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A similarity principle for locally solvable vector fields

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Abstract

This paper establishes a weak similarity principle for the class of locally solvable complex vector fields in the plane. The main tool is a local solvability result in an appropriate space of bounded mean oscillation functions.

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1. Introduction

In this article we study properties of solutions of first-order equations of the form:

$$Lw = Aw + B\overline{w}, \tag{1}$$

where w is a locally integrable function, A and B are bounded measurable functions and L is a planar complex vector field of class C^{1+r}, $0 < r < 1$. Equation (1) is a generalization of the classical elliptic equation:

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$$\frac{\partial w}{\partial \bar{z}} = Aw + B\overline{w}, \tag{2}$$

which has been the subject of many works (see, for example, [2–5,15,24]). Notice that (1) implies that Lw is locally integrable and satisfies the inequality

$$|Lw| \leqslant M|w| \tag{3}$$

for some positive constant M. Conversely, if w and Lw are locally integrable and satisfy (3) we may define A to be equal to the quotient Lw/w at points where w does not vanish and equal to zero otherwise; it then follows that A is measurable and bounded and w satisfies (1) with $B \equiv 0$. When $L = \bar{\partial}$ is the Cauchy–Riemann operator, solutions of (3) were called approximately analytic by L. Bers [3, p. 18].

Solutions of (2) are called pseudoanalytic functions or generalized analytic functions in the literature. Pseudoanalytic functions share many properties with analytic functions of a single complex variable. These properties follow from the similarity principle which is valid for solutions of (2). This principle says that locally every continuous solution w of (2) has the form

$$w = \mathrm{e}^{g} h,$$

for some holomorphic function h and Hölder continuous g. Thus, w and h are "similar" in the sense that both w/h and h/w are bounded away from zero on compact sets, in particular, the zero set of w is discrete.

Since in appropriate local coordinates, any elliptic vector field L becomes a multiple of $\bar{\partial} = \partial/\partial\bar{z}$ it turns out that the similarity principle holds as well for any elliptic vector field.

In a recent paper [12], A. Meziani studied the validity of the similarity principle for the following three nonelliptic vector fields:

$$L_1 = \frac{\partial}{\partial t} - 3\mathrm{i}t^2\frac{\partial}{\partial x}, \quad L_2 = \frac{\partial}{\partial t} - \mathrm{i}x\frac{\partial}{\partial x}, \quad \text{and} \quad M = \frac{\partial}{\partial t} - \mathrm{i}t\frac{\partial}{\partial x}.$$

There he proved, among other results, that an appropriate form of the similarity principle for L_1 and L_2 is valid, in the sense that if w is a solution of $L_j w = Aw + B\overline{w}$ $(j = 1, 2)$, then w has the form

$$w = \mathrm{e}^{g} h,$$

where $L_j h = 0$. It turns out that the main point about L_1 and L_2 is that they are locally solvable, a property that M does not share. Starting from this observation a weak form of the similarity principle was proved in [1] for a substantial class of locally solvable vector fields. Although the functions h and g involved in the representation of w might be no longer continuous for general L in this class, the connection it establishes between the zeros of w and h proves useful and can be applied, for instance, to obtain uniqueness in the Cauchy problem for certain types of semilinear equations with lowly regular weak solutions [1].

The main thrust of the present paper is to show that the weak similarity principle holds indeed for all locally solvable vector fields, i.e., those characterized by the Nirenberg–Treves condition $(\mathcal{P})$ [14]. Our techniques also allow us to reduce substantially the regularity assumptions on the coefficients of L for the associated local solvability result (class C^{1+r} for the principal part and class C^r for the zero-order term, $0 < r < 1$, suffices).

The local solvability of L enters in the picture as follows: given w, in order to find functions g and h such that $w = h\exp(g)$, the main step is to find g – after which h may be defined as $h = \mathrm{e}^{-g}w$ – so one must solve locally the equation $Lg = f$ for a bounded f. Furthermore, the solution g must be such $\exp(g)$ is locally integrable. While it is true that for any locally solvable vector field L and $1 < p < \infty$ the equation $Lg = f$ can locally be solved in L^p if f is in L^p [9,10], this is false, in general, for $p = \infty$ [11]. Clearly, finding a solution $g \in L^p$ for any $p < \infty$ when f is bounded is not good enough because $\exp g$ might not be locally integrable. This difficulty can be dealt with by introducing the space $X = L^\infty(\mathbb{R}_t; \mathrm{bmo}(\mathbb{R}_x))$ of measurable functions $u(x,t)$ such that, for almost every $t \in \mathbb{R}$, $x \mapsto u(x,t) \in \mathrm{bmo}(\mathbb{R})$ and $\|u(t,\cdot)\|_{\mathrm{bmo}} \leqslant C < \infty$ for a.e. $t \in \mathbb{R}$, where $\mathrm{bmo}(\mathbb{R})$ is a space of bounded mean oscillation functions, dual to the semilocal Hardy space $h^1(\mathbb{R})$ of Goldberg [6]. It was shown in [1] that for the class of vector fields L there considered, the equation $Lu = f$ can be locally solved with $u \in X$ if $f \in L^\infty$. Here we improve this result by showing that for any locally solvable vector field L the equation $Lu = f$ can be locally solved with $u \in X$ for any $f \in X$. This can be regarded as an ersatz for $p = \infty$ of the L^p local solvability valid for $1 < p < \infty$.

We now describe briefly the organization of the paper. In Section 2 we recall some facts about the semilocal Hardy space $h^1(\mathbb{R})$ where most of our analysis is carried out and state our main local solvability result, Theorem 2.2. This follows in a standard way from an a priori estimate (Theorem 2.1) whose rather long proof is presented in Sections 3–5. In Section 6 we derive a similarity principle for a vector field with C^{1+r} coefficients that satisfies $(\mathcal{P})$ and apply it to obtain uniqueness in the Cauchy problem for a semilinear equation involving a locally solvable vector field in any number of variables with rough coefficients. Finally, in Appendix A, we prove some facts that are important in the proof of Theorem 2.1 but they rather belong to the general theory of the space h^1.

2. A priori estimates in Hardy spaces

We recall some facts about the real Hardy space $H^1(\mathbb{R}) \subset L^1(\mathbb{R})$, a particular instance of the spaces introduced by Stein and Weiss in [19], and its semilocal version $h^1(\mathbb{R})$ introduced by Goldberg [6]. In many situations $H^1(\mathbb{R})$ is an advantageous substitute for $L^1(\mathbb{R})$ [18], as the latter does not behave well in many respects, for instance, concerning the continuity of singular integral operators. Let us choose a function $\Phi \geqslant 0 \in C_c^\infty([-1/2, 1/2])$, with $\int \Phi \,\mathrm{d}z = 1$. Write $\Phi_\varepsilon(z) = \varepsilon^{-1}\Phi(z/\varepsilon)$, $z \in \mathbb{R}$, and set

$$M_\Phi f(z) = \sup_{0<\varepsilon<\infty} \left|(\Phi_\varepsilon * f)(z)\right|;$$

then [18]

$$H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}):\ M_\Phi f \in L^1(\mathbb{R})\}.$$

A space of distributions is called semilocal if it is invariant under multiplication by test functions. The space $H^1(\mathbb{R})$ is not: ψu may not belong to $H^1(\mathbb{R})$ for $\psi \in C_c^\infty(\mathbb{R})$ and $u \in H^1(\mathbb{R})$. A way around this is the definition of the semilocal (or localizable) Hardy space – better suited for the study of PDEs – $h^1(\mathbb{R})$ [6,18] by means of the truncated maximal function,

$$m_\Phi f(z) = \sup_{0<\varepsilon\leqslant 1} \left|(\Phi_\varepsilon * f)(z)\right|, \qquad h^1(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}):\ m_\Phi f \in L^1(\mathbb{R})\},$$

which is stable under multiplication by test functions (we will systematically denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing function and by $\mathcal{S}'$ its dual, i.e., the space of tempered distributions). It turns out that if Φ is substituted in the definition of $h^1(\mathbb{R})$ by any other function $\Phi \in \mathcal{S}(\mathbb{R})$ only subjected to $\int \Phi \neq 0$, this will not change the space $h^1(\mathbb{R})$. Moreover, $h^1(\mathbb{R})$ is a Banach space with the norm

$$\|f\|_{h^1} = \|m_\Phi f\|_{L^1},$$

and $H^1 \subset h^1 \subset L^1$. Of course, this norm depends on the choice of Φ but different Φ will give equivalent norms, moreover, if $\mathcal{A} \subset \mathcal{S}$ is a bounded subset, there is a constant $C = C(\mathcal{A}) > 0$ such that $\|m_\phi f\|_{L^1} \leqslant C\|m_\Phi f\|_{L^1}$ for all $f \in \mathcal{S}$ and $\phi \in \mathcal{A}$. In fact more is true: denoting by $\mathcal{M}f(x) = \sup_{\phi\in\mathcal{A}} m_\phi f(x)$ the grand maximal function associated to $\mathcal{A}$ it follows that $\|\mathcal{M}f\|_{L^1} \leqslant C\|m_\Phi f\|_{L^1}$.

We now describe the atomic decomposition of h^1 [6,18]. An $h^1(\mathbb{R})$ atom is a bounded, compactly supported function $a(z)$ satisfying the following properties: there exists an interval I containing the support of a such that:

(1) $|a(z)| \leqslant |I|^{-1}$, a.e., with $|I|$ denoting the Lebesgue measure of I;
(2) if $|I| < 1$, we further require that $\int a(z)\,\mathrm{d}z = 0$.

Any $f \in h^1$ can be written as an infinite linear combination of h^1 atoms, more precisely, there exist scalars λ_j and h^1 atoms a_j such that the series $\sum_j \lambda_j a_j$ converges in h^1 to f. Furthermore, $\|f\|_{h^1} \sim \inf\sum_j |\lambda_j|$, where the infimum is taken over all atomic representations. Another useful fact is that the atoms may be assumed to be smooth functions. A simple consequence of the atomic decomposition is that $h^1(\mathbb{R})$ is stable under multiplication by Lipschitz functions $b(x)$: if a satisfies (1) with $|I| \geqslant 1$ it follows that $a(x)b(x)/\|b\|_{L^\infty}$ also does. If $|I| < 1$ and the center of I is x_0 we may write $a(x)b(x) = b(x_0)a(x) + (b(x) - b(x_0))a(x) = \beta_1(x) + \beta_2(x)$. Then $\beta_1(x)/\|b\|_{L^\infty}$ satisfies (1) and (2) (with the same I) while $\beta_2(x)/K$ satisfies (1) for the interval I' of center x_0 and length 1, where K is the Lipschitz constant of $a(x)$. It follows that $f \mapsto bf$ has norm $\leqslant \|a\|_{L^\infty} + K$ in $h_1(\mathbb{R})$. This argument can be pushed further to show that $h^1(\mathbb{R})$ is stable under multiplication by more general continuous functions including Hölder functions, as we now describe. Let ω be a modulus of continuity, meaning that $\omega : [0,\infty) \to \mathbb{R}^+$

is continuous, increasing, $\omega(0) = 0$ and $\omega(2t) \leqslant C\omega(t)$, $0 < t < 1$. Consider the Banach space $C^\omega(\mathbb{R}^n)$ of bounded continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$|f|_{C^\omega} \doteq \sup_{x \neq y} \frac{|f(y) - f(x)|}{\omega(|x-y|)} < \infty,$$

equipped with the norm $\|f\|_{C^\omega} = \|f\|_{L^\infty} + |f|_{C^\omega}$. Note that C^ω is only determined by the behavior of $\omega(t)$ for values of t close to 0. We will show in Lemma A.1 in Appendix A that if the modulus of continuity $\omega(t)$ satisfies

$$\frac{1}{h}\int_0^h \omega(t)\,\mathrm{d}t \leqslant C\left(1 + \ln\frac{1}{h}\right)^{-1}, \quad 0 < h < 1, \tag{4}$$

then $h^1(\mathbb{R})$ is stable under multiplication by functions $\in C^\omega(\mathbb{R})$. Note that the modulus of continuity $\omega(t) = t^r$, $0 < r < 1$, that defines the Hölder space C^r, satisfies (4).

Consider now a first-order linear differential operator in two variables,

$$L = \frac{\partial}{\partial t} + \mathrm{i}b(x,t)\frac{\partial}{\partial x} + c(x,t), \quad x, t \in \mathbb{R}. \tag{5}$$

We assume that:

(i) *$c(x,t) \in C^\omega(\mathbb{R}^2)$ where ω satisfies* (4);
(ii) *$b(x,t)$ is real and of class C^{1+r} for some $0 < r < 1$, i.e., for all multi-indexes $|\alpha| \leqslant 1$, $D^\alpha b$ is bounded and $D^\alpha b \in C^r(\mathbb{R}^2)$*;
(iii) *for any $x \in \mathbb{R}$ the function $t \mapsto b(x,t)$ does not change sign.*

We point out that (iii) means that the operator L given by (5) satisfies the Nirenberg–Treves condition $(\mathcal{P})$. We now introduce the space $Y = L^1(\mathbb{R}_t; h^1(\mathbb{R}_x))$ of measurable functions $u(x,t)$ such that, for almost every $t \in \mathbb{R}$, $x \mapsto u(x,t) \in h^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \left\|u(\cdot,t)\right\|_{h^1} \mathrm{d}t \leqslant C < \infty,$$

where $h^1(\mathbb{R})$ is the semilocal Hardy space $h^1(\mathbb{R})$ of Goldberg [6]. The dual of the space Y is the space X mentioned in the introduction.

Theorem 2.1. *Let the operator L given by* (5) *satisfy* (i), (ii) *and* (iii) *and let $a > 0$. Then there exist constants $C > 0$ and $T_0 > 0$ such that*

$$\|u\|_Y \leqslant CT\|Lu\|_Y, \tag{6}$$

for all $u \in C_c^\infty([-a,a] \times [-T,T])$, $0 < T \leqslant T_0$.

The a priori inequality (6) has a standard duality consequence which we now describe. The dual of $h^1(\mathbb{R})$, denoted by $\mathrm{bmo}(\mathbb{R})$, may be identified [6] with the space of locally integrable functions $f(x)$ such that $\sup_{|I|<1} |I|^{-1} \int_I |f - f_I| < \infty$ and $\sup_{|I|\geqslant 1} |I|^{-1} \int_I |f| < \infty$, where we have denoted by I an arbitrary interval and by f_I the mean of f on I. In particular, $\mathrm{bmo}(\mathbb{R})$ is contained in $\mathrm{BMO}(\mathbb{R})$, the space of bounded mean oscillation functions. Then, (6) implies local solvability in $L^\infty([-T, T], \mathrm{bmo}(\mathbb{R}_x))$ for the formal transpose L^t. Now, L and $-L^t$ have the same principal part, so L and $-L^t$ satisfy simultaneously the hypotheses of Theorem 2.1. Summing up, we obtain the following theorem:

Theorem 2.2. *Let the operator*

$$L = \frac{\partial}{\partial t} + \mathrm{i}b(x,t)\frac{\partial}{\partial x} + c(x,t)$$

satisfy (i)–(iii). *There is a neighborhood $U = (-a, a) \times (-T, T)$ of the origin such that for every function $f \in X = L^\infty(\mathbb{R}_t, \mathrm{bmo}(\mathbb{R}_x))$ there exists a function $u \in X$ which solves $Lu = f$ in U, with norm*

$$\|u\|_X \leqslant CT\|f\|_X.$$

In particular, the size of u can be taken arbitrary small by letting $T \to 0$.

We conclude this section by proving consequences of Theorems 2.1 and 2.2 that can be stated in a more invariant form that does depend on a special coordinate system. In Theorems 2.1 and 2.2 the operator L has a special form which is instrumental in obtaining a priori estimates with minimal assumptions on the regularity of the coefficients but, at least heuristically, after a suitable change of variables any first-order operator of principal type has this form. On the other hand, for operators with rough coefficients this change of variables imposes a loss of regularity on the coefficients of the transformed operator. One should also observe the loss of derivatives caused in the process of deriving estimates in terms of the original variables from estimates obtained in the new variables by the behavior of local Hardy norms under composition with diffeomorphisms. For this reason we now deal with operators having C^{2+r} coefficients in the principal part. Since we are dealing with mixed norms, the roles of t and x cannot be interchanged and we must consider change of variables that preserve the privileged role of t. Consider a general first-order operator defined in an open subset $\Omega \subset \mathbb{R}^2$ that contains the origin

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t),$$

with complex coefficients $A, B \in C^{2+r}(\Omega)$, $0 < r < 1$, $C \in C^\omega(\Omega)$. Assume that the lines $t = \text{const.}$ are noncharacteristic, which amounts to saying that $|A(x,t)| > 0$, $(x,t) \in \Omega$. Since the properties we are studying do not change if L is multiplied by a nonvanishing function of class C^{2+r}, we may assume without loss of generality that $A \equiv 1$, i.e.,

$$Lu = \frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t).$$

Write $B(x,t) = \tilde{a}(x,t) + \mathrm{i}\tilde{b}(x,t)$ with $\tilde{a}$ and $\tilde{b}$ real and choose $\rho > 0$ so that they are defined for $|x| < \rho$, $|t| < \rho$. Consider the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \tilde{a}(x,t), \quad x(0) = \xi, \qquad \frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad t(0) = 0,$$

with solution $(x(\xi,s), t(\xi,s))$ given by

$$x(\xi,s) = \xi + \int_0^s \tilde{a}\big(x(\xi,\sigma),\sigma\big)\,\mathrm{d}\sigma, \qquad t(\xi,s) = s.$$

Observe that $x(\xi,0) = \xi$ so $(\partial x/\partial\xi)(0,0) = 1$; also $(\partial t/\partial\xi)(0,0) = 0$ and $(\partial t/\partial s)(0,0) = 1$ so the Jacobian determinant $\det[\partial(x,t)/\partial(\xi,s)]$ assumes the value 1 at $x = s = 0$, granting that $(\xi,s) \mapsto (x,t)$ is, at least locally, a smooth change of variables. The chain rule gives:

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial t} + \tilde{a}(x,t)\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial \xi} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial x},$$

so in the new coordinates we have

$$\widetilde{L} = \partial_s + \mathrm{i}\big(\widetilde{B}/(\partial\xi/\partial x)\big)\partial_\xi + C\big(x(\xi,s),s\big) = \partial_s + \mathrm{i}b\partial_\xi + c,$$

where b is real of class C^{1+r} and $c \in C^\omega$. If L satisfies the Nirenberg–Treves condition $(\mathcal{P})$ so does $\widetilde{L}$, due to the well-known invariance of this property. Multiplying the coefficients b and c by a cut-off function $\chi \geqslant 0 \in C_c^\infty(\mathbb{R}^2)$ that is identically equal to 1 in neighborhood of the origin we have now an operator L', with coefficients defined globally in $\mathbb{R}^2$, that satisfies the hypotheses of Theorem 2.1 and agrees with $\widetilde{L}$ in a neighborhood of the origin. Thus, the a priori estimate (6) holds for L' in the variables (ξ,s). Let $u'(\xi,s) \in C_c^\infty(\mathbb{R}^2)$ be supported in a sufficiently small neighborhood of the origin and set $u(x,t) = u'(\xi(x,t),t)$, where $(x,t) \mapsto (\xi,s)$ is the inverse of $(\xi,s) \mapsto (x,t)$, thus of class C^{2+r}. Invoking the invariance of $h^1(\mathbb{R})$ under diffeomorphisms of class C^2 discussed in Proposition A.6 we conclude that if u' is supported in a convenient neighborhood of the origin we have:

$$C_1 \int_{\mathbb{R}} \big\|u(\cdot,t)\big\|_{h^1(\mathbb{R}_x)}\,\mathrm{d}t \leqslant \int_{\mathbb{R}} \big\|u'(\cdot,s)\big\|_{h^1(\mathbb{R}_\xi)}\,\mathrm{d}s \leqslant C_2 \int_{\mathbb{R}} \big\|u(\cdot,t)\big\|_{h^1(\mathbb{R}_x)}\,\mathrm{d}t$$

and this shows that the a priori estimate (6) for L' implies an analogous estimate for L, using the fact that $Lu(x,t) = L'u'(\xi(x,t),t)$.

Theorem 2.3. *Let L given by*:

$$Lu = A(x,t)\frac{\partial u}{\partial t} + B(x,t)\frac{\partial u}{\partial x} + C(x,t),$$

be defined in a neighborhood of the origin, with complex coefficients $A, B \in C^{2+r}(\Omega)$, $0 < r < 1$, $C \in C^{\omega}(\Omega)$. Assume that the level curves $t =$ const. are noncharacteristic for L and that L satisfies the Nirenberg–Treves condition $(\mathcal{P})$. Then there exist constants $a > 0$, $C > 0$ and $T_0 > 0$ such that

$$\|u\|_Y \leqslant CT\|Lu\|_Y$$

for all $u \in C_c^{\infty}([-a,a] \times [-T,T])$, $0 < T \leqslant T_0$. Hence, for every function $f \in X = L^{\infty}(\mathbb{R}_t, \mathrm{bmo}(\mathbb{R}_x))$ there exists a function $u \in X$ which solves $Lu = f$ in a neighborhood U of the origin, with norm

$$\|u\|_X \leqslant CT\|f\|_X.$$

The long proof of Theorem 2.1 will be presented in the next sections.

3. Beginning of the proof

Due to the hypothesis on $c(x,t)$ we have that $\|cu\|_Y \leqslant C\|u\|_Y$. This means that it is enough to prove (6) for the principal part $L_1 = \partial_t + \mathrm{i}b\partial_x$ of L, since in that case, writing $L = L_1 + c$, the perturbation introduced by the zero-order term may be absorbed by taking T small enough. In other words, we may assume from now on that $c(x,t) \equiv 0$ and we do so. Consider a test function $\chi \in C_c^{\infty}(-2,2)$ such that $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and set $1 - \chi(\xi) = \psi^+(\xi) + \psi^-(\xi)$ with

$$\psi^+(\xi) = \begin{cases} 1-\chi(\xi), & \text{if } \xi \geqslant 0, \\ 0, & \text{if } \xi \leqslant 0, \end{cases} \quad \text{and} \quad \psi^-(\xi) = \begin{cases} 0, & \text{if } \xi \geqslant 0, \\ 1-\chi(\xi), & \text{if } \xi \leqslant 0. \end{cases}$$

Given $\varphi \in \mathcal{S}(\mathbb{R}_x \times \mathbb{R}_t)$, for each fixed t we have a decomposition

$$\varphi(\cdot,t) = P_0\varphi(\cdot,t) + P^+\varphi(\cdot,t) + P^-\varphi(\cdot,t) = \varphi_0(\cdot,t) + \varphi^+(\cdot,t) + \varphi^-(\cdot,t), \tag{7}$$

where

$$P_0\varphi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\chi(\xi)\widehat{\varphi}(\xi,t)\,\mathrm{d}\xi,$$

$$P^+\varphi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\psi^+(\xi)\widehat{\varphi}(\xi,t)\,\mathrm{d}\xi, \tag{8}$$

$$P^-\varphi(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi}\psi^-(\xi)\widehat{\varphi}(\xi,t)\,\mathrm{d}\xi, \tag{9}$$

where we have denoted by $\widehat{\varphi}(\xi,t)$ the Fourier transform of the function $x \mapsto \varphi(x,\cdot)$ evaluated at ξ. Thus,

$$L\varphi = L\varphi_0 + L\varphi^+ + L\varphi^- = LP_0\varphi + LP^+\varphi + LP^-\varphi. \tag{10}$$

We fix once for all some $\phi \in \mathcal{S}(\mathbb{R})$, $\int \phi = 1$, set $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $0 < \varepsilon < 1$, and we consider

$$\begin{aligned}
\left|\phi_\varepsilon * \varphi_0(x,t)\right| &= \left|\int \phi_\varepsilon(x-x') \int_{-T}^{t} \frac{\partial \varphi_0}{\partial s}(x',s)\,\mathrm{d}s\,\mathrm{d}x'\right| \\
&\leqslant \int_{-T}^{t} \left(\left|\phi_\varepsilon * L\varphi_0(\cdot,s)(x)\right|\mathrm{d}s + \left|\phi_\varepsilon * \left(b(\cdot,s)\frac{\partial\varphi_0}{\partial x'}(\cdot,s)\right)(x)\right|\right)\mathrm{d}s \\
&\leqslant \int_{-T}^{T} m_\phi\big(L\varphi_0(\cdot,s)\big)(x) + m_\phi\left(b(\cdot,s)\frac{\partial\varphi_0}{\partial x'}(\cdot,s)\right)(x)\,\mathrm{d}s.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get $|\phi_\varepsilon * \varphi_0(\cdot,t)| \to |\varphi_0(\cdot,t)|$, so integrating in x from $-a$ to a we get:

$$\int_{-a}^{a} \left|\varphi_0(x,t)\right|\mathrm{d}x \leqslant \int_{-T}^{T} \left\|L\varphi_0(\cdot,s)\right\|_{h^1(\mathbb{R}_x)}\mathrm{d}s + \int_{-T}^{T} \left\|b(\cdot,s)\frac{\partial\varphi_0}{\partial x}(\cdot,s)\right\|_{h^1(\mathbb{R}_x)}\mathrm{d}s. \tag{11}$$

On the other hand, writing $L\varphi_0 = LP_0\varphi = P_0L\varphi + [L,P_0]\varphi$ and observing that $[\partial/\partial t, P_0] = 0$ we see that

$$[L,P_0]\varphi(\cdot,t) = \left[\mathrm{i}b(\cdot,t)\frac{\partial}{\partial x}, P_0\right]\varphi(\cdot,t) = \mathrm{i}b(\cdot,t)\frac{\partial P_0\varphi}{\partial x}(\cdot,t) - P_0\left(\mathrm{i}b\frac{\partial\varphi}{\partial x}(\cdot,t)\right),$$

so

$$\begin{aligned}
P_0\left(\mathrm{i}b\frac{\partial\varphi}{\partial x}(\cdot,t)\right) &= \mathcal{F}^{-1}(\chi) * \left(\mathrm{i}b\frac{\partial\varphi}{\partial x}(\cdot,t)\right) \\
&= \eta_1 * \left(\mathrm{i}\frac{\partial(b\varphi)}{\partial x}(\cdot,t)\right) - \eta_1 * \left(\mathrm{i}\frac{\partial b}{\partial x}\varphi(\cdot,t)\right) \\
&= \eta_2 * \big(\mathrm{i}b\varphi(\cdot,t)\big) - \eta_1 * \left(\mathrm{i}\frac{\partial b}{\partial x}\varphi(\cdot,t)\right),
\end{aligned}$$

where $\eta_1 = \mathcal{F}^{-1}(\chi)$ is the inverse Fourier transform of χ and $\eta_2 = \partial\eta_1/\partial x$. Observing that $(\partial P_0\varphi/\partial x)(x,t) = \mathcal{F}^{-1}(\mathrm{i}\xi\chi) * \varphi(\cdot,t)(x) = \eta_3 * \varphi(\cdot,t)$ with $\eta_3 = \mathcal{F}^{-1}(\mathrm{i}\xi\chi)$ and keeping in mind that $\{\phi_\varepsilon * \eta_j\}_{0<\varepsilon<1}$, $j = 1, 2, 3$, is a bounded family of rapidly decreasing functions, we get:

$$\left\| L\varphi_0(\cdot,t)\right\|_{h^1} \leqslant C\left(\left\| L\varphi(\cdot,t)\right\|_{h^1} + \left\| b\varphi(\cdot,t)\right\|_{h^1} + \left\| \frac{\partial b}{\partial x}\varphi(\cdot,t)\right\|_{h^1}\right), \tag{12}$$

$$\left\| \frac{\partial \varphi_0}{\partial x}(\cdot,t)\right\|_{h^1(\mathbb{R}_x)} \leqslant C\left\| \varphi(\cdot,t)\right\|_{h^1(\mathbb{R}_x)}. \tag{13}$$

At this point we recall that multiplication by a Lipschitz function – and this is the case of b – is a continuous operation in $h^1(\mathbb{R})$, a fact discussed in Section 2 right after the definition of atoms that we now state:

Lemma 3.1. *Assume that $b, b' \in L^\infty(\mathbb{R})$. There is a constant $C > 0$ such that*

$$\|bf\|_{h^1(\mathbb{R})} \leqslant C\|b\|_{\mathrm{Lip}}\|f\|_{h^1(\mathbb{R})}, \quad f \in h^1(\mathbb{R}),$$

where $\|b\|_{\mathrm{Lip}} = \max\{\|b'\|_\infty, \|b\|_\infty\}$.

Taking account of (12), (13), Lemma 3.1 and (11) we derive

$$\left\|\varphi_0(\cdot,t)\right\|_{L^1(-a,a)} \leqslant CT\left(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right),$$

which integrated with respect to t from $-T$ to T yields the following proposition:

Proposition 3.2. *There exists $C > 0$ such that*

$$\|\varphi_0\|_{L^1((-a,a)\times(-T,T))} \leqslant C\,T\left(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right) \tag{14}$$

for any $\varphi \in C_c^\infty((-a,a)\times(-T,T))$.

4. L^1 estimates for $\varphi^\pm$

The L^1 estimate for φ_0 is very simple, does not use condition $(\mathcal{P})$ and was only included for the sake of completeness. To obtain similar estimates for $\varphi^\pm$ we will use the Smith approach [17] that we now describe.

We first consider the operators:

$$L^+ = \frac{\partial}{\partial t} - b(x,t)|D_x| \quad \text{and} \quad L^- = \frac{\partial}{\partial t} + b(x,t)|D_x|, \tag{15}$$

$$\text{where} \quad |D_x|\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\xi}|\xi|\widehat{\varphi}(\xi)\,\mathrm{d}\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{16}$$

It is easily checked that if $\operatorname{supp}(\widehat{\varphi}) \subset (0,\infty)$ then $|D_x|\varphi = D_x\varphi$, while is $\operatorname{supp}(\widehat{\varphi}) \subset (-\infty,0)$ then $-|D_x|\varphi = D_x\varphi$, where com $D_x = -\mathrm{i}\,\partial_x$, $\mathrm{i} = \sqrt{-1}$. Thus,

$$L^+\varphi^+ = L\varphi^+, \qquad L^-\varphi^- = L\varphi^-. \tag{17}$$

From now on we concentrate on the operator L^+ since the handling of L^- is entirely analogous. Following Smith we associate to L^+ the real vector field in $\mathbb{R}^3$,

$$\ell = \partial_t + b(x,t)\partial_y$$

and for every point $(x,t,0)$ consider the integral curve of ℓ passing through $(x,t,0)$, i.e., the solution $\gamma(s) = (x(s), t(s), y(s))$ of the system of ODEs:

$$\begin{cases} x'(s) = 0, & x(0) = x, \\ t'(s) = 1, & t(0) = t, \\ y'(s) = b\big(x(s), t(s)\big), & y(0) = 0. \end{cases} \tag{18}$$

Thus $\gamma(s) = (x, s+t, y(s;t,x))$ with $y(s;x,t) = \int_0^s b(x, t+s')\,\mathrm{d}s'$.

Definition 4.1. The operator L^+ given by (15) is said to satisfy condition (Ψ^*) if $b(x,t)$ nowhere changes sign from $+$ to $-$ along the oriented integral curves of the system (18) as s increases, for any $(x,t) \in \mathbb{R}^2$.

Remark 4.2. If in the definition above one forbids sign changes from $-$ to $+$ instead of from $+$ to $-$, the operator L^+ is said to satisfy condition (Ψ). Thus, L^+ satisfies (Ψ) if and only if the transpose operator ${}^tL^+$ satisfies (Ψ^*).

Since $b(x, t+s')$ does not change sign because we assume that L satisfies condition $(\mathcal{P})$ it trivially follows that L^+ satisfies (Ψ^*). We will prove a priori estimates for L^+ assuming just (Ψ^*) which, of course, is weaker that assuming $(\mathcal{P})$. Assume that for some fixed x there is t_0 such that $b(x,t) \geqslant 0$ for $t \geqslant t_0$ and $b(x,t) \leqslant 0$ for $t \leqslant t_0$ (notice that condition (Ψ^*) prevents more than one change sign). It follows that $y(s;x,t) \geqslant 0$ for $s \geqslant 0$ if $t \geqslant t_0$ and $y(s;x,t) \geqslant 0$ for $s \leqslant 0$ if $t \leqslant t_0$. At any rate, we conclude that for any (x,t) either $y(s;x,t) \geqslant 0$ for all $s \geqslant 0$ or $y(s;x,t) \geqslant 0$ for all $s \leqslant 0$.

Let $U(x,t,y) = (P_y * \varphi^+(\cdot,t))(x) = \mathrm{e}^{-y|D_x|}\varphi^+(x,t)$ be the solution of the Dirichlet problem:

$$\begin{cases} \big(\partial_x^2 + \partial_y^2\big)U(x,t,y) = 0, & x \in \mathbb{R},\ y > 0, \\ U(x,t,0) = \varphi^+(x,t), \end{cases}$$

where $P(x) = \pi^{-1}(1+x^2)^{-1}$ is the upper plane Poisson kernel and $P_y(x) = y^{-1}P(x/y)$. Notice that

$$\partial_y \mathrm{e}^{-y|D_x|} = -|D_x|\mathrm{e}^{-y|D_x|} \quad \text{and} \quad \partial_t \mathrm{e}^{-y|D_x|} = \mathrm{e}^{-y|D_x|}\partial_t.$$

Thus, observing that $U(x,\pm T,y)=0$ we may express $\pm\varphi^+(x,t)=\pm U(x,t,0)$ as the line integral of ℓU along the integral curve of ℓ passing through $(x,t,0)$ as follows:

$$\begin{aligned}\mp\varphi^+(x,t) &= \pm\int_0^{\pm T-t}\frac{\mathrm{d}}{\mathrm{d}s}\big(U\big(x,s+t,y(s;x,t)\big)\big)\,\mathrm{d}s\\ &= \pm\int_0^{\pm T-t}\ell U\big(x,s+t,y(s;x,t)\big)\,\mathrm{d}s.\end{aligned}$$

For a given (x,t) we have chosen either $-T$ or T in order to achieve that $y(s;x,t)\geqslant 0$ as s varies on the interval of integration; this choice is essential to make sense of the formula as $U(x,t,y)$ is not defined for $y<0$. The substitution $s'=t+s$ in the last integral gives:

$$\mp\varphi^+(x,t)=\pm\int_t^{\pm T}\ell U\big(x,s',y(s'-t,x,t)\big)\,\mathrm{d}s',$$

implying

$$\big|\varphi^+(x,t)\big|\leqslant\int_{-T}^{T}\big|\ell U\big(x,s',y(s'-t;x,t)\big)\big|\,\mathrm{d}s'.$$

If $T>0$ is small we see that for $|t|,|s'|\leqslant T$ we have $0\leqslant y(s'-t;x,t)\leqslant 1$, so

$$\big|\varphi^+(x,t)\big|\leqslant\int_{-T}^{T}\sup_{0<y<1}\big|\ell U(x,s',y)\big|\,\mathrm{d}s' \tag{19}$$

(notice that φ^+ vanishes for $|t|>T$ so (19) is trivial for those values of t). On the other hand,

$$\ell U(x,s',y)=\mathrm{e}^{-y|D_x|}L\varphi^+(x,s')-\big[b,\mathrm{e}^{-y|D_x|}\big]D_x\varphi^+(x,s'), \tag{20}$$

so integrating by parts we obtain:

$$\begin{aligned}\big[b,\mathrm{e}^{-y|D_x|}\big]D_x\varphi^+(x,s') &= -\mathrm{i}\int\frac{b(x,s')-b(z,s')}{x-z}Q_y(x-z)\varphi^+(z,s')\,\mathrm{d}z\\ &\quad+P_y*\left(\frac{\partial b}{\partial x}\varphi^+(\cdot,s')\right)(x)\\ &= -\mathrm{i}Q_y*\big(\beta^x\varphi^+(\cdot,s')\big)(x)-\mathrm{i}P_y*\left(\frac{\partial b}{\partial x}\varphi^+(\cdot,s')\right)(x),\end{aligned} \tag{21}$$

where $Q_y(x) = x/y^2 P'(x/y)$ and

$$\beta^x(z,s') = \begin{cases} \dfrac{b(x,s') - b(z,s')}{x-z}, & \text{if } z \neq x, \\ b_x(x,s'), & \text{if } z = x. \end{cases} \tag{22}$$

We derive from (20) and (21) that

$$\begin{aligned} \ell_* U(x,s') &\doteq \sup_{0<y<1} \bigl|\ell U(s',x,y)\bigr| \\ &\leqslant \sup_{0<y<1} \bigl|P_y * L\varphi^+(x,s')\bigr| + \sup_{0<y<1} \left|P_y * \left(\frac{\partial b}{\partial x}\varphi^+(\cdot,s')\right)(x)\right| \\ &\quad + \sup_{0<y<1} \bigl|Q_y * \bigl(\beta^x \varphi^+(\cdot,s')\bigr)(x)\bigr|. \end{aligned} \tag{23}$$

Hence, (19) and (23) yield

$$\begin{aligned} \bigl|\varphi^+(x,t)\bigr| \leqslant \int_{-T}^{T} \sup_{0<y<1} \Biggl(\bigl|P_y * L\varphi^+(\cdot,s')(x)\bigr| + \left|P_y * \left(\frac{\partial b}{\partial x}\varphi^+(\cdot,s')\right)(x)\right| \\ + \bigl|Q_y * \bigl(\beta^x \varphi^+(\cdot,s')\bigr)(x)\bigr| \Biggr) \mathrm{d}s'. \end{aligned}$$

Integrating this inequality with respect to x from $-a$ to a we obtain:

$$\begin{aligned} \bigl\|\varphi^+(\cdot,t)\bigr\|_{L^1(-a,a)} &\leqslant \Bigl\| \sup_{0<y<1} \bigl|P_y * L\varphi^+\bigr| \Bigr\|_{L^1((-a,a)\times(-T,T))} \\ &\quad + \left\| \sup_{0<y<1} \left|P_y * \left(\frac{\partial b}{\partial x}\varphi^+\right)\right| \right\|_{L^1((-a,a)\times(-T,T))} \\ &\quad + \Bigl\| \sup_{0<y<1} \bigl|Q_y * \bigl(\beta^x \varphi^+\bigr)\bigr| \Bigr\|_{L^1((-a,a)\times(-T,T))}. \end{aligned} \tag{24}$$

To estimate the terms on the right of the last inequality we need some lemmas. The first of them is concerned with the standard (nonlocal) Hardy space H^1.

Lemma 4.3. *Let $Q \in C^\infty(\mathbb{R})$ with $|Q^{(n)}(x)| \leqslant C_n/(1+|x|)^{n+2}$, $n = 0,1,2,3,\ldots$. Then*

$$\int_{\mathbb{R}} \sup_{y>0} \bigl|Q_y * f(x)\bigr| \,\mathrm{d}x \leqslant C\|f\|_{H^1(\mathbb{R})}, \quad f \in H^1(\mathbb{R}).$$

Proof. Let $\phi \in C_c^\infty(-1,1)$ satisfy $\phi(x) = 1$ for $|x| \leqslant 1/2$. Thus,

$$1 = \phi(x) + \sum_{k=1}^{\infty}\big(\phi\big(2^{-k-1}x\big) - \phi\big(2^{-k}x\big)\big) \quad \text{and}$$

$$Q(x) = \phi(x)Q(x) + \sum_{k=1}^{\infty} Q(x)\big(\phi\big(2^{-k-1}x\big) - \phi\big(2^{-k}x\big)\big) = \sum_{k=0}^{\infty} 2^{-k}\Phi_{2^k}^{(k)}(x)$$

with $\Phi^{(0)}(x) = \phi(x)Q(x)$ and $\Phi^{(k)}(x) = 2^{2k}(\phi(2^{-1}x) - \phi(x))Q(2^k x)$ for $k \geqslant 1$ (we are using as always the notation $\Phi_\varepsilon(x) = \varepsilon^{-1}\Phi(\varepsilon^{-1}x)$ for any $\varepsilon > 0$).

Since $\phi(2^{-1}x) - \phi(x)$ is supported in $1/2 \leqslant |x| < 2$, the estimates satisfied by Q and its derivatives show that the collection $\{\Phi^{(k)}\}_{k\in\mathbb{N}}$ constitutes a bounded subset of $\mathcal{S}(\mathbb{R})$. Therefore,

$$\begin{aligned}\sup_{y>0}\big|f * Q_y(x)\big| &\leqslant \sum_k 2^{-k}\sup_{y>0}\big|\Phi_{y2^k}^{(k)} * f(x)\big| \\ &\leqslant \sum_k 2^{-k}\sup_{s>0}\big|\Phi_s^{(k)} * f(x)\big| \leqslant C\mathcal{M}f(x),\end{aligned}$$

where $\mathcal{M}f$ is the grand maximal function associated to $\{\Phi^{(k)}\}_{k\in\mathbb{N}}$, i.e., $\mathcal{M}f(x) = \sup_{k\in\mathbb{N}} M_{\Phi^{(k)}}f(x)$ and C is a constant. □

We return to the semilocal Hardy space h^1 in the next lemma.

Lemma 4.4. *Let $0 < \alpha < \infty$, let P be the Poisson kernel in $\mathbb{R}^2_+$ and let Q be a function satisfying $|Q^{(n)}(x)| \leqslant C_n/(1+|x|)^{n+2}$, $n = 0, 1, 2, 3, \ldots$, as in the previous lemma. There exists $C > 0$ such that*

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1}\big|P_y * f(x)\big|\,\mathrm{d}x \leqslant C\|f\|_{h^1(\mathbb{R})}, \quad f \in h^1(\mathbb{R}),$$

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1}\big|Q_y * f(x)\big|\,\mathrm{d}x \leqslant C\|f\|_{h^1(\mathbb{R})}, \quad f \in h^1(\mathbb{R}).$$

Proof. To prove the first inequality we need only show that there exists $C > 0$ such that

$$\Big\|\sup_{0<y<1}|P_y * a|\Big\|_{L^1(-\alpha,\alpha)} \leqslant C$$

for all h^1-atoms a. Let a be an h^1-atom supported in the interval $I = (x_0 - r, x_0 + r)$. If $r \leqslant 1/2$ the atom a must satisfy the moment condition and it is also an H^1-atom so the inequality is well known and valid even for $\alpha = \infty$. If $r > 1/2$ we observe that

$$\sup_{0<y<1}\big|P_y * a(x)\big| \leqslant \sup_{0<y<1}\|a\|_{L^\infty}\|P_y\|_{L^1} \leqslant |I|^{-1}\|P\|_{L^1} \leqslant \|P\|_{L^1} = 1.$$

The proof for Q is similar, it uses Lemma 4.3 for H^1-atoms that can also be considered as h^1-atoms supported in small intervals and the fact that $\|Q\|_{L^1} < \infty$ to deal with atoms that do not satisfy the moment condition. □

Lemma 4.5. *Let* $0 < \alpha < \infty$. *Assume that* Q *satisfies the hypotheses of the previous lemma,* $\beta \in L^\infty(\mathbb{R}^2)$ *is such that for some* $K > 0$,

$$\left|\beta(x,y) - \beta(x,x_0)\right| \leqslant K\frac{|x_0 - y|}{|x - x_0|}, \quad \text{if } |x - x_0| \geqslant 2|y - x_0|.$$

Then there exists $C > 0$ *such that for every* $f \in h^1(\mathbb{R})$ *with* $\operatorname{supp}(f) \subset (-\alpha, \alpha)$ *holds the inequality*

$$\int_{-\alpha}^{\alpha} \sup_{0<y<1} \left|Q_y * \left(\beta^x f\right)(x)\right| \mathrm{d}x \leqslant C\|f\|_{h^1(\mathbb{R})},$$

where $\beta^x(y) = \beta(x,y)$.

Proof. Since $\operatorname{supp}(f) \subset (-\alpha, \alpha)$ we may, in view of Lemma A.3, expand f as a linear combination of atoms supported in $(-\alpha - 1, \alpha + 1)$ and reduce the estimate to the case of atoms with this property. Let a be an h^1-atom, with $s(a) \subset I \subset (-\alpha - 1, \alpha + 1)$, $I = (x_0 - r, x_0 + r)$. If $r > 1$ we have:

$$\begin{aligned}
\int_{-\alpha}^{\alpha} \sup_{0<y<1} \left|Q_y * \left(\beta^x a\right)(x)\right| \mathrm{d}x &= \int_{-\alpha}^{\alpha} \sup_{0<y<1} \left|\int Q_y(x-z)\beta^x(z)a(z)\,\mathrm{d}z\right| \mathrm{d}x \\
&\leqslant \int_{-\alpha}^{\alpha} \|\beta\|_{L^\infty}\|a\|_{L^\infty}\|Q_y\|_{L^1}\,\mathrm{d}x \\
&\leqslant 2\alpha\|\beta\|_{L^\infty}\|Q\|_{L^1}.
\end{aligned}$$

Let us next assume that $r \leqslant 1$. We recall the decomposition of Q used in the proof of Lemma 4.3 and observe that the functions $\Phi_{2^k}^{(k)}$ are supported in the set

$$D_k = \left\{2^k \leqslant |x| \leqslant 2^{k+1}\right\}.$$

Since $\operatorname{supp}(a) \subset (-\alpha - 1, \alpha + 1)$, it follows that

$$\operatorname{supp}\left(\Phi_{2^k}^{(k)} * a\right) \cap (-\alpha, \alpha) \subset \left[D_k + (-\alpha - 1, \alpha + 1)\right] \cap (-\alpha, \alpha) = \emptyset \quad \text{for large } k.$$

Hence, we may write

$$Q_y * \left(\beta^x a\right)(x) = \sum_0^n 2^{-k}\phi_{2^k y}^{(k)} * \left(\beta^x a\right)(x), \quad 0 < y < 1,\ x \in (-\alpha, \alpha),$$

with n depending only on α. Since the family $\mathcal{A} = \{\Phi_{2^k}^{(k)}\}_{0 \leqslant k \leqslant n}$ is finite, thus a bounded subset of $\mathcal{S}(\mathbb{R})$, the integral of the grand maximal function associated to $\mathcal{A}$ is majorized by the integral of the maximal function of a single convenient function. In other words, it will be enough to show that

$$\int_{\mathbb{R}} \sup_{0<\varepsilon<1} \left|\phi_\varepsilon * \left(\beta^x a\right)(x)\right| \mathrm{d}x \leqslant C,$$

for some fixed $\phi \in C_c^\infty(-1,1)$, $\int \phi = 1$. First we note that

$$\int_{I^*} \sup_{0<\varepsilon<1} \left|\phi_\varepsilon * \left(\beta^x a\right)(x)\right| \mathrm{d}x \leqslant C\|\beta\|_{L^\infty}.$$

On the other hand, let $x \notin I^*$, $y \in I$ (in particular, $|x - x_0| > 2|y - x_0|$). Since $\operatorname{supp}(\phi_\varepsilon) \subset (-\varepsilon, \varepsilon)$, it follows that whenever $|x - x_0| > 2\varepsilon$ we will have that $|x - y| > |x - x_0| - |x_0 - y| > \varepsilon$ implying that $\phi_\varepsilon * (\beta^x a)(x) = 0$. Therefore, we need only worry with those values of x for which $|x - x_0| < 2\varepsilon$. In that case, keeping in mind that $\int a(y)\,\mathrm{d}y = 0$ and $0 < \varepsilon < 1$, we get:

$$\begin{aligned}
\left|\phi_\varepsilon * \left(\beta^x a\right)(x)\right| &\leqslant \int \left|\phi_\varepsilon(x-y)\beta^x(y) - \phi_\varepsilon(x-x_0)\beta^x(x_0)\right|\left|a(y)\right| \mathrm{d}y \\
&\leqslant \int \left\{\frac{1}{\varepsilon^2}\|\phi'\|_{L^\infty}\|\beta\|_{L^\infty}|x_0 - y| + \frac{K}{\varepsilon}\|\phi\|_{L^\infty}\frac{|x_0 - y|}{|x - x_0|}\right\}\left|a(y)\right| \mathrm{d}y \\
&\leqslant C(\beta, K)\int \frac{|x_0 - y|}{|x - x_0|^2}\left|a(y)\right| \mathrm{d}y,
\end{aligned}$$

which yields

$$\left|\phi_\varepsilon * \left(\beta^x a\right)(x)\right| \leqslant C(\beta, K)\frac{r}{|x - x_0|^2}\int_I \left|a(y)\right| \mathrm{d}y \leqslant C(\beta, K)\frac{r}{|x - x_0|^2}.$$

Thus,

$$\int_{{}^cI^*} \sup_{0<\varepsilon<1} \left|\phi_\varepsilon * \left(\beta^x a\right)(x)\right| \mathrm{d}x \leqslant C(\beta, K)\, r\int_{{}^cI^*} \frac{1}{|x - x_0|^2}\,\mathrm{d}x \leqslant C(\beta, K),$$

as we wished to prove. □

We observe that the function Q that appears in (24) satisfies the hypotheses of Lemmas 4.4 and 4.5. Furthermore, Lemma 4.5 can be applied to the function $\beta(x, y) = \beta^x(y)$ defined in (22), since $\|\beta\|_{L^\infty} \leqslant \|b'\|_{L^\infty}$ and, for $|x - x_0| \geqslant 2|y - x_0|$, we have:

$$\left|\beta^x(y) - \beta^x(x_0)\right| \leqslant \frac{|b(x) - b(y)||x_0 - y|}{|x - y||x - x_0|} + \frac{|b(y) - b(x_0)|}{|x - x_0|} \leqslant 2\,\|b'\|_{L^\infty}\frac{|x_0 - y|}{|x - x_0|}.$$

Thus, estimate (24) and its analogue for φ^- now give:

Proposition 4.6. *There is a constant $C > 0$ such that*

$$\|\varphi^\pm\|_{L^1((-a,a)\times(-T,T))} \leqslant CT\big(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\big) \tag{25}$$

for all $\varphi \in C_c^\infty((-a,a)\times(-T,T))$.

Proof. Applying Lemmas 4.4 and 4.5 to the right-hand side of (24) and its analogue for φ^- we obtain:

$$\begin{aligned}&\|\varphi^\pm(\cdot,t)\|_{L^1((-a,a)\times(-T,T))}\\ &\quad\leqslant C\big(\|L\varphi^\pm\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi^\pm\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\big),\end{aligned} \tag{26}$$

which can be integrated with respect to t from $-T$ to T in order to get

$$\begin{aligned}&\|\varphi^\pm\|_{L^1((-a,a)\times(-T,T))}\\ &\quad\leqslant C\,T\,\big(\|L\varphi^\pm\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi^\pm\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\big).\end{aligned} \tag{27}$$

Next we write, recalling (17),

$$L\varphi^\pm = LP^\pm\varphi = P^\pm L\varphi + [L, P^\pm]\varphi. \tag{28}$$

Observe that $P^\pm$ is a pseudo-differential operator of order 0 (and type $(\rho,\delta) = (1,0)$) acting in the x variable, so it is bounded in $h^1(\mathbb{R})$. That would also be the case of $[L, P^+] = [b(\cdot,t)D_x, P^+]$ should b be smooth with bounded derivatives of all orders, but since we are only assuming that b is of class C^{1+r} we will invoke instead Proposition A.5 in Appendix A.3 to grant the continuity of $[b(\cdot,t)D_x, P^\pm]$ in $h^1(\mathbb{R})$. Thus, (27) implies (25). □

5. End of the proof

In view of (7), (10), (14) and (25) we may state the:

Proposition 5.1. *There exists a constant $C > 0$ such that*

$$\|\varphi\|_{L^1((-a,a)\times(-T,T))} \leqslant C\,T\big(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\big) \tag{29}$$

for all $\varphi \in C_c^\infty((-a,a)\times(-T,T))$.

Notice that the error term $\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}$ on the right-hand side of (29) cannot be absorbed by taking T small because the norm on the left-hand side is weaker. To circumvent this difficulty we need to derive a stronger inequality, analogous to (29) but with the stronger norm $\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}$ on its left-hand side. To achieve this we make use of the mollified Hilbert transform $\widetilde{H}$ defined by $\widehat{\widetilde{H}f} = (1-\chi)\widehat{Hf}$, where H denotes the usual Hilbert transform, $\chi \in C_c^\infty(-2,2)$, $\phi = 1$, for $|\xi| \leqslant 1$. The usefulness of $\widetilde{H}$, which is a pseudo-differential operator of order zero, derives mainly from the fact that it can be used to define an equivalent norm on $h^1(\mathbb{R})$ without appealing to maximal functions, as granted by the following estimates (cf. [6]):

$$C_1\|\widetilde{H}f\|_{h^1} \leqslant \|f\|_{h^1} \leqslant C_2\big(\|f\|_{L^1} + \|\widetilde{H}f\|_{L^1}\big), \quad f \in h^1(\mathbb{R}).$$

Another ingredient is the following lemma:

Lemma 5.2. *Let $r(D)$ be a pseudo-differential of order zero with symbol $r(x,\xi) = r(\xi)$ independent of x. Assume that for some $C > 0$ the following inequality holds*:

$$\|f\|_{h^1} \leqslant C\big(\|f\|_{L^1} + \|r(D)f\|_{L^1}\big), \quad f \in h^1.$$

Let K be the kernel of $r(D)$ and for each $\varepsilon > 0$ write

$$\begin{aligned} r(D)f(x) &= \big\langle \chi\big(\varepsilon(x-\cdot)\big)K, f\big\rangle + \big\langle\big(1-\chi\big(\varepsilon(x-\cdot)\big)\big)K, f\big\rangle \\ &= r_1^\varepsilon(D)f(x) + r_2^\varepsilon(D)f(x), \end{aligned}$$

where $\chi \in C_c^\infty(-2,2)$ with $\chi(y) = 1$ for $|y| \leqslant 1$. Then there exists ε_0 such that for all $0 < \varepsilon \leqslant \varepsilon_0$ there exist constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon) > 0$ such that

$$\|f\|_{h^1} \leqslant C_1\big(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}\big) \leqslant C_2\|f\|_{h^1}. \tag{30}$$

Proof. For each $\varepsilon > 0$, $r_1^\varepsilon(D)$ is a pseudo-differential operator of order zero, thus bounded in h^1, so

$$\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1} \leqslant \|f\|_{h^1} + \|r_1^\varepsilon(D)f\|_{h^1} \leqslant C_2(\varepsilon)\|f\|_{h^1}.$$

On the other hand, $\|r_2^\varepsilon(D)f\|_{L^1} \leqslant \|K_2^\varepsilon\|_{L^1}\|f\|_{L^1}$ and $\|K_2^\varepsilon\|_{L^1} \to 0$ as $\varepsilon \to 0$. Therefore, there exists $\varepsilon_0 > 0$ such that $\|K_2^\varepsilon\|_{L^1} \leqslant 1/2C$ for $0 < \varepsilon \leqslant \varepsilon_0$. Thus

$$\begin{aligned} \|f\|_{h^1} &\leqslant C\big(\|f\|_{L^1} + \|r(D)f\|_{L^1}\big) \\ &\leqslant C\left(\|f\|_{L^1(\mathbb{R})} + \|r_1^\varepsilon(D)f\|_{L^1(\mathbb{R})} + \frac{1}{2C}\|f\|_{L^1}\right) \\ &\leqslant C\big(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}\big) + \frac{1}{2}\|f\|_{h^1}, \end{aligned}$$

which implies

$$\|f\|_{h^1} \leqslant 2C\big(\|f\|_{L^1} + \|r_1^\varepsilon(D)f\|_{L^1}\big). \qquad \square$$

Remark 5.3. Notice that $r_1^\varepsilon(D)$ is given by convolution with a distribution supported in the interval $(-2/\varepsilon, 2/\varepsilon)$, in particular if $u \in \mathcal{E}'([-r, r])$ – i.e., if u is distribution supported in the interval $[-r, r]$ – $r_1^\varepsilon(D)u$ is supported in the interval $[-r - 2\varepsilon^{-1}, r + 2\varepsilon^{-1}]$.

We are now able to complete the proof of Theorem 2.1. We must show that there exist constants C and $T_0 > 0$ such that for any $0 < T \leqslant T_0$ and $\varphi \in C_c^\infty((-a, a) \times (-T, T))$,

$$\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} \leqslant CT\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}. \tag{31}$$

Proof. Given a function $\phi \in C_c^\infty((-a, a) \times (-T, T))$, set

$$\widehat{\widetilde{H}\varphi(\cdot, t)}(\xi) = (1 - \chi)(\xi)\widehat{H\varphi(\cdot, t)}(\xi), \tag{32}$$

where H is the Hilbert transform and $\chi \in C_c^\infty(-2, 2)$, $\chi(\xi) = 1$ for $|\xi| = 1$. The symbol of $\widetilde{H}$ is equal to $h(\xi) = \psi^+(\xi) - \psi^-(\xi)$, where ψ^+ and ψ^- were defined at the beginning of Section 2. We see that $\widetilde{H}$ is a pseudo-differential operator satisfying the hypotheses of Lemma 5.2 and we may write it as a sum $\widetilde{H} = \widetilde{H}_1^\varepsilon + \widetilde{H}_2^\varepsilon$, where $\widetilde{H}_1^\varepsilon : \mathcal{E}'((-a, a)) \to \mathcal{E}'((-a', a'))$ satisfies (30), i.e.,

$$\|\varphi(\cdot, t)\|_{h^1(\mathbb{R}_x)} \leqslant C\left(\|\varphi(\cdot, t)\|_{L^1(-a,a)} + \|\widetilde{H}_1^\varepsilon\varphi(\cdot, t)\|_{L^1(-a',a')}\right) \tag{33}$$

for some $C > 0$. Since $H_1^\varepsilon\varphi(x, t) \in C_c^\infty((-a', a') \times (-T, T))$, applying (29) (with a' in the place of a) to $H_1^\varepsilon\varphi$, we get:

$$\begin{aligned} &\|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))} \\ &\quad\leqslant CT\left(\|L\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right). \end{aligned} \tag{34}$$

Since $L\widetilde{H}_1^\varepsilon = \widetilde{H}_1^\varepsilon L + [L, \widetilde{H}_1^\varepsilon]$ and, invoking once again Proposition A.5 in the Appendix, $\widetilde{H}_1^\varepsilon$ and $[L, \widetilde{H}_1^\varepsilon]$ are bounded operators in $h^1(\mathbb{R}_x)$, it follows from (34) that

$$\begin{aligned} &\|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))} \\ &\quad\leqslant CT\left(\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right). \end{aligned} \tag{35}$$

Integrating (33) with respect to t and using (35) we see that

$$\begin{aligned} \|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} &\leqslant C\left(\|\varphi\|_{L^1((-T,T)\times(-a,a))} + \|\widetilde{H}_1^\varepsilon\varphi\|_{L^1((-T,T)\times(-a',a'))}\right) \\ &\leqslant CT\left(\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} + \|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}\right). \end{aligned}$$

It is now enough to choose T_0 such that $CT \leqslant 1/2$ if $T \leqslant T_0$ to get

$$\|\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))} \leqslant 2CT\|L\varphi\|_{L^1((-T,T),h^1(\mathbb{R}_x))}$$

as desired. □

6. Applications

6.1. Generalized similarity principle

Throughout this section we consider a vector field defined in some open rectangle $\Omega = I_1 \times I_2$ of the plane:

$$L = \frac{\partial}{\partial t} + \mathrm{i}b(x,t)\frac{\partial}{\partial x}, \quad t, x \in \mathbb{R}, \tag{36}$$

and assume that

(i) *$b(x,t)$ is real and of class C^{1+r} for some $0 < r < 1$, i.e., $D^\alpha b$ is bounded for all multi-indexes $|\alpha| \leqslant 1$ and $|D^\alpha b(p) - D^\alpha b(q)| \leqslant C|p-q|^r$ for all $p, q \in \mathbb{R}^2$, $|\alpha| = 1$;*
(ii) *for any $x \in I_1$ the function $I_2 \ni t \mapsto b(x,t)$ does not change sign.*

Assume also that A is an L^∞ function, $\omega \in L^p_{\mathrm{loc}}(\Omega)$ for some $1 < p < \infty$, and that

$$L\omega = A\omega \tag{37}$$

in the sense of distributions. We will also be interested in solutions of the homogeneous equation

$$Lh = 0. \tag{38}$$

The next theorem describes a factorization for ω involving the space $X = L^\infty(\mathbb{R}_t; \mathrm{bmo}(\mathbb{R}_x))$ of measurable functions $u(x,t)$ such that for almost every $t \in \mathbb{R}$ $x \mapsto u(x,t) \in \mathrm{bmo}(\mathbb{R})$ and $\|u(t,\cdot)\|_{\mathrm{bmo}} \leqslant C < \infty$ for a.e. $t \in \mathbb{R}$. Observe that X is invariant under multiplication by test functions and $u \in X \Rightarrow |u| \in X$, because $\mathrm{bmo}(\mathbb{R})$ already has these properties.

Theorem 6.1. *Let L given by* (36) *satisfy* (i) *and* (ii) *and assume that $1 < p < \infty$, $A \in L^\infty(\Omega)$.*

(a) *If $\omega \in L^p_{\mathrm{loc}}(\Omega)$ satisfies* (37)*, every point of Ω has a neighborhood Ω' where ω may be written as*

$$\omega = \mathrm{e}^g h,$$

where $h \in L^{p'}_{\mathrm{loc}}(\Omega')$ satisfies (38) *in Ω', $g \in X$ and $\mathrm{e}^g \in L^{q'}_{\mathrm{loc}}(\Omega')$ for some $p' \in [1, p]$ and $q' \geqslant p'/(p'-1)$. In addition, p' may be chosen arbitrarily close to p.*
(b) *Conversely, if $h \in L^p_{\mathrm{loc}}(\Omega)$ satisfies* (38)*, every point of Ω has a neighborhood Ω' where h may be written as*

$$h = \mathrm{e}^{-g}\omega,$$

where $\omega \in L^{p'}_{\mathrm{loc}}(\Omega')$ *satisfies* (37) *in* Ω', $g \in X$ *and* $\mathrm{e}^{-g} \in L^{q'}_{\mathrm{loc}}(\Omega')$ *for some* $p' \in [1, p]$ *and* $q' \geqslant p'/(p'-1)$. *Again,* p' *may be chosen arbitrarily close to* p.

Corollary 6.2. *Let* L *be as above,* $1 < p < \infty$, $A, B \in L^{\infty}(\Omega)$ *and assume that* $\omega \in L^{p}_{\mathrm{loc}}(\Omega)$ *satisfies*:

$$L\omega = A\omega + B\overline{\omega}. \tag{39}$$

Every point of Ω *has a neighborhood* Ω' *where* ω *may be written as*

$$\omega = \mathrm{e}^{g} h,$$

with $h \in L^{p'}_{\mathrm{loc}}(\Omega')$ *satisfying* $Lh = 0$ *in* Ω', $g \in X$ *and* $\mathrm{e}^{g} \in L^{q'}_{\mathrm{loc}}(\Omega')$ *for some* $p' \in [1, p]$ *and* $q' \geqslant p'/(p'-1)$. *In addition,* p' *may be chosen arbitrarily close to* p.

Notice that the relationship between p' and q' in part (a) of the theorem (respectively in part (b)) shows that the product of e^{g} and h (respectively e^{-g} and ω) is locally integrable. Corollary 6.2 extends the similarity principle presented in [1] in two ways. First, $b(x,t)$ is only subjected to (i) and (ii), in particular, it is allowed to change sign in an appropriate way prescribed by condition $(\mathcal{P})$, second, only low regularity is assumed on $b(x,t)$.

Example 6.3. If $b(x,t) = x|x|^{r}$, $0 < r < 1$, then L given by (36) satisfies the hypotheses of Theorem 6.1.

The proof of Theorem 6.1 is essentially the same as the proof of the similarity principle given in [1] and we only include it for the sake of completeness; the only new ingredient is our stronger local solvability result. In particular, it depends on the following lemma stated and proved in [1].

Lemma 6.4. (i) *Let* $p, q \in (1, \infty)$, $1/p + 1/q = 1$, $u, f \in L^{p}_{\mathrm{loc}}(\Omega)$, $v, g \in L^{q}_{\mathrm{loc}}(\Omega)$, *and assume that* $Lu = f$ *and* $Lv = g$. *Then*

$$L(uv) = fv + ug. \tag{40}$$

(ii) *Let* $p \in (1, \infty]$ *and assume that* $g \in X$ *satisfies* $Lg \in L^{p}(\Omega)$. *If* $\|g\|_X$ *is sufficiently small,*

$$L(\mathrm{e}^{g}) = \mathrm{e}^{g}\, Lg \quad \text{in } \Omega. \tag{41}$$

Now, we return to the proof of Theorem 6.1.

Proof. Consider a neighborhood Ω' of a given point of Ω where we may solve the equation:

$$Lg = A \quad \text{in } \Omega'. \tag{42}$$

The right-hand side is bounded. Therefore, Theorem 2.3 implies that shrinking Ω' we may solve (42) with $\|g\|_X$ as small as we wish. Then, if we set $h = \mathrm{e}^{-g}\omega$ and use the Leibniz and chain rules (40) and (41) provided by Lemma 6.4 we get:

$$Lh = \mathrm{e}^{-g}(L\omega - \omega Lg) = \mathrm{e}^{-g}(A\omega - \omega A) = 0.$$

Thus, $\omega = \mathrm{e}^{g}h$ as we wished to prove. It is a consequence of the John–Nirenberg inequality that by shrinking Ω' we may take $\mathrm{e}^{-g} \in L^{q'}$ with q' arbitrarily large and this implies that $h \in L^{p'}(\Omega')$ with $p' < p$ arbitrarily close to p. This proves (a). Similarly, to prove (b) one defines ω as $\omega = \mathrm{e}^{g}h$ with g solving (42) and then checks that $L\omega = A\omega$ and the other required properties are valid in a sufficiently small neighborhood of the given point. □

The corollary follows from the theorem, part (a). Indeed if ω is a solution of (39) it satisfies as well

$$L\omega = \widetilde{A}\omega, \quad \text{where } \widetilde{A} = A + B\frac{\chi}{\omega}\overline{\omega} \tag{43}$$

and χ is the characteristic function of the set $\{\omega(x) \neq 0\}$. It is clear that $\widetilde{A}$ is measurable and bounded so part (a) of the theorem gives the required representation for ω.

We see that Theorem 6.1 establishes a one-to-one correspondence between the germs – at a given point – of solutions in $\bigcup_{1<p<\infty} L^p$ of (37) and the germs of solutions in $\bigcup_{1<p<\infty} L^p$ of (38). Let us now discuss briefly to what extent is the Nirenberg–Treves condition $(\mathcal{P})$ necessary for Theorem 6.1 to hold.

Example 6.5. Let L be any vector field L of the form (36) with $b(x,t)$ smooth and let ω be a locally integrable function that satisfies the equation

$$L\omega = \omega \tag{44}$$

in a neighborhood of the origin, which amounts to taking $A \equiv 1$. A simple computation shows that solutions of (44) are of the form $\omega = \mathrm{e}^{t}h$ with $Lh = 0$ and conversely, for any solution of $Lh = 0$ there exists a solution of (44) such that $\omega = \mathrm{e}^{t}h$. For instance, if L is the vector field constructed by L. Nirenberg in [13] with the property that any solution h of $Lh = 0$ defined in a disk centered at the origin must be constant, it follows that all solutions of (44) are of the form $\omega = c\mathrm{e}^{t}$, $c = \text{const}$. Thus, we may say that there is a correspondence between solutions of (44) and solutions of the homogeneous equation $Lh = 0$ in spite of the fact that L may not satisfy condition $(\mathcal{P})$.

Of course, the trick in the example above was to choose $A \equiv 1$ which is in the range of L for whichever L. On the other hand, we have the following fact:

Proposition 6.6. *Let L be given by* (36) *with b smooth. Suppose that, for any smooth function A, all smooth solutions h of* (38) *may be locally written as $h = \mathrm{e}^{-g}\omega$, where ω, g, Lg, e^{g} and e^{-g} are locally integrable, ω satisfies* (37) *and the chain rule $L\mathrm{e}^{g} = \mathrm{e}^{g}Lg$ holds. Then, L satisfies condition $(\mathcal{P})$.*

Proof. Fix a smooth function A in a neighborhood of some point in Ω and take $h \equiv 1$, so $Lh = 0$. Then there is a local solution ω of (37) such that $\omega = \mathrm{e}^g$. Thus, $L\omega = A\omega$ or $\mathrm{e}^g Lg = A\,\mathrm{e}^g$. Since $0 < |\mathrm{e}^g| < \infty$ a.e., we conclude that $Lg = A$ showing that L is locally solvable at an arbitrary point of Ω. This implies $(\mathcal{P})$. $\square$

6.2. Uniqueness in the Cauchy problem

Consider a vector field defined in some neighborhood $\Omega = \Omega_1 \times (-T, T)$ of the origin in $\mathbb{R}^{n+1}$:

$$L = \frac{\partial}{\partial t} + \mathrm{i}\sum_{k=1}^{n} b_k(x,t)\frac{\partial}{\partial x_k},$$

where each b_k is real-valued, of class C^{1+r}, $0 < r < 1$. Assume furthermore that L satisfies condition $(\mathcal{P})$, which in this context means that *for each $x \in \Omega_1$, the vector-valued function*

$$t \mapsto \big(b_1(x,t), \ldots, b_{n-1}(x,t)\big) = \vec{b}(x,t)$$

never changes direction. Consider next a bounded, measurable complex valued function $f(x,t,\zeta): \Omega \times \mathbb{C} \to \mathbb{C}$ satisfying a Lipschitz condition in ζ, i.e.,

$$\big|f(x,t,\zeta) - f(x,t,\zeta')\big| \leqslant K|\zeta - \zeta'|, \quad (x,t,\zeta), (x,t,\zeta') \in \Omega \times \mathbb{C}.$$

Finally, let $u(x,t), w(x,t) \in L^p(\Omega)$, $p \geqslant 2$, satisfy, in the weak sense,

$$Lu = f(x,u), \qquad Lw = f(x,w) \qquad \text{in } \Omega \quad \text{and} \quad u(x,0) = w(x,0).$$

The fact that Lu is bounded implies that for any test function $\phi(x) \in C_c^\infty(\Omega_1)$ the integrable function $(-T,T) \ni t \mapsto \int u(x,t)\phi(x)\,\mathrm{d}x$ is a continuous function of t which can be evaluated at $t = 0$ and the same can be said about w. This lends a meaning to the requirement $u(x,0) = v(x,0)$.

Proposition 6.7. *Under the above conditions, $u \equiv w$ in a neighborhood of the origin.*

Assuming that the coefficients of L are smooth, a better result – in the sense that it was only required that $u, w \in L^p$, $p > 1$ – was proved in [1] as an application of the similarity principle; here we demand that $p \geqslant 2$ but work instead with rough coefficients.

Proof. Since we are working locally we may as well assume that $p = 2$. The arguments in [1] can be adapted without changes to reduce the situation to the case of an operator in two variables $L = \partial_t + \mathrm{i}b(x,t)\partial_x$ with $0 \leqslant b(x,t) \in C^{1+r}$ and solutions u and v which coincide for $t \leqslant 0$. The difference $u - v$ satisfies an inequality $|L(u-v)| \leqslant M|u-v|$ so using the similarity principle given by Theorem 6.1 we may write $u - v = \mathrm{e}^g h$ and the uniqueness property for the original equation is further reduced to that of the homogeneous equation

$Lh = 0$. The latter follows from an L^2-type Carleman estimate first proved in [20] which is known to hold when $b \in C^1$ [25, p. 9]. $\square$

The first result linking condition $(\mathcal{P})$ to uniqueness in the Cauchy problem for C^1 solutions of the linear equation $Lu = 0$ is due to Strauss and Treves [20]. Their Carleman estimate proves as well uniqueness for solutions of $|Lu| \leqslant M|u|$. Methods later developed based on the Baouendi–Treves approximation scheme [22,23] give uniqueness for solutions of $Lu = 0$ in the class of distributions but cannot handle directly solutions of $|Lu| \leqslant M|u|$. Here we have used the similarity principle to reduce uniqueness of $|Lu| \leqslant M|u|$ to the study of the homogeneous equation $Lu = 0$. Finally, we notice that, if $n \geqslant 2$ condition $(\mathcal{P})$ is essentially necessary if uniqueness in the Cauchy problem for the inequality $|Lu| \leqslant M|u$ is to hold. For instance, if condition $(\mathcal{P})$ is violated strongly at the origin in the sense that $\vec{b}(0,0)$ and $\vec{b}_t(0,0)$ are linearly independent, there exist smooth functions u and c supported on $t \geqslant 0$ and not vanishing identically in any neighborhood of the origin, such that $Lu + cu = 0$, in particular, the inequality $|Lu| \leqslant M|u|$ is valid in a neighborhood of the origin. We refer to [16,25] on the subject of counterexamples to uniqueness based on the methods of geometrical optics.

Appendix A. Hardy space lemmas

A.1. Multipliers in h^1

Consider a modulus of continuity $\omega(t)$ that satisfies

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t \leqslant K\left(1+\ln\frac{1}{h}\right)^{-1}, \quad 0 < h < 1, \tag{A.1}$$

and the corresponding space $C^\omega(\mathbb{R}^n)$.

Lemma A.1. *Let $b \in C^\omega(\mathbb{R}^n)$ and $f \in h^1(\mathbb{R}^n)$. Then $bf \in h^1(\mathbb{R}^n)$ and there exists $C > 0$ such that*

$$\|bf\|_{h^1} \leqslant C\|b\|_{C^\omega}\|f\|_{h^1}, \quad b \in C^\omega\big(\mathbb{R}^n\big),\ f \in h^1\big(\mathbb{R}^n\big).$$

Proof. Let $b(x) \in C^\omega$. It is enough to check that $\|bf\| \leqslant C\|b\|_{C^\omega}$ for every h^1-atom a with C an absolute constant. This fact is obvious for atoms supported in balls B with radius $\rho \geqslant 1$ without moment condition because b is bounded so $ba/\|b\|_{L^\infty}$ is again an atom without moment condition. If $B = B(x_0, \rho)$, $\rho < 1$, we may write $a(x)b(x) = b(x_0)a(x) + (b(x) - b(x_0))a(x) = \beta_1(x) + \beta_2(x)$. Then $\beta_1(x)/\|b\|_{L^\infty}$ is again an atom while $\beta_2(x)$ is supported in B and satisfies

$$\|\beta_2\|_{L^\infty} \leqslant 2\|b\|_{L^\infty}\|a\|_{L^\infty} \leqslant \frac{C}{\rho^n},$$

$$\|\beta_2\|_{L^1} \leqslant C\|a\|_{L^\infty} \int\limits_B \omega\big(|x-x_0|\big)\,\mathrm{d}x \leqslant \frac{C'}{(1+|\ln\rho|)}.$$

We wish to conclude that $\|m_\Phi\beta_2\|_{L^1} < \infty$. Let $B^* = B(x_0, 2\rho)$. Since $m_\Phi\beta_2(x) \leqslant M\beta_2(x)$, where M is the Hardy–Littlewood function, we have:

$$J_1 = \int\limits_{B^*} m_\Phi\beta_2(x)\,\mathrm{d}x \leqslant |B^*|^{1/2}\|M\beta_2\|_{L^2} \leqslant C\rho^{n/2}\|\beta_2\|_{L^2} \leqslant C'.$$

It remains to estimate

$$J_2 = \int\limits_{\mathbb{R}\setminus B^*} m_\Phi\beta_2(x)\,\mathrm{d}x = \int\limits_{2\rho\leqslant|x-x_0|\leqslant 2} m_\Phi\beta_2(x)\,\mathrm{d}x \tag{A.2}$$

(observe that $m_\Phi\beta_2$ is supported in $B(x_0, 2)$ because $\operatorname{supp}\Phi \subset B(0,1)$). If $0 < \varepsilon < 1$ and $\Phi_\varepsilon * \beta_2(x) \neq 0$ for some $|x-x_0| \geqslant 2\rho$ it is easy to conclude that $\varepsilon \geqslant |x-x_0|/2$, which implies

$$\big|\Phi_\varepsilon * \beta_2(x)\big| \leqslant \left|\int \Phi_\varepsilon(y)\beta_2(x-y)\,\mathrm{d}y\right| \leqslant \frac{C\|\beta_2\|_{L^1}}{\varepsilon^n} \leqslant \frac{C'|x-x_0|^{-n}}{(1+|\ln\rho|)},$$

so

$$m_\Phi\beta_2(x) \leqslant \frac{C'}{|x-x_0|^n(1+|\ln\rho|)} \quad \text{for } |x-x_0| \geqslant 2\rho. \tag{A.3}$$

It follows from (A.2) and (A.3) that

$$J_2 \leqslant \int\limits_{2\rho\leqslant|x-x_0|\leqslant 2} \frac{C'}{|x-x_0|^n(1+|\ln\rho|)}\,\mathrm{d}x \leqslant C''$$

which leads to

$$\|ba\|_{h^1} \leqslant \|\beta_1\|_{h^1} + \|\beta_2\|_{h^1} \leqslant C_1 + J_1 + J_2 \leqslant C_2.$$

Inspection of the proof shows that C_2 may be estimated by $C\|b\|_{C^\omega}$. $\square$

Example A.2. Suppose that a modulus of continuity $\omega(t)$ satisfies conditions:

$$\omega(t)/t^n \text{ is a decreasing function of } t, \tag{A.4}$$

$$D = \int_0^1 \frac{\omega(t)}{t}\,\mathrm{d}t < \infty. \tag{A.5}$$

A short and elegant argument shows (cf. [21, p. 25]) that under these conditions $h^1(\mathbb{R}^n)$ is stable under multiplication by elements of $C^\omega(\mathbb{R}^n)$. On the other hand, (A.5) alone already implies that

$$\omega(h)\ln\frac{1}{h} = \int_h^1 \frac{\omega(h)}{t}\,\mathrm{d}t \leqslant \int_h^1 \frac{\omega(t)}{t}\,\mathrm{d}t \leqslant D, \quad 0 < h < 1,$$

which keeping in mind the obvious estimate

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t \leqslant \frac{\omega(h)}{n},$$

shows that the modulus of continuity ω satisfies (A.1) and Lemma A.1 can be applied, proving the mentioned stability of $h^1(\mathbb{R}^n)$ under multiplication by elements of $C^\omega(\mathbb{R}^n)$.

Consider now a modulus of continuity $\omega(t)$ such that

$$\omega(t) = \frac{1 - n\ln t}{\ln^2 t} \quad \text{for } 0 < t < 1/2.$$

Since $\omega(t) \geqslant |\ln t|^{-1}$, it follows that $\int_0^{1/2}(\omega(t)/t)\,\mathrm{d}t = \infty$ and the Dini condition (A.5) is not satisfied. On the other hand,

$$\frac{1}{h^n}\int_0^h \omega(t)t^{n-1}\,\mathrm{d}t = \left(\ln\frac{1}{h}\right)^{-1} \approx \left(1 + \ln\frac{1}{h}\right)^{-1}, \quad \text{as } h \to 0,$$

so criterium (A.1) is satisfied. This shows that (A.5) is strictly more stringent than (A.1).

A.2. A local atomic decomposition

Lemma A.3. *Let $f \in h^1(\mathbb{R}^1)$ be supported in an interval $(-\alpha, \alpha)$. There exists an atomic decomposition $f = \sum_j \lambda_j a_j$ with h^1-atoms a_j supported in $(-\alpha - 1, \alpha + 1)$ and $\|f\|_{h^1} \sim \sum_j |\lambda_j|$.*

Proof. We start from some atomic decomposition $f = \sum \lambda_j a_j + \sum \Lambda_k B_k$ with $\|f\|_{h^1} \sim \sum_j |\lambda_j| + |\Lambda_j|$ and atoms satisfying

$$\operatorname{supp}(B_k) \subset J_k = (y_k - s_k, y_k + s_k), \quad s_k \geqslant 1/2 \quad \text{and}$$
$$\operatorname{supp}(a_j) \subset I_j = I(x_j - r_j, x_j + r_j), \quad r_j < 1/2.$$

Let $0 \leqslant \chi \leqslant 1 \in C_c^\infty(-\alpha - 1/2, \alpha + 1/2)$ satisfy $\chi(x) = 1$ for $|x| \leqslant \alpha$, and set $M = \sup|\chi'|$. We have:

$$\begin{aligned} f &= \sum \lambda_j \chi a_j + \sum \Lambda_k \chi B_k \\ &= \sum \lambda_j \chi(x_j) a_j + \sum M\lambda_j \big(\chi - \chi(x_j)\big)(a_j/M) + \sum \Lambda_k \chi B_k \\ &= \sum \lambda_j \tilde{a}_j + \sum M\lambda_j \widetilde{A}_j + \sum \Lambda_k \widetilde{B}_k, \end{aligned}$$

where all terms with $I_j \cap \operatorname{supp}(\chi) = \emptyset$ or $J_k \cap \operatorname{supp}(\chi) = \emptyset$ have been discarded. This gives the desired decomposition. Indeed, $\widetilde{B}_k = \chi B_k$ is clearly an h^1-atom with $\operatorname{supp}(\widetilde{B}_k) \subset (-\alpha - 1, \alpha + 1)$ for any k. Furthermore, $\widetilde{A}_j = (\chi - \chi(x_j)) a_j / M$ is also an h^1-atom because $\operatorname{supp}(\widetilde{A}_j) \subset I_j \subset (x_j - 1/2, x_j + 1/2)$ and

$$\|\widetilde{A}_j\|_\infty \leqslant r_j \|\chi'\|_\infty \|a_j\|_\infty / M \leqslant 1.$$

Observe that no moment conditions are required for $\widetilde{A}_j$ and $\widetilde{B}_k$. Finally, $\tilde{a}_j = \chi(x_j) a_j$ has mean equal to zero and thus it is an h^1-atom with $\operatorname{supp}(\tilde{a}_j) \subset I_j$. Since $r_j < 1/2$ and $I_j \cap \operatorname{supp}(\chi) \neq \emptyset$ we see that

$$I_j \subset \operatorname{supp}(\chi) + (-1/2, 1/2) \subset (-\alpha - 1, \alpha + 1)$$

and we conclude that all atoms are supported as we wished. Furthermore,

$$\|f\|_{h^1} \leqslant C \Big\{ \sum M |\lambda_j| + \sum |\Lambda_k| \Big\} \leqslant C' \|f\|_{h^1}. \qquad \square$$

A.3. Commutators

We consider now a bounded smooth function $\psi(\xi)$, $\xi \in \mathbb{R}$, such that

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}x^k} \psi(\xi) \right| \leqslant C_k \frac{1}{(1 + |\xi|)^k}, \quad \xi \in \mathbb{R},\ k = 0, 1, 2, \ldots .$$

Then $\psi(\xi)$ is a symbol of order zero and defines the pseudo-differential operator:

$$\psi(D) u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi} \psi(\xi) \widehat{u}(\xi)\, \mathrm{d}\xi, \quad u \in \mathcal{S}(\mathbb{R}).$$

In particular, $\psi(D)$ is bounded in $h^1(\mathbb{R})$. The Schwartz kernel of $\psi(D)$ is the tempered distribution $k(x - y)$ defined by $\widehat{k}(\xi) = \psi(\xi)$ which is smooth outside the diagonal $x \neq y$. Moreover $k(x - y)$ may be expressed as

$$k(x - y) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int \mathrm{e}^{\mathrm{i}(x-y)\xi - \varepsilon|\xi|^2} \psi(\xi)\, \mathrm{d}\xi = \lim_{\varepsilon \to 0} k_\varepsilon(x - y),$$

where the limit holds both in the sense of $\mathcal{S}'$ and pointwise for $x \neq y$. Furthermore, the approximating kernels $k_\varepsilon(x-y)$ satisfy uniformly in $0 < \varepsilon < 1$ the pointwise estimates

$$\left|k_\varepsilon(x-y)\right| \leqslant \frac{C_j}{|x-y|^j}, \quad j = 1, 2, \ldots, \tag{A.6}$$

which of course also hold for $k(x-y)$ itself when $x \neq y$.

We consider a function $b(x)$ of class $C^{1+\sigma}$, $0 < \sigma < 1$, and wish to prove that the commutator $[\psi(D), b\partial_x]$ is bounded in $h^1(\mathbb{R})$. A simple standard computation that includes an integration by parts gives:

$$\left[\psi(D), b\partial_x\right]u(x) = \int k'(x-y)\big(b(y) - b(x)\big)u(y)\,\mathrm{d}y - \psi(D)(b'u),$$

where the integral should be interpreted as the pairing

$$\left\langle k'(x-\cdot)\big(b(\cdot) - b(x)\big), u(\cdot)\right\rangle$$

between a distribution depending on the parameter x and a test function u. Since multiplication by b' is bounded in $h^1(\mathbb{R})$ with norm controlled by $\|b'\|_{C^\sigma}$, we need only worry with the remaining integral term that can be rewritten as

$$\begin{aligned} Tu(x) &= \int (y-x)k'(x-y)\frac{b(x)-b(y)}{x-y}u(y)\,\mathrm{d}y \\ &= \int k_1(x-y)\beta(x,y)u(y)\,\mathrm{d}y, \end{aligned} \tag{A.7}$$

$$\text{where} \quad \beta(x,y) = \int_0^1 b'\big(\tau x + (1-\tau)y\big)\,\mathrm{d}\tau \quad \text{and} \quad k_1(x) = -xk'(x).$$

Observe that $\beta \in C^\sigma(\mathbb{R}^2)$.

Lemma A.4. *Assume T is given by* (A.7) *with kernel $K(x,y) = k_1(x-y)\,\beta(x,y)$. Then T is bounded in $h^1(\mathbb{R})$.*

Proof. It follows that $\widehat{k_1}(\xi) = (\xi k(\xi))' = k(\xi) + \xi\, k'(\xi)$. In other words, $\widehat{k_1}(\xi) = \psi_1(\xi)$ is a symbol of order zero and T has kernel $k_1(x-y)\beta(x,y)$. We may write $\beta(x,y) = b'(x) + |x-y|^\sigma r(x,y)$ with $r(x,y) \in L^\infty(\mathbb{R}^2)$ so

$$\begin{aligned} Tu(x) &= b'(x)\psi_1(D)u(x) + \int k_1(x-y)\,|x-y|^\sigma r(x,y)u(y)\,\mathrm{d}y \\ &= T_1u(x) + T_2u(x). \end{aligned}$$

The first operator T_1 is obviously bounded in h^1 because it is the composite of $\psi_1(D)$ with multiplication by a C^σ function. To analyze T_2 we check – writing $k_1 = \lim_{\varepsilon\to 0} k_{1,\varepsilon}$ and using (A.6) for $k_{1,\varepsilon}$ – that its Schwartz kernel is a locally integrable distribution given by the integrable function $k_2(x, y) = k_1(x - y)\,|x - y|^\sigma r(x, y)$. Hence,

$$\left|k_2(x, y)\right| \leqslant C_1\left|k_1(x - y)\right|\,|x - y|^\sigma = k_3(x - y).$$

Observe that $k_3(x) \leqslant C\,\min(|x|^{\sigma-1}, |x|^{-2})$ so $k_3 \in L^1(\mathbb{R})$. We will now show that

$$m_\Phi k_3(x) = \sup_{0<\varepsilon<1}\left|\Phi_\varepsilon * k_3(x)\right| \in L^1(\mathbb{R}),$$

where $\Phi \geqslant 0 \in C_c^\infty([-1/2, 1/2])$, $\int \Phi\,\mathrm{d}z = 1$, $\Phi_\varepsilon(x) = \varepsilon^{-1}\Phi(x/\varepsilon)$. Since $m_\Phi k_3$ is pointwise majorized by the restricted Hardy–Littlewood maximal function

$$mk_3(x) = \sup_{0<\varepsilon<1}\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} k_3(t)\,\mathrm{d}t,$$

we start by observing that

$$\sup_{0<\varepsilon<1}\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t \leqslant \frac{|x|^{\sigma-1}}{\sigma}. \tag{A.8}$$

In doing so we may assume that $x > 0$. If $0 < \varepsilon \leqslant x$, we have:

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t = \frac{(x+\varepsilon)^\sigma - (x-\varepsilon)^\sigma}{2\varepsilon\sigma} \leqslant \frac{(x+\varepsilon)^{\sigma-1}}{\sigma} \leqslant \frac{x^{\sigma-1}}{\sigma},$$

where we have used the elementary inequality

$$\frac{b^\sigma - a^\sigma}{b - a} \leqslant b^{\sigma-1}, \quad 0 \leqslant a < b,\ 0 < \sigma < 1.$$

Similarly, if $0 < x < \varepsilon$,

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |t|^{\sigma-1}\,\mathrm{d}t = \frac{(x+\varepsilon)^\sigma + (x-\varepsilon)^\sigma}{2\varepsilon\sigma} \leqslant \frac{(x+\varepsilon)^{\sigma-1}}{\sigma} \leqslant \frac{x^{\sigma-1}}{\sigma}.$$

This proves (A.8). Thus,

$$m_\Phi k_3(x) \leqslant C\,mk_3(x) \leqslant C'\,|x|^{\sigma-1},$$

which shows that $m_\Phi k_3$ is locally integrable. For large $|x|$ the inequality $k_3(x) \leqslant C|x|^{-2}$ easily implies $m_\Phi k_3(x) \leqslant C|x|^{-2}$ and we conclude that $m_\Phi k_3 \in L^1$. Finally, we see that

$$\left|\Phi_\varepsilon * T_2 u(x)\right| \leqslant \Phi_\varepsilon * k_3 * |u|(x) \leqslant m_\Phi k_3 * |u|(x),$$

so $m_\Phi T_2 u(x) \leqslant m_\Phi k_3 * |u|(x)$ which implies that $\|T_2 u\|_{h^1} \leqslant C\|u\|_{L^1} \leqslant C\|u\|_{h^1}$. This proves that $T = T_1 + T_2$ is bounded in $h^1(\mathbb{R})$. $\square$

Summing up, we have proved the

Proposition A.5. *If $\psi(\xi)$, $\xi \in \mathbb{R}$, is a smooth symbol of order 0 and $b(x) \in C^{1+\sigma}(\mathbb{R})$, $0 < \sigma < 1$, the commutator $[\psi(D), b\partial_x]$ is bounded in $h^1(\mathbb{R})$.*

A.4. Change of variables

Consider a real function $F \in C^2(\mathbb{R})$ such that for some $K \geqslant 1$,

$$\frac{1}{K} \leqslant F'(x) \leqslant K, \quad \left|F''(x)\right| \leqslant K, \quad x \in \mathbb{R}.$$

Proposition A.6. *The map $h^1(\mathbb{R}) \ni u \mapsto u \circ F$ is bounded in $h^1(\mathbb{R})$.*

Proof. It is enough to show that there is a constant $C > 0$ such that $\|a \circ F\|_{h^1} \leqslant C$ for all h^1-atoms $a(x)$, i.e., $\|m_\Phi(a \circ F)\|_{L^1} \leqslant C$ where $\Phi \geqslant 0 \in C_c^\infty([-1/2, 1/2])$ such that $\int \Phi \,\mathrm{d}z = 1$ has been fixed. If a is supported in an interval I then $A = a \circ F$ is supported in $J = F^{-1}(I)$ and $K^{-1}|I| \leqslant |J| \leqslant K|I|$. Thus, if $|I| \geqslant 1$ and $\|a\|_{L^\infty} \leqslant |I|^{-1}$ it follows that A is supported in some interval J with $|J| = K|I| \geqslant K \geqslant 1$ and $\|A\|_{L^\infty} = \|a\|_{L^\infty} \leqslant K|J|^{-1}$ so A/K is an atom and $\|a \circ F\|_{h^1} \leqslant C$. Let us now assume that $|I| \leqslant 1$ and $\int a(x)\,\mathrm{d}x = 0$. Choose J containing the support of $A = a \circ a$ such that $|J| = K|I|$. Note that $m_\Phi A$ is supported in $\tilde{J} = [-1/2, 1/2] + J$ which has lenght $|\tilde{J}| \leqslant K + 1$. We write

$$\int_{\mathbb{R}} m_\Phi A \,\mathrm{d}x = \int_{J^*} m_\Phi A \,\mathrm{d}x + \int_{\tilde{J}\setminus J^*} m_\Phi A \,\mathrm{d}x = L_1 + L_2,$$

where $J = [x_0 - \ell, x_0 + \ell]$ and $J^* = [x_0 - 2\ell, x_0 + 2\ell]$. We have

$$L_1 \leqslant \|m_\Phi A\|_{L^\infty}|J^*| \leqslant 2K\|a\|_{L^\infty}|I| \leqslant 2K.$$

To estimate L_2 we study $m_\Phi A(x)$ for $|x - x_0| \geqslant 2\ell$. In this case

$$\Phi_\varepsilon * A(x) = \int_{x_0-\ell}^{x_0+\ell} \Phi_\varepsilon(x - y)A(y)\,\mathrm{d}y$$

vanishes if $\varepsilon \leqslant |x - x_0|/2$, so we may restrict our attention to values of $\varepsilon > |x - x_0|/2$. The Taylor formula of order one for $y \mapsto \Phi_\varepsilon(x - y)$ around $x - x_0$ gives

$$\Phi_\varepsilon(x - y) = \Phi_\varepsilon(x - x_0) + \frac{x_0 - y}{\varepsilon^2} r(x, y),$$

where the remainder $r(x, y)$ is bounded. We may write

$$\begin{aligned} \Phi_\varepsilon * A(x) &= \Phi_\varepsilon(x - x_0) \int A(y)\,\mathrm{d}y + \int \frac{x_0 - y}{\varepsilon^2} r(x, y) A(y)\,\mathrm{d}y \\ &= g_\varepsilon(x) + h_\varepsilon(x) \end{aligned}$$

and

$$\bigl|\Phi_\varepsilon * A(x)\bigr| \leqslant \sup_{(|x-x_0|/2)<\varepsilon<1} \bigl|g_\varepsilon(x)\bigr| + \sup_{(|x-x_0|/2)<\varepsilon<1} \bigl|h_\varepsilon(x)\bigr| = g(x) + h(x).$$

Since $|x_0 - y| \leqslant \ell$ when y belongs to the support of A and $\varepsilon > |x - x_0|/2$ we obtain $|h_\varepsilon(x)| \leqslant C\ell |x - x_0|^{-2}$ which yields

$$\int_{\mathbb{R}\setminus J^*} h(x)\,\mathrm{d}x \leqslant C.$$

To estimate $g_\varepsilon(x)$ we introduce the change of variables $z = F(y)$ to get

$$\int A(y)\,\mathrm{d}y = \int a\bigl(F(y)\bigr)\,\mathrm{d}y = \int a(z)\delta(z)\,\mathrm{d}z \quad \text{where } \delta(z) = \bigl[F^{-1}\bigr]'(z).$$

Set $z_0 = F(x_0) \in I$ and write $\delta(z) = \delta(z_0) + (z - z_0)r(z)$, where $\|r\|_{L^\infty} \leqslant \|[F^{-1}]''\|_{L^\infty} \leqslant K^3$. Since a has vanishing mean we get, recalling that $|I| \leqslant K^{-1}|J| \leqslant 2\ell$,

$$\left|\int A(y)\,\mathrm{d}y\right| = \left|\int a(z)(z - z_0)r(z)\,\mathrm{d}z\right| \leqslant C\ell.$$

The variable factor in the expression of $g_\varepsilon(x)$ is $\Phi_\varepsilon(x - x_0)$ that may be estimated by $C/|x - x_0|$ on $\mathbb{R} \setminus J^*$, so

$$\int_{\tilde{J}\setminus J^*} g(x)\,\mathrm{d}x \leqslant C\ell \int_{2\ell}^{K+1} \frac{\mathrm{d}s}{s} \leqslant C\ell \ln\frac{K+1}{2\ell} \leqslant C_1,$$

as $2\ell = |J| \leqslant K$. Since $m_\Phi A(x) \leqslant g(x) + h(x)$ we see that L_2 is bounded by a constant that depends only on K. $\square$

Additional literature [7,8].

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