinear product $\star$ on $H_\pi(N^*C)[\![\epsilon]\!]$ deforming the graded commutative product and such that $\epsilon^{-1}(a \star b - (-1)^{|a|\cdot|b|} b \star a)$ is the Poisson bracket modulo ϵ. It seems that this is impossible in general. What one always has is a quantisation of the P_∞-algebra $\Gamma(C, N^*C)$ for any submanifold C as an A_∞-algebra. From this result we then solve the original quantisation problem if suitable obstructions vanish.

Theorem 3.2. *Let $C \subset M$ be a submanifold of a Poisson manifold M and let $A = \Gamma(C, \wedge NC)$ with a P_∞-structure λ induced by the Poisson structure on M. Then there is an $\mathbb{R}[\![\epsilon]\!]$-linear A_∞-structure on $A_\epsilon = A[\![\epsilon]\!]$ inducing λ on $A_\epsilon/\epsilon A_\epsilon$. Moreover, if C is coisotropic (so that $\lambda_0 = 0$) then $\mu_0 = O(\epsilon^2)$.*

In the case where C is an affine subspace in $\mathbb{R}^n$ there is an explicit Feynman diagram expansion describing this A_∞-structure, see [5].

Theorem 3.2 is proved below as a consequence of the relative formality Theorem 4.8.

Corollary 3.3. *If C is coisotropic and $H_\pi^2(N^*C) = 0$ then the A_∞-structure in the preceding theorem may be chosen as a flat A_∞-structure $\mu = (\mu_1, \mu_2, \ldots)$. In particular, μ_2 induces an associative product on the cohomology $H_{\pi,\epsilon}(N^*C)$ of the complex $(A[\![\epsilon]\!], \mu_1)$. If additionally $H_\pi^1(N^*C) = 0$, then there is an isomorphism of $\mathbb{R}[\![\epsilon]\!]$-modules $H_{\pi,\epsilon}^0(N^*C) \to H_\pi^0(N^*C)[\![\epsilon]\!]$ sending μ_2 to a star-product on the Poisson algebra $H_\pi^0(N^*C)$.*

Proof. Theorem 3.2 gives an A_∞-structure $(\mu_n)_{n\geqslant 0}$ with $\mu_0 = O(\epsilon^2)$ and $\mu_n = O(\epsilon)$ for $n \neq 2$. The problem is to find an $a \in \epsilon A[\![\epsilon]\!]$ of degree 1 such that

$$\sum_{n=0}^{\infty} \mu_n(a, a, \ldots, a) = 0. \tag{1}$$

Suppose for the moment that we have such an a. Let Q be the coderivation of $F(A[1])$ associated to μ and denote by $T \in \mathrm{End}(F(A[1]))$ the unique coalgebra automorphism with vanishing Taylor components $T_j = p_1 \circ T|_{A[1]^{\otimes j}}$ except $T_0 = a$ and $T_1(x) = x$, $x \in A[1]$. Then $\hat{Q} = T^{-1} \circ Q \circ T$ is a coderivation of $F(A[1])$ defining a new A_∞-structure $\hat{\mu}$ obeying the properties of Theorem 3.2, but with $\hat{\mu}_0 = 0$. Indeed,

$$\hat{\mu}_0 = p_1 \circ \hat{Q}|_{V^{\otimes 0}} = \sum p_1 \circ T_1^{-1} \circ Q \circ (T_0 \otimes \cdots \otimes T_0) = \sum_{n \geqslant 0} Q(a, \ldots, a) = 0.$$

Equation (1) for a power series $a = \epsilon a_1 + \epsilon^2 a_2 + \cdots \in \epsilon \Gamma(C, NC)[\![\epsilon]\!]$ can then be solved in a standard recursive way as a cohomological problem for the Lie algebroid differential $d = \epsilon^{-1}\mu_1|_{\epsilon=0} = \lambda_1$. Since a is odd and μ_2 is commutative to lowest order, we have $\mu_2(a, a) = O(\epsilon^3)$, so if $\mu_0 = \epsilon^2 F + O(\epsilon^3)$, to lowest order the equation is $F + da_1 = 0$. To lowest order, $[\mu, \mu] = 0$ implies $dF = 0$ so there is a solution a_1. Then at each step one has to solve an equation of the form $d(x) = b$ for given b which is shown recursively to be d-closed, as a consequence of $[\mu, \mu] = 0$.

As $\mu_1 = \epsilon d + O(\epsilon^2)$, where d is the Lie algebroid differential, we have an $\mathbb{R}$-linear map $p : H^0_{\pi,\epsilon}(N^*C) \to H^0_\pi(N^*C)$ sending $f_0 + \epsilon f_1 + \cdots \in \mathrm{Ker}(\mu_1)|_{C^\infty(C)}$ to $f_0 \in \mathrm{Ker}(d)$. If $H^1_\pi(N^*C)$ vanishes there is a right inverse $\sigma : f_0 \mapsto f$ to p obtained by solving recursively for $f_1, f_2, \ldots$ the equation $df = 0$ with $f = f_0 + \epsilon f_1 + \cdots$. By induction, at each step the equation is of the form $df_j = C_j(f_0, \ldots, f_{j-1})$ with closed right-hand side and has a unique solution $f_j \in K$. The "quantisation map" σ is then extended by $\mathbb{R}[\![\epsilon]\!]$-linearity to an injective homomorphism $\sigma : H^0_\pi(N^*C)[\![\epsilon]\!] \to H^0_{\pi,\epsilon}(N^*C)$ of $\mathbb{R}[\![\epsilon]\!]$-modules. The map σ is surjective. Indeed, if $f = f_0 + \epsilon f_1 + \cdots \in H^0_{\pi,\epsilon}(N^*C)$ then $f = \sigma(g_0 + \epsilon g_1 + \cdots)$ with g_j recursively defined by

$$g_0 = f_0 f - \sigma\big(g_0 + \cdots + \epsilon^j g_j\big) = \epsilon^{j+1} g_{j+1} + O\big(\epsilon^{j+2}\big).$$

Since, by construction, $\sigma(f_0) = f_0 + O(\epsilon)$, the product μ_2 induces a deformation of the product on $H^0(N^*C)$ whose skew-symmetric part at order ϵ is, by Theorem 2.2, is the given Poisson bracket. □

Remark. If $H^2_\pi(N^*C)$ vanishes, one can also construct a quantisation of $C^\infty(\underline{C}) = H^0_\pi(N^*C)$ by the BRST method, see [3].

Remark. From the explicit construction of the A_∞-structure one sees that in some cases the obstruction vanishes even if $H^2 \neq 0$. For example, if $\mathfrak{h} \subset \mathfrak{g}$ is an inclusion of finite dimensional real Lie algebras, then the subspace $C = \mathfrak{h}^\perp = (\mathfrak{g}/\mathfrak{h})^*$ of linear functions on $\mathfrak{g}$ vanishing on $\mathfrak{h}$ is a coisotropic submanifold of the Poisson manifold $\mathfrak{g}^*$ with Kostant–Kirillov bracket. In this case the anomaly μ_0 vanishes [5] even when $H^2_\pi(N^*C) = H^2_{\mathrm{Lie}}(\mathfrak{h}; C^\infty(\mathfrak{h}^\perp)) \neq 0$.

4. The relative formality theorem

4.1. The Gerstenhaber algebra of multiderivations

Let A be a graded commutative algebra. Recall that a derivation of degree d of a graded algebra A is a linear map $\varphi : A \to A$ of degree d such that $\varphi(ab) = \varphi(a)b + (-1)^{d\cdot|a|} a\varphi(b)$, $a \in A^{|a|}$. Derivations form a graded left A-module $\mathrm{Der}(A)$ with a Lie bracket $[\varphi, \psi] = \varphi \circ \psi -$

$(-1)^{|\varphi|\cdot|\psi|}\psi \circ \varphi$. On the graded commutative algebra $S_A(\mathrm{Der}(A)[-1])$ (the graded symmetric algebra of the A-module $\mathrm{Der}(A)[-1]$) we then have a Gerstenhaber structure, namely a (super) Lie bracket of degree -1 compatible with the product. The Lie bracket is the extension of [,] on $\mathrm{Der}(A)$ to all of $S_A(\mathrm{Der}(A)[-1])$ by the rule

$$[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + (-1)^{(\deg(\beta)+1)\cdot\deg(\gamma)}[\alpha, \gamma]\beta. \tag{2}$$

Here deg denotes the degree in the Lie algebra of multiderivations

$$\mathcal{T}(A) = S_A\big(\mathrm{Der}(A)[-1]\big)[1],$$

for which the Lie bracket has degree 0 (and the product degree 1). By definition

$$\deg(\alpha_1 \cdot \dots \cdot \alpha_n) = \sum_{j=1}^{n} |\alpha_j| + n - 1, \quad \text{for } \alpha_j \in \mathrm{Der}^{|\alpha_j|}(A).$$

The signs are then

$$\alpha\beta = (-1)^{(\deg(\alpha)-1)(\deg(\beta)-1)}\beta\alpha, \qquad [\alpha, \beta] = -(-1)^{\deg(\alpha)\deg(\beta)}[\beta, \alpha]$$

and the Jacobi identity is

$$(-1)^{\deg(\alpha)\deg(\gamma)}\big[[\alpha, \beta], \gamma\big] + \text{cycl.} = 0.$$

To make contact with the Hochschild complex it will be useful to view multiderivations as multilinear maps on A. First of all we have a map $\sigma : S_A(\mathrm{Der}(A)[-1]) \to \bigoplus^j \wedge_A^j(\mathrm{Der}(A))[-j]$ given by

$$\sigma(\varphi_1 \cdot \dots \cdot \varphi_j) = (-1)^{\sum_{\alpha=1}^{n}(\alpha-1)|\varphi_\alpha|}\varphi_1 \wedge \dots \wedge \varphi_j.$$

If $\mu : A^{\otimes n} \to A$ denotes the product in A, we have a map $\tau : \wedge_A^j \mathrm{Der}(A) \to \mathrm{Hom}(A^{\otimes j}, A)$:

$$\tau(\varphi_1 \wedge \dots \wedge \varphi_j) = \mu \circ \mathrm{Alt}_j(\varphi_1 \otimes \dots \otimes \varphi_j).$$

The graded alternation map $\mathrm{Alt}_j \in \mathrm{End}(A^{\otimes j})$ is $(1/j!)\sum_{\sigma\in S_j} \mathrm{sign}(\sigma)\sigma$, for the natural action (with Koszul signs) of the symmetric group on the tensor algebra of the graded vector space A.

The *Hochschild–Kostant–Rosenberg map* is then the injective homomorphism

$$\varphi_{\mathrm{HKR}} = \tau \circ \sigma : S_A\big(\mathrm{Der}(A)[-1]\big) \mapsto \bigoplus_{j=0}^{\infty} \mathrm{Hom}_k\big(A^{\otimes j}, A\big). \tag{3}$$

Remark. In a more natural setting one would avoid the use of exterior algebras and only use the symmetric algebras, which are always graded commutative. The advantage would be that many signs would be simpler: for example, we would not have the sign coming from the definition of σ and would define the HKR map as a map to $\bigoplus \mathrm{Hom}(A[1]^{\otimes j}, A[1])$. However other things would become more exotic: for example, in this setting an associative product (solution of the

Maurer–Cartan equation) would be a map of degree 1 obeying a "graded associativity" property, with signs.

4.2. Fourier transform

For the relative formality theorem two algebras and the corresponding Gerstenhaber algebras are relevant. Let $C \subset M$ be a submanifold of a manifold M which we may assume to be the total space of a vector bundle $p : E \to C$ of rank r with C embedded as the zero section. Let $A = \bigoplus_0^r A^j$, with $A^j = \Gamma(C, \wedge^j E)$ the graded commutative algebra of sections of the exterior algebra of E. Let $B = \bigoplus B^j$, with $B^0 = \Gamma(C, S(E^*))$ and $B^j = 0$ for $j \neq 0$, be the algebra of polynomial functions on E, considered as a graded algebra concentrated in degree zero. Then the Gerstenhaber algebra $S_B(\mathrm{Der}(B)[-1])$ may be identified with the algebra of multivector fields on $M = E$ which are polynomial along the fibres.

Theorem 4.1. *The two Gerstenhaber algebras* $S_A(\mathrm{Der}(A)[-1])$ *and* $S_B(\mathrm{Der}(B)[-1])$ *are canonically isomorphic up to choice of sign. In particular, the Lie algebras of multiderivations* $\mathcal{T}(A)$ *and* $\mathcal{T}(B)$ *are canonically isomorphic up to choice of sign.*

Let us first suppose that E is a trivial bundle $C \times V$ over an open subset C of $\mathbb{R}^n$ with coordinates $x^1, \ldots, x^n$. Let $\theta_1, \ldots, \theta_r$ be a basis of V. Then A is freely generated by its degree zero component $A^0 = C^\infty(C)$ and θ_μ of degree 1 ($\mu = 1, \ldots, r$). Then $\mathrm{Der}(A)$ is a free A-module generated by $\xi_i = \partial/\partial x^i$ ($i = 1, \ldots, n$) of degree 0 and $\psi^\mu = \partial/\partial\theta_\mu$ ($\mu = 1, \ldots, r$) of degree -1. Thus, as a graded algebra, $S_A(\mathrm{Der}(A)[-1])$ is the free graded commutative A^0-algebra $A^0[\theta_\mu, \psi^\mu, \xi_i]$ with generators θ_μ of degree 1, ψ^μ of degree 0, and ξ_i of degree 1. The Lie bracket is defined by the relations

$$[\xi_i, f] = \frac{\partial f}{\partial x^i}, \quad \left[\psi^\mu, f\right] = [\theta_\mu, f] = 0, \qquad f \in C^\infty(C),$$
$$\left[\psi^\mu, \theta_\nu\right] = \delta^\mu_\nu,$$

and the remaining brackets between generators vanish. Similarly, B is generated by $C^\infty(C)$ and the dual basis elements y^μ of V^*. We then have $S_B(\mathrm{Der}(B)[-1]) = A^0[y^\mu, \eta_\mu, \xi_i]$ with $\eta_\mu = \partial/\partial y^\mu$ of degree 1. The Lie bracket is

$$[\xi_i, f] = \frac{\partial f}{\partial x^i}, \quad [\eta_\mu, f] = \left[y^\mu, f\right] = 0, \qquad f \in C^\infty(C),$$
$$\left[\eta_\mu, y^\nu\right] = \delta_{\mu,\nu}.$$

The isomorphism is then the isomorphism of graded commutative algebras over A^0 that on generators is defined by

$$\xi \mapsto \xi, \qquad \theta_\mu \mapsto -\eta_\mu, \qquad \psi^\mu \mapsto y^\mu.$$

To prove the theorem for general vector bundles, we show that both algebras $S_A(\mathrm{Der}(A)[-1])$, $S_B(\mathrm{Der}(B)[-1])$ are (non-canonically) isomorphic, as graded commutative algebras, to $R = \bigoplus_j R^j$, where

$$R^j = \bigoplus_{p,q} \Gamma\big(C, \wedge^p E \otimes S^q E^* \otimes \wedge^{j-p} TC\big).$$

The isomorphisms depend on the choice of a connection ∇ on E. This connection induces a connection, also denoted by ∇ on $\wedge E$ and a dual connection ∇^* on E^* and on SE^*. Note that we have canonical inclusions of A and B into R. The isomorphism $j_A^\nabla : R \to S_A(\mathrm{Der}(A)[-1])$ sends A to A, $\psi \in \Gamma(C, S^1 E^*)$ to the inner multiplication $\iota_\psi \in \mathrm{Der}(A)$ and $\xi \in \Gamma(C, TC)$ to $\nabla_\xi \in \mathrm{Der}(A)$. The isomorphism $j_B^\nabla : R \to S_B(\mathrm{Der}(B)[-1])$ sends B to B, $\eta \in \Gamma(C, \wedge^1 E)$ to $-\iota_\eta$ and $\xi \in \Gamma(C, TC)$ to $\nabla_\xi^* \in \mathrm{Der}(B)$.

Lemma 4.2. *The composition of isomorphisms $j_B^\nabla \circ (j_A^\nabla)^{-1}$ is independent of the choice of connection ∇ and respects the Lie brackets.*

Proof. It is sufficient to show that $j_B^\nabla \circ (j_A^\nabla)^{-1}$ sends $\nabla'_\xi \in \mathrm{Der}(A)$ to ∇'^*_ξ for any other connection ∇'. The difference between two connections is a 1-form with values in $\mathrm{End}(E) = E \otimes E^*$. So we can write $\nabla'_\xi s = \nabla_\xi s + \sum a_i \wedge \iota_{b_i} s$ for some $b_i \in \Gamma(C, E^*)$, $a_i \in \Gamma(C, E)$ (depending on ξ) and any $s \in A$. Thus the isomorphism maps ∇'_ξ to $\nabla^*_\xi - \sum b_i \iota_{a_i}$ which is precisely ∇'^*_ξ on $\Gamma(C, E^*)$ and thus also on B.

With this result it is now easy to show that the isomorphism respects the Lie bracket: as it is sufficient to prove this locally, we may choose a connection which is locally the trivial connection on the trivial bundle and use the local calculation above. □

4.3. The Lie algebra of multidifferential operators

Let A be a graded commutative algebra, $C(A, A)$ the Hochschild cochain complex of A (see A.3). The shifted complex $C(A, A)[1]$ is a differential graded Lie algebra with respect to the Gerstenhaber bracket. It has a subalgebra $\mathcal{D}(A)$ consisting of multidifferential operators, namely sums of cochains of the form $(a_1, \dots, a_p) \mapsto \prod \varphi_i(a_i)$, where φ_i are compositions of derivations. The HKR map $\mathcal{T}(A) \to \mathcal{D}(A)$ induces a homomorphism of Gerstenhaber algebras on the cohomology ($\mathcal{T}(A)$ is considered as a complex with zero differential).

Lemma 4.3. *If $A = \bigoplus_j \Gamma(C, \wedge^j E)$ for a vector bundle $E \to C$, then the HKR map* (3)*, viewed as a map from $\mathcal{T}(A)$ with zero differential to $\mathcal{D}(A)$ with Hochschild differential, induces an isomorphism on cohomology.*

We prove this lemma in Appendix A, see Lemma A.2

4.4. Completions

For our application, in the above construction we take E to be the normal bundle NC to a submanifold $C \subset M$ with vanishing ideal $I(C) = \{f \in C^\infty(M) \mid f|_C = 0\}$. The relevant Lie algebra is then $\mathcal{T}(M, C) = \varprojlim \mathcal{T}(M)/I(C)^n \mathcal{T}(M)$ of multivector fields on a formal neighbourhood of C. Let us fix an identification of NC with a tubular neighbourhood of C as in 2.5. Introduce the graded commutative algebras $A = \bigoplus A^j$, $A^j = \Gamma(C, \wedge^j E)$ and $B = \Gamma(C, S(E^*))$ concentrated in degree 0. Then $\mathcal{T}(M, C)$ may be viewed as the completion $\hat{\mathcal{T}}(B) = \varprojlim \mathcal{T}(B)/I_B^n \mathcal{T}(B)$ of the B-module $\mathcal{T}(B)$. Here I_B is the ideal $\Gamma(C, \bigoplus_{j>0} S^j(E^*))$ of B. Then there is a completion $\hat{\mathcal{T}}(A)$ defined by requiring the isomorphism of Theorem 4.1 to extend to an isomorphism of the completed Lie algebras. This completion is defined using the same ideal $\Gamma(C, \bigoplus_{j>0} S^j(E^*))$ of B, which is now realised as the space of $C^\infty(C)$-multilinear multiderivations of A with values in $C^\infty(C)$. In both cases we have a Gerstenhaber algebra $G = \bigoplus_{j=0}^\infty G^j$ with non-negative grading and a sequence of ideals (for the algebra structure) $I^n \subset G^0$ such that $[I^n, G] \subset I^{n-1}$. It follows that the inverse limit is still a Gerstenhaber algebra.

Proposition 4.4. *The image of a Poisson bracket on a formal neighbourhood of C in M under the* (*completed*) *isomorphism of Theorem* 4.1

$$\mathcal{T}^1(M, C) = \hat{\mathcal{T}}^1(B) \to \hat{\mathcal{T}}^1(A)$$

is the P_∞-structure of Proposition 2.1

Proof. This can be proved in local coordinates using the trivial connection on trivial bundles to describe the isomorphism: the result is that the components of the P_∞-structure are the Taylor expansion coefficients in the transverse coordinates of the components of the Poisson bivector field. □

Now we need to find a completion of the Lie algebra of multidifferential operators in such a way that the HKR map remains an isomorphism.

Definition. The completed Lie algebra of multidifferential operators of $A = \Gamma(C, \wedge E)$ is $\hat{\mathcal{D}}(A) = \bigoplus_n \hat{\mathcal{D}}^n(A)$, where

$$\hat{\mathcal{D}}^n(A) = \prod_{p+q-1=n} \mathrm{Hom}^p\big(A^{\otimes q}, A\big)$$

is the direct product.

The Gerstenhaber Lie bracket of two homogeneous elements $\phi = (\phi_{p,q})_{p+q-1=n}$, $\psi = (\psi_{p,q})_{p+q-1=m}$ has (p,q)-component

$$[\phi, \psi]_{p,q} = \sum [\phi_{p',q'}, \psi_{p'',q''}],$$

where the range of the sum is $p' + p'' = p$, $q' + q'' = q - 1$, $p' + q' - 1 = n$, $p'' + q'' - 1 = m$, $q', q'' \geqslant 0$, so we have a finite sum. Clearly the HKR map extends naturally to an injective map $\hat{\mathcal{T}}(A) \to \hat{\mathcal{D}}(A)$.

Lemma 4.5. *The HKR map extends to a quasiisomorphism $\hat{\mathcal{T}}(A) \to \hat{\mathcal{D}}(A)$.*

Proof. A cochain in $\hat{\mathcal{D}}(A)$ of degree n is a sequence $\phi = (\phi_{p,q})_{p+q-1=n}$ with $\phi_{p,q} \in \mathrm{Hom}^p(A^{\otimes q}, A)$. As the Hochschild differential only shifts q, ϕ is a cocycle if and only if

$b\phi_{p,q} = 0$ for all p, q. By the HKR theorem for $\mathcal{D}(A)$, $\phi_{p,q} = \psi_{p,q} \bmod b\,\mathrm{Hom}^p(A^{\otimes(q-1)}, A)$, for a unique $\psi_{p,q}$ in the image of the HKR map. This implies that $\phi = \psi +$ exact for a unique ψ in the image of the extension of the HKR map to $\hat{\mathcal{T}}(A)$. $\square$

4.5. A graded version of Kontsevich's theorem

Let $E \to C$ be a vector bundle on a smooth manifold C and $A = \bigoplus_{j=0}^{\infty} \Gamma(C, \wedge^j E)$. Let $\mathcal{T}(A) = S_A(\mathrm{Der}(A)[-1])[1]$ be the differential graded Lie algebra of multiderivations of A with Schouten–Nijenhuis bracket. Let $\mathcal{D}(A)$ be the subcomplex of the shifted Hochschild complex $C(A, A)[1]$ consisting of multidifferential operators, with Lie algebra structure given by the Gerstenhaber bracket. In the language of supermanifolds, A is (by definition) the algebra of smooth functions on the supermanifold ΠE^* obtained from E by changing the parity of the fibres and $\mathcal{T}(A)$ is the Lie algebra of multivector fields on ΠE^*.

Theorem 4.6. *There exists an L_∞-quasiisomorphism $U : \mathcal{T}(A) \to \mathcal{D}(A)$ whose first order term U_1 is the Hochschild–Kostant–Rosenberg map* (3)

$$\gamma_1 \cdots \gamma_p \mapsto \frac{1}{p!}\mu \circ \sum_{\sigma \in S_p} (-1)^{\sum_{i<j,\sigma(i)>\sigma(j)} \deg(\gamma_i)\deg(\gamma_j)} \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(p)}.$$

This theorem is implicitly stated in [10]. We give a construction of an L_∞-quasiisomorphism in Appendix A. Composing this L_∞-isomorphism with the Lie algebra isomorphism of 4.2, we obtain the proof of Theorem 4.8.

We need a completed version of this theorem. For this the following property of the L_∞-morphism is important. Suppose $\gamma_1, \ldots, \gamma_n \in \mathcal{T}(A)$ are of order $p_1, \ldots, p_n$, i.e., $\gamma_i \in S^{p_i}(\mathrm{Der}(A)[1])[-1]$. Let $f_1, \ldots, f_m \in A$. Then $U_n(\gamma_1, \ldots, \gamma_n)(f_1, \ldots, f_m)$ vanishes unless

$$\sum_{i=1}^{n} p_i = 2n + m - 2.$$

This condition expresses the fact that the degree of the differential forms appearing in the definition of the weights entering U_n coincides with the dimension of the configuration spaces over which these differential forms are integrated. It follows that for given n and m there are only finitely many values of $(p_1, \ldots, p_n)$ giving a non-trivial contribution and thus all Taylor components U_n are well-defined on $\hat{\mathcal{T}}(A)$. We thus obtain:

Theorem 4.7. *There exists an L_∞-quasiisomorphism $U : \hat{\mathcal{T}}(A) \to \hat{\mathcal{D}}(A)$ whose first order term U_1 is the Hochschild–Kostant–Rosen berg map.*

4.6. The relative formality theorem

Theorem 4.8. *Let $C \subset M$ be a submanifold of a smooth manifold M with vanishing ideal $I(C)$, $A = \Gamma(C, \wedge NC)$ the graded commutative algebra of sections of the exterior algebra of the normal bundle, $\mathcal{T}(M) = \Gamma(M, \wedge TM)$ the DGLA of multivector fields with Nijenhuis–Schouten bracket and zero differential, $\mathcal{T}(M, C) = \varprojlim \mathcal{T}(M)/I(C)^n\mathcal{T}(M)$ the DGLA of multivector fields in an infinitesimal neighbourhood of C. Then there is an L_∞-quasiisomorphism*

$$U : \mathcal{T}(M, C) \to \hat{\mathcal{D}}(A),$$

whose first order term U_1 is the composition

$$\mathcal{T}(M,C) \simeq \hat{\mathcal{T}}(B) \to \hat{\mathcal{T}}(A) \xrightarrow{\varphi_{\mathrm{HKR}}} \hat{\mathcal{D}}(A),$$

where the middle arrow is the Fourier transform isomorphism of Theorem 4.1 *and φ_{HKR} is the HKR map* (3).

Theorem 4.8 follows from Theorems 4.1 and 4.7.

The L_∞-quasiisomorphism induces a bijection between deformation functors (see [10]). In particular, a Poisson bivector field π on M defines a solution $\epsilon\pi$ of the Maurer–Cartan equation $[\epsilon\pi, \epsilon\pi] = 0$ in the pronilpotent Lie algebra $\mathcal{T}(M,C) \otimes \epsilon\mathbb{R}[\![\epsilon]\!]$. This solution is mapped by U to a solution μ of the Maurer–Cartan equation $2b\mu + [\mu, \mu] = 0$ in $\epsilon\hat{\mathcal{D}}(A)[\![\epsilon]\!]$, i.e., a deformation of the product on A as an A_∞-algebra. This proves Theorem 3.2.

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Appendix A. Formality theorem for supermanifolds

This section contains a graded version of Kontsevich's formality theorem, stating that the differential graded Lie algebra of multidifferential operators on a graded super vector spaces is L_∞-quasiisomorphic to its cohomology, the graded Lie algebra of multivector fields. The proof is the same as Kontsevich's proof in the case of ordinary vector spaces. Our contribution is to write all signs and develop a formulation in which the signs (which are already non-trivial in the case of ordinary vector spaces) appear in a transparent way.

A.1. Notations and conventions

We work in the tensor category of graded vector spaces over a field or more generally of graded left modules over a graded commutative ring R with unit. All graded modules shall be meant to be $\mathbb{Z}$-graded and shall be considered as super vector spaces with the induced $\mathbb{Z}/2\mathbb{Z}$-grading. The word super shall usually be omitted. Thus R is a $\mathbb{Z}$-graded commutative ring $R = \bigoplus_{j\in\mathbb{Z}} R^j$ and an object is a $\mathbb{Z}$-graded left R-module $V = \bigoplus V^j$. Morphisms from V to W form a $\mathbb{Z}$-graded left R-module $\mathrm{Hom}(V,W) = \bigoplus_d \mathrm{Hom}^d(V,W)$. We denote by $|a|$ the degree of a homogeneous element a. The Koszul sign rule holds. Thus a homogeneous morphism $\phi \in \mathrm{Hom}(V,W)$ is an additive map obeying $\phi(rv) = (-1)^{|r||\phi|} r\phi(v)$, $r \in R$, $v \in V$; the tensor product $V \otimes W$ of objects is defined as the quotient of the tensor product over $\mathbb{Z}$ by the relation $rv \otimes w = (-1)^{|r||v|} v \otimes rw$, $r \in R$; the tensor products of morphisms $\phi \in \mathrm{Hom}(V,V')$, $\psi \in \mathrm{Hom}(W,W')$ is $\phi \otimes \psi(v \otimes w) = (-1)^{|\psi||v|}\phi(v) \otimes \psi(w)$, $v \in V$, $w \in W$.

For a graded R-module V let $V[n]$ be the graded R-module such that $V[n]^j = V^{n+j}$. We have a tautological map (the identity) $s^n : V[n] \to V$ of degree n.

We often denote by $(v_1, \dots, v_n)$ the element $v_1 \otimes \dots \otimes v_n \in V^{\otimes n} = V \otimes \dots \otimes V$. The symmetric group S_n acts on $V^{\otimes n}$ with signs: so the transposition $s_i = (i, i+1)$ acts as

$$s_i(v_1, \dots, v_n) = (-1)^{|v_i||v_{i+1}|}(v_1, \dots, v_{i+1}, v_i, \dots, v_n).$$

The product of symmetric groups S acts on the tensor algebra $T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$ (with $V^{\otimes 0} = R$) and we have the algebras of coinvariants for the ordinary action $S(V) = T(V)/(x - \sigma x), \sigma \in S$ and for the alternating action $\wedge(V) = T(V)/(x - \operatorname{sign}(\sigma)\sigma x), \sigma \in S$.

A.2. Tensor coalgebras

Let V be a free graded R-module over a commutative unital ring R ($= \mathbb{R}$ or $\mathbb{R}[\![\epsilon]\!]$ in our application). We set $V^{\otimes 0} = R$ and $V^{\otimes j} = V \otimes \dots \otimes V$. The graded counital tensor coalgebra generated by V is the graded R-module $F(V) = \bigoplus_{j \geqslant 0} V^{\otimes j}$ with coproduct

$$\Delta(\gamma_1, \dots, \gamma_n) = \sum_{j=0}^{n} (\gamma_1, \dots, \gamma_j) \otimes (\gamma_{j+1}, \dots, \gamma_n).$$

In the first and last term we have $(\ ) = 1 \in R$. The counit is the canonical projection onto $V^{\otimes 0} = R$. The spaces of invariant tensors $I_n(V) = \{v \in V^{\otimes n} \mid \sigma v = v\ \forall \sigma \in S_n\}$ form a commutative sub-coalgebra $C(V) = \bigoplus_{n \geqslant 0} I_n(V)$ of $F(V)$. It is the symmetric coalgebra generated by V. The quotient $F^0(V)$ by the coideal $R = V^{\otimes 0}$ can be described as the coalgebra $\bigoplus_{j \geqslant 1} V^{\otimes j}$ without counit and whose coproduct is given by the formula above without the first and last term. Similarly we have the coalgebra $C^0(V) = C(V)/R$. The coalgebras $C^0(V)$, $F^0(V)$ are freely generated by V. For $C^0(V)$ this means that if C is a cocommutative coalgebra without counit so that, for each $x \in C$, the iterated coproduct $\Delta^n(x)$ vanishes for n large enough, every linear map $U : C \to V$ is uniquely the composition of a map of coalgebras $\bar{U} : C \to C^0(V)$ with the canonical projection $p_1 : C^0(V) \to V$ on the first direct summand $V = I_1(V)$. The formula for the composition of $\bar{U}$ with the canonical projection $p_n : C^0(V) \to I_n(V)$ is

$$p_n \circ \bar{U}(x) = U \otimes \dots \otimes U\bigl(\Delta^n(x)\bigr).$$

For $F(V)$, $C(V)$ we will need the following infinitesimal version of this fact in a special case.

Lemma A.1. *Let $Q : F(V) \to V$ be a linear map, $p_n : F(V) \to V^{\otimes n}$ the canonical projection onto the nth summand. Then there is a unique coderivation $\bar{Q} : F(V) \to F(V)$ such that $p_1 \circ \bar{Q} = Q$. The same holds for in the cocommutative case for $C(V)$ with projections $p_n : C(V) \to I_n(V)$.*

The formula for the components of $\bar{Q}_n = \bar{Q}|_{V^{\otimes n}}$ (or $\bar{Q}|_{I_n(V)}$) in terms of the components of Q is

$$\bar{Q}_n = \sum_{m=0}^{n} \sum_{l=0}^{n-m} 1^{\otimes l} \otimes Q_m \otimes 1^{\otimes n-m-l},$$

where $1^{\otimes l}$ denotes the identity on $V^{\otimes l}$.

Closely related to $C(V)$ is the shuffle coalgebra $S(V)$, the R-module of coinvariants $\bigoplus V^{\otimes j}/\{\sigma v - v, \sigma \in S_j\}$ with shuffle coproduct

$$\Delta_{\mathrm{sh}}(\gamma_1, \ldots, \gamma_n) = \sum_{p+q=n} \sum_{(p,q)\text{-shuffles}} \pm(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(p)}) \otimes (\gamma_{\sigma(p+1)}, \ldots, \gamma_{\sigma(n)}).$$

The sum is over permutations such that $\sigma(1) < \cdots < \sigma(p)$, $\sigma(p+1) < \cdots < \sigma(n)$ with sign

$$\pm = \varepsilon(\sigma, \gamma_1, \ldots, \gamma_n) = (-1)^{\sum_{i<j, \sigma(i)>\sigma(j)} |\gamma_i|\cdot|\gamma_j|}. \tag{4}$$

The map $S(V) \to C(V)$ sending $(\gamma_1, \ldots, \gamma_n)$ to $\sum_{\sigma \in S_n} \pm(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$ is then an isomorphism of coalgebras.

Although the language of coalgebras is technically convenient, it is better for the intuition to think in terms of the dual algebras. If R is the field of real or complex numbers, the dual space $C(V)^* = \mathrm{Hom}(C(V), R)$ is the algebra of jets at zero of functions on the (super)manifold V and $C^0(V)^*$ the subalgebra of functions vanishing at 0 and a coderivation of $C(V)$ is a formal vector field. The algebras $F(V)^*$, $F^0(V)^*$ are the corresponding non-commutative analogues.

A.3. The Hochschild complex of a graded algebra

Let $A = \bigoplus_{j \in \mathbb{Z}} A^j$ be a graded associative algebra with unit over a field k. The Hochschild complex $C(A, A)$ with values in A, is the complex $C(A, A) = \bigoplus_n C^n(A, A)$ where

$$C^n(A, A) = \bigoplus_{m+d=n} C^{d,m}(A, A), \qquad C^{d,m}(A, A) = \mathrm{Hom}^d\big(A^{\otimes m}, A\big).$$

The Hochschild differential of $\phi \in C^{|\phi|,m}(A, A)$ is

$$\begin{aligned} b\phi(a_1, \ldots, a_{m+1}) = (-1)^{|\phi||a_1|} a_1 \phi(a_2, \ldots, a_{m+1}) \\ + \sum_{j=1}^{m} (-1)^j \phi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{m+1}) \\ + (-1)^{m+1} \phi(a_1, \ldots, a_m) a_{m+1}. \end{aligned} \tag{5}$$

The shifted Hochschild complex $C(A, A)[1]$ is a differential graded Lie algebra whose Lie bracket is (a graded version of) the Gerstenhaber bracket: let $\phi \in C^{|\phi|,m_1}(A, A)$, $\psi \in C^{|\psi|,m_2}(A, A)$.

$$[\phi, \psi]_{\mathrm{G}} = \phi \bullet \psi - (-1)^{(|\phi|+m_1-1)(|\psi|+m_2-1)} \psi \bullet \phi, \tag{6}$$

with Gerstenhaber product[3]

$$\phi \bullet \psi = (-1)^{(|\psi|+m_2-1)(m_1-1)} \sum_{l=0}^{m_1-1} (-1)^{l(m_2-1)} \phi \circ \big(1^{\otimes l} \otimes \psi \otimes 1^{\otimes(m_1-1-l)}\big).$$

[3] In the ungraded case, this product differs by a factor $(-1)^{(m_1-1)(m_2-1)}$ from the product defined in [7]. With our convention we obtain a more standard bracket on multivector fields.

We also have the cup product on $C(A,A)$: $\phi_1 \cup \phi_2 = \mu \circ \phi_1 \otimes \phi_2$, where μ is the product in A.

The simplest way to prove that the Gerstenhaber bracket is a Lie bracket is to use its interpretation as a commutator of coderivations [17]. This also explains the origin of the signs: a coderivation of degree d of a coalgebra C is a linear endomorphism $\phi \in \mathrm{Hom}^d(C,C)$ obeying $\Delta \circ \phi = (\phi \otimes 1 + 1 \otimes \phi) \circ \Delta$. Coderivations of C form a graded Lie algebra $\mathrm{Coder}(C)$ with respect to the graded commutator $\phi \circ \psi - (-1)^{|\phi||\psi|} \psi \circ \phi$. Let $C = F(A[1])$. Then a coderivation ϕ of C is uniquely determined by its composition $p_1 \circ \phi$ with the canonical projection onto $A[1]$ and every map $F(A[1]) \to A[1]$ extends to a coderivation. Thus we can identify derivations of C with maps $F(A[1]) \to A[1]$. Under this identification, the Lie bracket is

$$[\phi, \psi] = \sum_{l=0}^{m_1-1} \phi \circ \big(1^{\otimes l} \otimes \psi \otimes 1^{\otimes(m_1-1-l)}\big) - (-1)^{|\phi||\psi|}(\phi \leftrightarrow \psi).$$

if $\phi \in \mathrm{Hom}(A[1]^{\otimes m_1}, A[1])$ and $\psi \in \mathrm{Hom}(A[1]^{\otimes m_2}, A[1])$. Let $s : A[1] \to A$ be the tautological map (of degree 1) and introduce $\tilde{\phi}$ by $\phi = s^{-1} \circ \tilde{\phi} \circ (s \otimes \cdots \otimes s)$. Then the Gerstenhaber bracket is

$$[\tilde{\phi}, \tilde{\psi}]_G = \widetilde{[\phi, \psi]},$$

and the signs are obtained by the Koszul rule when letting the maps s go past ϕ, ψ and other maps s.

The Hochschild differential can also be expressed in terms of the bracket: let $\mu : A \otimes A \to A$ denote the product in A. Then the associativity is the relation $[\mu, \mu]_G = 0$. It follows that $[\mu, \cdot]$ is a differential and indeed

$$b\phi = (-1)^{|\phi|}[\mu, \phi] = -[\phi, \mu].$$

The cohomology of $C(A,A)$ is denoted by $HH(A,A)$. The Gerstenhaber bracket induces a graded Lie algebra structure on $HH(A,A)[1]$. In terms of homological algebra, $HH(A,A) = \mathrm{Ext}_{A-A}(A,A)$ is the Ext group of A in the category of $A-A$-bimodules over the graded algebra A. Indeed $C(A,A) \simeq \mathrm{Hom}_{A-A}(B(A), A)$, where $B(A) = \bigoplus_j (A \otimes A^{\otimes j} \otimes A)$ is the bar resolution, with degree assignment

$$|a \otimes a_1 \otimes \cdots \otimes a_j \otimes b| = |a| + \sum_{i=1}^{j} |a_i| + |b| - j.$$

The sign in (5) comes from the Koszul rule for morphisms $\phi : M \to N$ of $A-A$-bimodules:

$$\phi(amb) = (-1)^{|a||\phi|} a\phi(m)b, \quad a, b \in A,\ m \in M.$$

A.4. HKR cocycles

Let A be a graded commutative algebra over a field k of characteristic zero, $C(A,A) = \bigoplus_{n \geqslant 0} \mathrm{Hom}(A^{\otimes n}, A)$ the Hochschild cochain complex of A.

A cochain $\phi \in C(A, A)$ is called an *HKR-cocycle* if (i) the map $a \mapsto \phi(a_1, \dots, a_{n-1}, a)$ is a derivation of A for any homogeneous $a_1, \dots, a_{n-1} \in A$ and (ii) ϕ is alternating in the graded sense, i.e.,

$$\phi(a_1, \dots, a_i, a_{i+1}, \dots, a_n) = -(-1)^{|a_i||a_{i+1}|}\phi(a_1, \dots, a_{i+1}, a_i, \dots, a_n).$$

It is easy to check that cochains obeying (i) and (ii) are indeed cocycles. They thus form a subcomplex $C_{\mathrm{HKR}}(A, A)$ with zero differential that can be identified with $\wedge_A \mathrm{Der}(A)$ via the map $\phi_1 \wedge \dots \wedge \phi_n \mapsto \mu \circ \mathrm{Alt}(\phi_1 \otimes \dots \otimes \phi_n)$ for derivations ϕ_j; here $\mu : A^{\otimes n} \to A$ is the product.

A cochain $\phi \in C(A, A)$ is called *multidifferential operator* if it is a sum of terms of the form $a_1 \otimes \dots \otimes a_n \mapsto D_1(a_1) \cdots D_n(a_n)$, for some differential operators (compositions of derivations) D_i. Multidifferential operators form a subcomplex $C_{\mathrm{Diff}}(A, A)$ of $C(A, A)$ containing $C_{\mathrm{HKR}}(A, A)$.

Let N be a graded supermanifold. For us this means that the ground field k is $\mathbb{R}$ (or $\mathbb{C}$) and $N = E^*$ is the total space of the dual of a graded vector bundle $E \to C$ so that $C^\infty(N) = \Gamma(C, S(E))$, where $S(E) \to C$ is the graded symmetric algebra of E (thus $S(E) = S(E^{\mathrm{even}}) \otimes \wedge(E^{\mathrm{odd}})$).

Lemma A.2. *If $A = C^\infty(N)$ for a graded supermanifold N the HKR map $\wedge_A \mathrm{Der}(A) \simeq C_{\mathrm{HKR}}(A, A) \hookrightarrow C_{\mathrm{Diff}}(A, A)$ is a quasiisomorphism of complexes.*

In the ungraded case ($E = 0$) a version of this theorem can be found in [19], see also [10]. It is an analogue for smooth functions of the original HKR theorem [8], which deals with regular affine algebras.

The proof is the same as the proof in the ungraded case (see [10, 4.4.1.1]) but with some twists. First one uses the filtration by the total order of multidifferential operators to pass to the associated graded complexes of principal symbols. These complexes are sections of vector bundles and the differential is $C^\infty(N)$-linear, so the problem is reduced to proving a version of the HKR theorem for each fibre. If $T = T_xM \oplus E_x$ is a tangent space to N, the complex of principal symbols at a point $x \in M$ is $\bigoplus_{n \geqslant 0} S(T)^{\otimes n}$ with degree assignment $|D_1 \otimes \dots \otimes D_n| = \sum |D_i| + n$, $D_i \in S(T)$. An element of $S(T)$, considered as a differential operator with constant coefficient, defines a linear function on the algebra $S(T^*)$ of polynomial functions on the graded vector space T. Thus we obtain an embedding $\bigoplus_{n \geqslant 0} S(T)^{\otimes n} \to \mathrm{Hom}_k(S(T^*)^{\otimes n}, k)$ as a subcomplex. The differential on $\mathrm{Hom}_k(S(T^*)^{\otimes n}, k)$ is

$$\begin{aligned} d\varphi(f_1, \dots, f_{n+1}) = {} & (-1)^{|f_1||\varphi|}\epsilon(f_1)\varphi(f_2, \dots, f_{n+1}) \\ & + \sum_{j=1}^{n} (-1)^j \varphi(f_1, \dots, f_j f_{j+1}, \dots, f_{n+1}) \\ & - (-1)^n \varphi(f_1, \dots, f_n). \end{aligned}$$

Here $\epsilon(f) = f(0)$.

Lemma A.3. *The map of complexes $(\wedge T, 0) \to (\bigoplus_{n \geqslant 0} S(T)^{\otimes n}, d)$ sending $t_1 \wedge \dots \wedge t_n$ to $\mathrm{Alt}(t_1 \otimes \dots \otimes t_n)$ is a quasiisomorphism.*

Proof. Let $S = S(T^*)$ and view both complexes as subcomplexes of $(\bigoplus_j C^j(S,k), d)$,

$$C^j(S,k) = \prod_{p+q=j} \mathrm{Hom}^p\big(S^{\otimes q}, k\big).$$

In particular, $(\wedge T, 0)$ is identified with the subcomplex $C_{\mathrm{HKR}}(S,k)$ consisting of cochains obeying (i) and (ii) above. We show that the embedding of this subcomplex is a quasiisomorphism. Since the complex $C(S,k)$ is a direct product of subcomplexes consisting of multidifferential operators of fixed total order, it then follows that the same statement holds if we replace $C(S,k)$ by its subcomplex $S(T)$.

To compute the cohomology of $C(S,k)$ we first notice that it is $\mathrm{Ext}_{\text{mod-}S}(k,k)$, where k is considered as a right S-module via ϵ and has a free resolution $\cdots \to S^{\otimes 3} \to S^{\otimes 2} \to S \to k$ with differential

$$a_1 \otimes \cdots \otimes a_{n+1} \to \epsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1} - \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1},$$

inducing the differential above. This Ext group (in the category of graded right S-modules) can be computed using a graded version of the Koszul resolution of the S-module k: let $v_1, \ldots, v_n$ be a homogeneous basis of the graded vector space T^* so that S is the graded polynomial algebra $k[v_1, \ldots, v_n]$. Let $K = S[u_1, \ldots, u_n] = k[v_1, \ldots, v_n, u_1, \ldots, u_n]$ be the differential graded commutative algebra with u_i of degree $|v_i| + 1$ and differential ∂ such that $\partial u_i = v_i$, $\partial v_i = 0$. Then K is a free S-module and the map $(K, \partial) \to (k[u_1, \ldots, u_n], 0)$ is a quasiisomorphism. The proof of the latter statement is similar to the one in the ungraded case (see, e.g., [14, VII.2]): since K is the (graded) tensor product of algebras $k[v_i, u_i]$, it is sufficient to check this for $n = 1$. In this case there is a homotopy $h : K \to K$ of degree 1 obeying $h\partial + \partial h = \mathrm{id} - \epsilon$ from which the claim follows immediately: if $x = v_i$ is even, $h(x^p) = x^{p-1}u$ $(p \geqslant 1)$, $h(1) = 0 = h(x^p u)$; if x is odd, $h(u^p) = u^{p+1}/(p+1)$, $h(xu^p) = 0$. Thus $\mathrm{Ext}_S(k,k)$ is the cohomology of

$$\mathrm{Hom}_S(K,k) = \mathrm{Hom}_k\big(k[u_1, \ldots, u_n], k\big).$$

Since the induced differential vanishes identically (v_i acts by zero on k), we obtain

$$\mathrm{Ext}^j_{S(T^*)}(k,k) = \mathrm{Hom}^j_k\big(k[u_1, \ldots, u_n], k\big).$$

This space may be identified with $\wedge T$. To find the map, we need to write the map between the two resolutions, which is known to exist from abstract nonsense. Its explicit expression is

$$u_{i_1} \cdots u_{i_n} a \mapsto (-1)^{\sum_{\alpha=1}^{n} (\alpha-1) d(i_\alpha)} \mathrm{Alt}(v_{i_1} \otimes \cdots \otimes v_{i_n}) \otimes a, \quad a \in S,$$

where $d(i) = |v_i|$ is the degree of v_i. The claim of the lemma then follows from the fact that the restriction to $C^j_{\mathrm{HKR}}(S,k)$ of the dual map $\bigoplus_{p+q=j} \mathrm{Hom}^p(S^{\otimes q}, k) \to \mathrm{Hom}^j(k[u_1, \ldots, u_n], k)$ is an isomorphism.

A.5. Expressions in local coordinates

Let $A = S(V)$ be the algebra of polynomial functions on a finite dimensional graded real vector space V^*. If $(x_i)_{i=1}^d$ is a homogeneous basis of degrees $\epsilon_i = |x_i|$, then $S(V)$ is the free graded commutative algebra $\mathbb{R}[x_1, \dots, x_d]$ generated by the x_i's. The Gerstenhaber algebra $S_A(\mathrm{Der}(A)[-1])$ may then be identified $\tilde{A} = S(V \oplus V[1]^*) = k[x_1, \dots, x_n, \theta_1, \dots, \theta_n]$ where θ_i is the dual basis with degrees $|\theta_i| = 1 - \epsilon_i$. Write a general element of $\tilde{A}^{|\gamma|} = \tilde{A}[1]^{|\gamma|-1}$ with the summation convention as

$$\gamma = \gamma^{i_1 \cdots i_m} \theta_{i_1} \cdots \theta_{i_m}, \qquad |\gamma| = |\gamma^{i_1 \cdots i_m}| + m - \sum \varepsilon_{i_\alpha},$$

with $\gamma^{\dots,i,j,\dots} = (-1)^{(1-\varepsilon_i)(1-\varepsilon_j)} \gamma^{\dots,j,i,\dots} \in A$. The HKR map is then

$$\gamma \mapsto (-1)^{\sum_{\alpha=1}^m (\alpha-1)\varepsilon_{i_\alpha}} \gamma^{i_1 \cdots i_m} \mu \circ (\partial_{i_1} \otimes \cdots \otimes \partial_{i_m}).$$

Here $\mu(f_1, \dots, f_m) = f_1 \cdots f_m$ is the product in A.

The Lie algebra structure on $S(V) \otimes S(V[1]^*) = S(V \oplus V[1]^*)$ induced by the HKR homomorphism may be understood geometrically as the Poisson structure on the functions on the degree-shifted cotangent bundle $T^*[1]M$ of the supermanifold $M = V^*$ with its canonical odd symplectic structure. Here is the explicit description. On $\tilde{A}$ there is a Poisson bracket of degree -1:

$$[\gamma_1, \gamma_2] = \sum_{i=1}^d \bigl(\gamma_1 \overleftarrow{\partial}_{\theta_i} \overrightarrow{\partial}_{x_i} \gamma_2 - \gamma_1 \overleftarrow{\partial}_{x_i} \overrightarrow{\partial}_{\theta_i} \gamma_2\bigr).$$

The operator $\overleftarrow{\partial}_x$ is the right partial derivative with respect to x acting on the argument on its left as a right derivation $(ab)\overleftarrow{\partial}_x = a(b\overleftarrow{\partial}_x) + (-1)^{|b||x|}(a\overleftarrow{\partial}_x)b$. The left derivatives $\overrightarrow{\partial}_x$ are defined in the same way and act to the right. In more standard notation (using only left derivatives),

$$[\gamma_1, \gamma_2] = \sum_{i=1}^d (-1)^{(1-\varepsilon_i)(|\gamma_1|-1)} \partial_{\theta_i} \gamma_1 \partial_{x_i} \gamma_2 - (-1)^{\varepsilon_i(|\gamma_1|-1)} \partial_{x_i} \gamma_1 \partial_{\theta_i} \gamma_2.$$

Shifting degrees we obtain thus a Lie algebra $\mathcal{T}(A) = \tilde{A}[1]$, the graded Lie algebra of *multivector fields*.

A.6. Q-manifolds and L_∞-algebras

We use the language of (formal, pointed) Q-manifolds. Let $V = \bigoplus_{j \in \mathbb{Z}} V^j$ be a graded real vector space. Let $C^0(V) = \bigoplus_{j=1}^\infty I_j(V)$ be the free cocommutative coalgebra without counit generated by V. Its dual is the algebra of functions in an infinitesimal neighbourhood of 0 in V. A (formal, pointed) Q-manifold is a graded vector space V with a coderivation Q of $C^0(V)$ of degree 1 obeying $[Q, Q] = 0$. Dually, Q may be thought of as a vector field of degree 1 defined on a formal neighbourhood of 0 in the supermanifold V and vanishing at 0. A morphism $U : (V, Q) \to (V', Q')$ of Q-manifolds is a coalgebra morphism $C^0(V) \to C^0(V')$ of degree 0 obeying $Q' \circ U = U \circ Q$.

In explicit terms, a coderivation Q of $C^0(V)$ is uniquely determined by its composition $p_1 \circ Q$ with the canonical projection $p_1 : C(V) \to V$ sending $I_j(V)$ to 0 for $j \neq 1$, see Lemma A.1. The restriction of $p_1 \circ Q$ to $I_j(V) = (V^{\otimes j})^{S_j}$ is a map $Q_j : I_j(V) \to V$ of degree 1, the jth Taylor component. The condition $[Q, Q] = 0$ is then equivalent to

$$\sum_{j+k=n} \sum_{l=0}^{j-1} Q_j \circ \left(1^{\otimes l} \otimes Q_k \otimes 1^{\otimes (j-l-1)}\right) = 0,$$

on $I_n(V)$, $n = 1, 2, \ldots$. Similarly, a coalgebra morphism U is uniquely determined by its Taylor components $U_j = p_1 \circ U|_{I_j(V)})$ $(j \geqslant 1)$. The Q-manifold morphism property is then

$$\sum_{j_1+\cdots+j_k=n} Q_k \circ (U_{j_1} \otimes \cdots \otimes U_{j_k}) = \sum_{j+k-1=n} \sum_{l=0}^{j-1} U_j \circ \left(1^{\otimes l} \otimes Q_k \otimes 1^{\otimes (j-l-1)}\right)$$

on $I_n(V)$.

If $\mathfrak{g}$ is a differential graded Lie algebra, then $\mathfrak{g}[1]$ (with $\mathfrak{g}[1]^i = \mathfrak{g}^{i+1}$) is a Q-manifold: the Taylor components of the coderivation vanish except Q_1 and Q_2, which are given in terms of the differential d and the bracket by

$$Q_1 = d, \qquad Q_2(\gamma_1, \gamma_2) = (-1)^{|\gamma_1|}[\gamma_1, \gamma_2], \quad \gamma_i \in \mathfrak{g}[1]^{|\gamma_i|} = \mathfrak{g}^{|\gamma_i|+1}$$

(a more pedantically correct notation for the right-hand side of the equation for Q_2 would be $(-1)^{|\gamma_1|} s^{-1}[s\gamma_1, s\gamma_2]$).

Definition. A flat L_∞-algebra structure on a vector space (or R-module) $\mathfrak{g}$ is a Q manifold structure on $\mathfrak{g}[1]$.

It is convenient to express a flat L_∞-algebra in terms of the structure maps $\tilde{Q}_j \in \mathrm{Hom}^{2-j}(\wedge^j \mathfrak{g}, \mathfrak{g})$ (differential and higher Lie brackets) of $\mathfrak{g}$. They are the Q_j up to sign. The precise relation is most easily expressed using the tautological map $s : \mathfrak{g}[1] \to \mathfrak{g}$ of degree 1. We then have $\tilde{Q}_j = s^{-1} \circ Q_j \circ (s \otimes \cdots \otimes s)$. Note that $s^{\otimes j}$ intertwines the action of S_n on $\mathfrak{g}[1]^{\otimes j}$ with the alternating action of S_n on $\mathfrak{g}^{\otimes j}$: $s^{\otimes j}\sigma = \mathrm{sign}(\sigma)\sigma s^{\otimes j}$, $s \in S_j$. Thus if Q_j are symmetric $\tilde{Q}_j$ are skew-symmetric. Explicitly,

$$Q_j(\gamma_1, \ldots, \gamma_j) = (-1)^{\sum_{i=1}^{j}(j-i)|\gamma_i|} s^{-1} \tilde{Q}_j(s\gamma_1, \ldots, s\gamma_j), \quad \gamma_i \in \mathfrak{g}[1].$$

A.7. The local formality theorem

Theorem A.4. *Let $A = \mathbb{R}[x_1, \ldots, x_d]$ be the graded commutative algebra generated by x_i of degree ε_i, $i = 1, \ldots, d$, or its completion $A = \mathbb{R}[\![x_1, \ldots, x_d]\!]$. Then there exists a morphism of formal pointed Q-manifolds $U : \mathcal{T}(A)[1] \to \mathcal{D}(A)[1]$ such that U_1 is the HKR quasiisomorphism.*

A.8. The construction of the L_∞-morphism

The condition for the Taylor components $(U_n)_{n\geqslant 1}$ of a morphism of Q-manifolds U from $\mathcal{T}(A)[1] \to \mathcal{D}(A)[1]$ are simplified if we add a component $U_0 = \mu_A$, the product in A. Geometrically, this means that we shift the origin of the Q-manifold $\mathcal{D}(A)$ by U_0 to a point where the vector field Q is purely quadratic. The conditions for $(U_n)_{n\geqslant 0}$ to be satisfied are then

$$\sum_{n_1+n_2=n} Q_2 \circ (U_{n_1} \otimes U_{n_2}) = \sum_{l=0}^{n-2} U_{n-1} \circ \big(1^{\otimes l} \otimes Q_2 \otimes 1^{\otimes(n-l-2)}\big), \quad n \geqslant 1, \tag{7}$$

on the space of symmetric tensors $I_n(\mathcal{T}(A)[1])$. Replacing $\bigoplus I_n$ by the isomorphic shuffle coalgebra (see A.2), we can write this as

$$\begin{aligned} &\sum_{p+q=n} \sum_{(p,q)\text{-shuffles}} \pm Q_2\big(U_p(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(p)}), U_q(\gamma_{\sigma(q+1)}, \dots, \gamma_{\sigma(n)})\big) \\ &\quad = \sum_{i<j} (-1)^{\varepsilon_{ij}} U_{n-1}\big(Q_2(\gamma_i, \gamma_j), \gamma_1, \dots, \hat{\gamma}_i, \dots, \hat{\gamma}_j, \dots, \gamma_n\big), \\ &\varepsilon_{ij} = \sum_{k=1}^{i-1} |\gamma_i| \cdot |\gamma_k| + \sum_{k=1}^{j-1} |\gamma_j| \cdot |\gamma_k| - |\gamma_i| \cdot |\gamma_j|, \end{aligned} \tag{8}$$

and $\pm$ is the same sign as in (4). The maps U_n are sums of integrals over configuration spaces on the upper half-plane H. Let n, m be non-negative integers such that $2n + m \geqslant 2$. Let $\mathrm{Conf}^+_{n,m} = \{(z,x) \in H^n_+ \times \mathbb{R}^m \mid z_i \neq z_j\ (i \neq j),\ x_1 < \cdots < x_m\}$, with orientation form $d^2z_1 \cdots d^2z_n\, dx_1 \cdots dx_m$, where $d^2z_i = d\,\mathrm{Re}(z_i)\, d\,\mathrm{Im}(z_i)$. The group G_2 of affine transformations $z \to \lambda z + a$, $\lambda > 0, a \in \mathbb{R}$ acts freely on $\mathrm{Conf}^+_{n,m}$ since $2n + m \geqslant 2$ and preserves the orientation. Let the orientation of G_2 be defined by the volume form $da \wedge d\lambda$. The quotient $\mathcal{C}^+_{n,m}$ is oriented as in [1] in such a way that any trivialisation $G_2 \times \mathcal{C}^+_{n,m} \to \mathrm{Conf}^+_{n,m}$ of the left principal G_2-bundle $\mathrm{Conf}_{n,m}$ is orientation preserving. Here is an explicit description: if $n \geqslant 1$, $\mathcal{C}^+_{n,m}$ may be identified with the submanifold of $\mathrm{Conf}^+_{n,m}$ consisting of points with $z_1 = i$ and orientation form $d^2z_2 \cdots d^2z_n\, dx_1 \cdots dx_m$. If $m \geqslant 2$ it can be identified with the submanifold given by $x_1 = 0$, $x_m = 1$, with orientation form $(-1)^m d^2z_1 \cdots d^2z_n\, dx_2 \cdots dx_{m-1}$. Let $\mathcal{G}_{n,m}$ be the set of graphs $\Gamma = (V_\Gamma, E_\Gamma)$ with the following properties: the set of vertices $V_\Gamma = \{1, \dots, n, \bar{1}, \dots, \bar{m}\}$ consists of vertices of the first type $1, \dots, n$ and vertices of the second type $\bar{1}, \dots, \bar{m}$; the edges $(i,j) \in E_\Gamma \subset V_\Gamma \times V_\Gamma$ are such that i is always of the first type and $i \neq j$.

Let moreover $\tau \in \mathrm{End}(\tilde{A} \otimes \tilde{A})$ be the endomorphism

$$\tau = \sum_{\alpha=1}^{d} (-1)^{\varepsilon_\alpha} \partial_{\theta_\alpha} \otimes \partial_{x_\alpha}.$$

To each graph $\Gamma \in \mathcal{G}_{n,m}$ we associate an element ω_Γ of the graded algebra $\Omega(\mathcal{C}^+_{n,m}) \otimes \mathrm{End}(A_{n,m})$ of differential forms with values in the endomorphisms of

$$A_{n,m} = \tilde{A}^{\otimes n} \otimes A^{\otimes m}.$$

Let the factors in the tensor product $A_{n,m}$ be numbered $1, \dots, n, \bar{1}, \dots, \bar{m}$ and for $i, j \in V_\Gamma$ let $\tau_{ij} \in \mathrm{End}(A_{n,m})$ the endomorphism acting as τ on the factors i and j and as the identity on the other factors:

$$\tau_{ij} = \sum_\alpha (-1)^{\varepsilon_\alpha} 1 \otimes \cdots \otimes 1 \otimes \partial_{\theta_\alpha} \otimes 1 \otimes \cdots \otimes 1 \otimes \partial_{x_\alpha} \otimes 1 \otimes \cdots \otimes 1.$$

Set $d\phi_{ij} \in \Omega^1(\mathcal{C}^+_{n,m})$ the differential of the Kontsevich angle function $\phi(z_i, z_j) = \arg(z_i - z_j) - \arg(\bar{z}_i - z_j)$ (with $z_{\bar{k}} = x_k$ if $\bar{k}$ is of the second type). Both τ_{ij} and $d\phi_{ij}$ are elements of the algebra $\Omega(\mathcal{C}^+_{n,m}) \otimes \mathrm{End}(A_{n,m})$ of degree 1 and -1, respectively. Thus their product is of degree 0 and

$$\omega_\Gamma = \prod_{(i,j)\in E_\Gamma} d\phi_{ij} \tau_{ij}$$

is independent of the choice of ordering of factors.

Let U_n be the map $\bigoplus_{m\geqslant 0}(\tilde{A}^{\otimes n} \otimes A^{\otimes m}) \to A$

$$U_n = \sum_{m\geqslant 0} (-1)^{(\sum |\gamma_i| - 1)m} \sum_{\Gamma \in \mathcal{G}_{n,m}} U_\Gamma, \tag{9}$$

where

$$U_\Gamma = \mu \int_{\mathcal{C}^+_{n,m}} \prod_{(i,j)\in E_\Gamma} d\phi_{ij} \tau_{ij}$$

is the composition

$$\left(\tilde{A}^{\otimes n} \otimes A^{\otimes m}\right) \xrightarrow{\omega_\Gamma} \Omega\left(\mathcal{C}^+_{n,m}\right) \otimes \left(\tilde{A}^{\otimes n} \otimes A^{\otimes m}\right) \to \tilde{A}^{\otimes n} \otimes A^{\otimes m} \xrightarrow{\epsilon\mu} A.$$

The second map is the integration $\omega \otimes a \to (\int \omega) a$ (of degree $-2n - m + 2$) and is defined to be zero on differential forms of the wrong degree. Thus U_Γ on $\tilde{A}^{\otimes n} \otimes A^{\otimes m}$ vanishes unless the number of edges is

$$|E_\Gamma| = 2n + m - 2.$$

The map $\epsilon\mu : \tilde{A}^{\otimes n} \otimes A^{\otimes m} \to A$ is the product in $\tilde{A}$ followed by the projection $\epsilon : \tilde{A} \to A$ sending θ_i to 0.

Proposition A.5. *The maps U_n are Taylor components of a morphism U with the properties stated in Theorem* A.4.

The proof of this theorem is based on the Stokes theorem as in [10]. The quadratic relation (7) are obtained from a sequence of relations for integrals over configuration spaces associated to graphs. Let $\Gamma \in G_{n,m}$ be a graph such that

$$|E_\Gamma| = 2n + m - 3.$$

Then $d\omega_\Gamma$ (which vanishes) is a form of degree $2n + m - 2$ and we have the Stokes theorem on the Kontsevich compactification $\bar{C}^+_{n,m}$ of $C^+_{n,m}$:

$$0 = \int_{\bar{C}^+_{n,m}} d\omega_\Gamma = \sum_i \int_{\partial_i \bar{C}^+_{n,m}} \omega_\Gamma.$$

The sum is over the faces of the manifold with corners $\bar{C}^+_{n,m}$. The faces contributing non-trivially are of two types.

(a) Faces of the first type are diffeomorphic to $C^+_{n',m'} \times C^+_{n'',m''}$ with $n' + n'' = n$ and $m' + m'' = m + 1$ and correspond to limiting configurations where n' points in H and m' consecutive points with labels $\overline{l+1}, \dots, \overline{l+m'}$ on the real line converge to a single point. The orientation from the Stokes theorem differs from the product orientation by a factor $(-1)^{lm'+l+m'}$, as computed in [1].

(b) Faces of the second type are diffeomorphic to $C_{n'} \times C^+_{n'',m}$. with $n' + n'' = n - 1$ and correspond to limiting configurations where n' points in H converge to the same point in the upper half plane. Here the relative position of these n' collapsing points is parametrised by the manifold $C_{n'}$ ($n' \geqslant 2$), the quotient of $\mathbb{C}^{n'} \setminus \bigcup_{i<j}\{z_i = z_j\}$ by the group G_3 of affine transformations $z \mapsto \lambda z + a$ with $\lambda > 0$ and $a \in \mathbb{C}$ and orientation form $d^2 a \wedge d\lambda$. By Kontsevich's lemma (see [10, Lemma 6.6]) the integrals over these faces vanish except for $n' = 2$. In this case the induced orientation on the face differs by the product orientation by a factor -1, see [1].

The faces of the first type will contribute to the expression on the left-hand side of (8) as in [10]. We need to keep track of the signs. Let us consider the case of a face of the first type in which the n' points in the upper half-plane collapsing to a point on the real axis are the last n' points; this corresponds to the trivial shuffle in (8). The remaining shuffles are treated similarly or can be related to the trivial one by permutation symmetry considerations. Let us denote accordingly

$$|\gamma''| = \sum_{i=1}^{n''} |\gamma_i|, \qquad |\gamma'| = \sum_{i=n''+1}^{n} |\gamma_i|, \qquad |\gamma| = |\gamma'| + |\gamma''| = \sum_1^n |\gamma_i|.$$

The sign with which the integral of ω_Γ over this face contributes to the term

$$U_{n''}(\gamma_1, \dots, \gamma_{n''})\bigl(1^{\otimes l} \otimes U_{n'}(\gamma_{n''+1}, \dots, \gamma_n) \otimes 1^{\otimes m''-1}\bigr), \tag{10}$$

appearing (with a certain sign we give below) in the left-hand side of (8) is

$$(-1)^{lm'+m'+l}(-1)^{|\gamma''|m'}(-1)^{-(|\gamma'|-1)m'-(|\gamma''|-1)m''}.$$

The first sign of this product comes from comparing the orientations, as discussed above, the second from moving γ_i, $i \leqslant n''$, to the left of $\prod_{ij\in E_{\Gamma'}} \tau_{ij}$ ($|E_{\Gamma'}| \equiv m' \bmod 2$), the third appears in the definition of $U_{n'}$, $U_{n''}$. The same term (10) appears in the left-hand side of (8) with a sign

$$(-1)^{|\gamma''|-1}(-1)^{(|\gamma'|-1)(m''-1)+(m'-1)l},$$

which is the product of the sign coming from comparing Q_2 to the Gerstenhaber bracket and the sign appearing in the definition of the Gerstenhaber bracket. The ratio between these signs is

$$(-1)^{|\gamma|(m'+m''-1)} = (-1)^{|\gamma|m},$$

which is the sign with which the considered face contributes to the left-hand side of (8). Let us turn to the right-hand side: the face in which the first two points in the upper half-plane collapse contributes to the term right-hand side of (8) with $i = 1$, $j = 2$ with the same sign $(-1)^{|\gamma|m}$ which is the sign appearing in the definition of U_{n-1} by taking into account the fact that $|[\gamma_1, \gamma_2]| + \sum_{i\geqslant 3} |\gamma_i| = |\gamma| - 1$. The orientation sign (-1) is used to write the term on the right-hand side, and no other sign appears since the expression

$$\mu \circ \tau(\gamma_1 \otimes \gamma_2) = (-1)^{|\gamma_1|-1} \sum_i \gamma_1 \overleftarrow{\partial}_{\theta_i} \overrightarrow{\partial}_{x_i} \gamma_2$$

obtained from Stokes has the same sign as the corresponding term in $Q_2(\gamma_1, \gamma_2) = (-1)^{|\gamma_1|-1}[\gamma_1, \gamma_2]$.

There remains to show that U_1 is the HKR map. Let

$$\gamma = \gamma^{i_1\cdots i_m} \theta_{i_1} \cdots \theta_{i_m}.$$

Then there is only one graph in $\mathcal{G}_{1,m}$ contributing to $U_1(\gamma)$. Its edges are $(1, \bar{j})$, $1 \leqslant j \leqslant m$. We then have for any $f \in A^{\otimes m}$.

$$\begin{aligned}
U_1(\gamma) f &= (-1)^{(|\gamma|-1)m} \int\limits_{C^+_{1,m}} \prod_{i=1}^m \omega_{1,\bar{i}}(\gamma \otimes f) \\
&= (-1)^{(|\gamma|-1)m} \frac{1}{(2\pi)^m} \int\limits_{0<\varphi_1<\cdots<\varphi_m<2\pi} \prod_{i=1}^m d\varphi_i \tau_{1\bar{i}}(\gamma \otimes f) \\
&= (-1)^{(|\gamma|-1)m+m(m-1)/2} \frac{1}{m!} \tau_{1\bar{1}} \cdots \tau_{1\bar{m}}(\gamma \otimes f) \\
&= (-1)^{(|\gamma|-1)m+m(m-1)/2} (-1)^{|\gamma|m+\sum_{\alpha=1}^m (\alpha(1-\epsilon_{i_\alpha})+\epsilon_{i_\alpha})} \\
&\quad \times \gamma^{i_1\cdots i_m} \mu \circ (\partial_{i_1} \otimes \cdots \otimes \partial_{i_m}) f \\
&= (-1)^{\sum_{\alpha=1}^m (\alpha-1)\epsilon_{i_\alpha}} \gamma^{i_1\cdots i_m} \mu \circ (\partial_{i_1} \otimes \cdots \otimes \partial_{i_m}) f.
\end{aligned}$$

A.9. Globalisation

The L_∞-morphism of Proposition A.5 obeys all the additional properties of [10] needed to go from a local to a global L_∞-morphism. In particular the non-trivial fact that U_n vanishes if one of its arguments is a linear vector field is valid here because of the vanishing of the same integrals over configuration spaces as in [10]. One can then deduce the existence of a morphism of Q-manifolds $U : \mathcal{T}(A)[1] \to \mathcal{D}(A)[1]$ for the algebra A of functions on any supermanifold along the lines of [9,10] or [6].

The explicit construction goes as follows. To fix notations, let $E \to M$ be the graded vector bundle that realizes the given supermanifold; viz., $A = \Gamma(M, \hat{S}E^*)$. Using a connection, we may identify $\mathcal{T}(A)$ with $\Gamma(M, ST[-1]M \hat{\otimes} \hat{S}E^* \hat{\otimes} \hat{S}E[-1])$ as GLAs.

We may get a Fedosov resolution thereof following [6]. A simplification is obtained by using an idea contained in [2]. Namely, we consider the complex $\Omega^\bullet(\mathcal{T}) := \Gamma(\Lambda^\bullet T^*M \otimes \mathcal{T})$, with

$$\mathcal{T} = S(T^*M)\,\hat{\otimes}\, ST[-1]M \,\hat{\otimes}\, \hat{S}E^* \,\hat{\otimes}\, \hat{S}E[-1]$$

as a GLA. This is the only difference with the construction of [6]. The rest goes exactly the same way. Namely, one constructs a compatible differential D with cohomology concentrated in degree zero and equal to $\mathcal{T}(A)$ as follows. First one considers the (globally well-defined) differential $\delta := [dx^i \frac{\partial}{\partial y^i},\ ]$, with $\{x^i\}$ local coordinates on M and $\{y^i\}$ the induced local coordinates on the tangent fibers. Then one picks up a torsion-free affine connection which together with the already chosen connection on E defines a connection on $\mathcal{T}$. The induced covariant derivative ∇ commutes with δ since the connection is torsion free. One then kills the curvature constructing by induction, exactly as in [6], an element A of $\Omega^1(\mathcal{T})$ such that $D := \nabla - \delta + A$ squares to zero and has the wished-for properties. Similarly one gets a Fedosov resolution $\Omega^\bullet(\mathcal{D})$ of $\mathcal{D}(A)$.

Next, on a coordinate neighbourhood W, one defines a splitting $D = d + \mathsf{B}$ with $d = dx^i \frac{\partial}{\partial x^i}$. The local L_∞-quasiisomorphism defined in the previous subsection, may be extended to an L_∞-quasiisomorphism

$$U_W : \big(\Omega^\bullet(\mathcal{T})|_W, d, [\,,\,]\big) \rightsquigarrow \big(\Omega^\bullet(\mathcal{D})|_W, d + \partial, [\,,\,]\big),$$

with ∂ the Hochschild differential. Now one observes that B is a MC element in $\Omega^\bullet(\mathcal{T})|_W$ (as a vector field) and $\Omega^\bullet(\mathcal{D})|_W$ (as a first-order differential operator) and that it is mapped to itself by U_W since it is a vector field. So one can localize U_W at B and get a new L_∞-quasiisomorphism

$$\mathbf{U}_W : \big(\Omega^\bullet(\mathcal{T})|_W, D, [\,,\,]\big) \rightsquigarrow \big(\Omega^\bullet(\mathcal{D})|_W, D + \partial, [\,,\,]\big).$$

Since B transforms from one coordinate neighbourhood to another by the addition of a linear vector field and since higher components of the L_∞-quasiisomorphism vanish on linear vector fields, one realizes that $\mathbf{U}_W$ does not really depend on a choice of local coordinates, so it extends to a global L_∞-quasiisomorphism $\mathbf{U}$. Finally, one may modify $\mathbf{U}$ in such a way that its image lies in the DGLA of zero D-cochains, which may be identified with $\mathcal{D}(A)$.

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