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Polaroid type operators under quasi-affinities

Pietro Aiena [a,*], Muneo Chō [b], Manuel González [c]

[a] Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy
[b] Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan
[c] Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain

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ABSTRACT

In this paper we study the preservation of some polaroid conditions under quasi-affinities. As a consequence, we derive several results concerning the preservation of Weyl type theorems and generalized Weyl type theorems under quasi-affinities.

1. Preliminaries

Two operators $T \in L(X)$, $S \in L(Y)$, X and Y Banach spaces, are said to be *intertwined* by $A \in L(X, Y)$ if $SA = AT$; and A is said to be a *quasi-affinity* if it has a trivial kernel and dense range. If T and S are intertwined by a quasi-affinity then T is called a *quasi-affine transform* of S, and we write $T \prec S$. In [4], it has been studied the preservation of Weyl type theorems between two operators intertwined by a quasi-affinity A, and the more general case in which T and S are asymptotically intertwined by A. An operator $T \in L(X)$ is said to be *polaroid* if every isolated point of the spectrum is a pole of the resolvent of T. It is known that the polaroid condition on T and the single-valued extension property (SVEP) for T, or for its dual T^*, imply that both T and T^* satisfy Weyl's theorem (see next Theorem 2.3). Moreover, if T^* has SVEP (respectively, T has SVEP) then Weyl's theorem for T (respectively, for T^*) is equivalent to several other variants of it, see [3]. The SVEP is preserved if T and S are intertwined by an injective map (see next Lemma 3.1). For this reason it is interesting to study the preservation of the polaroid condition from S to T (or also some other related conditions) in the case where T and S are intertwined or asymptotically intertwined by a quasi-affinity. In this paper we determine some sufficient conditions which ensure this preservation and, as a consequence, we extend in several directions many of the results established in [4], concerning the transmission of Weyl type theorems from S to T.

We first fix the terminology here used. Let X be an infinite-dimensional complex Banach space and let $T \in L(X)$. We denote by $\alpha(T)$ the dimension of the kernel $\ker T$ and by $\beta(T)$ the codimension of the range $T(X)$. Recall that the operator $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and the range $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) < \infty$. If either T is upper or lower semi-Fredholm then T is said to be a *semi-Fredholm operator*, while if T is both upper and lower semi-Fredholm then T is said to be a *Fredholm operator*. If T is semi-Fredholm then the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. An operator $T \in L(X)$ is said to be a *Weyl operator*, $T \in W(X)$, if T is a Fredholm operator having index 0. We also consider the set $W_+(X) := \{T \in \Phi_+(X)\colon \operatorname{ind} T \leqslant 0\}$.

* Corresponding author.
E-mail addresses: paiena@unipa.it (P. Aiena), chiyom01@kanagawa-u.ac.jp (M. Chō), gonzalem@unican.es (M. González).

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The *Weyl spectrum* and the *upper semi-Weyl spectrum* are defined, respectively

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin W(X)\},$$
$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin W_+(X)\}.$$

The *ascent* of an operator $T \in L(X)$ is defined as the smallest non-negative integer $p := p(T)$ such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* of T is defined as the smallest non-negative integer $q := q(T)$ such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, if $\lambda \in \mathbb{C}$ then $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent of T. In this case λ is an eigenvalue of T and an isolated point of the spectrum $\sigma(T)$, see [25, Proposition 50.2]. An operator $T \in L(X)$ is said to be *Browder* if $T \in \Phi(X)$ and $p(T) = q(T) < \infty$. We denote by $B(X)$ the class of Browder operators and $B_+(X) := \{T \in \Phi_+(X)\colon\ p(T) < \infty\}$. The *Browder spectrum* and the *upper semi-Browder spectrum* of T are defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin B(X)\},$$
$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C}\colon\ \lambda I - T \notin B_+(X)\}.$$

Clearly, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

Let $\sigma_{\mathrm{a}}(T)$ denote the classical *approximate point spectrum* of T and let $\sigma_{\mathrm{s}}(T)$ be the *surjectivity spectrum* of T. Define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T)\colon\ 0 < \alpha(\lambda I - T) < \infty\}$$

and

$$\pi_{00}^{a}(T) := \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)\colon\ 0 < \alpha(\lambda I - T) < \infty\},$$

where $\operatorname{iso} A$ is the set of isolated points of A. Let $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, i.e. $p_{00}(T)$ is the set of all poles of the resolvent of T having finite rank. Clearly, for every $T \in L(X)$ we have

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^{a}(T). \tag{1}$$

We now introduce the Weyl type theorems:

Definition 1.1. An operator $T \in L(X)$ is said to satisfy (W), *Weyl's theorem*, if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. T is said to satisfy (aW), *a-Weyl's theorem*, if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^{a}(T)$. T is said to satisfy property (w) if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$.

The concept of semi-Fredholm operator has been generalized by Berkani [15,17] in the following way: for every $T \in L(X)$ and a non-negative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi-B-Fredholm* (resp. *B-Fredholm*, *upper semi-B-Fredholm*, *lower semi-B-Fredholm*) if for some integer $n \geqslant 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator with $\operatorname{ind} T_n = \operatorname{ind} T_m$ for all $m \geqslant n$ [17]. This enables one to define the index of a semi-B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

An operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi-B-Weyl*) if for some integer $n \geqslant 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl). These classes of operators generate the *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$ and the *upper B-Weyl spectrum* $\sigma_{\mathrm{usbw}}(T)$. Analogously, $T \in L(X)$ is said to be *B-Browder* (respectively, *upper semi-B-Browder*) if for some integer $n \geqslant 0$ $T^n(X)$ is closed and $T_{[n]}$ is Browder (respectively, upper semi-Browder). The *B-Browder spectrum* is denoted by $\sigma_{\mathrm{bb}}(T)$ and the *upper semi-B-Browder spectrum* by $\sigma_{\mathrm{usbb}}(T)$.

If $T \in L(X)$ define

$$E(T) := \{\lambda \in \operatorname{iso}\sigma(T)\colon\ 0 < \alpha(\lambda I - T)\},$$

and

$$E^{a}(T) := \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)\colon\ 0 < \alpha(\lambda I - T)\}.$$

Evidently, $E(T) \subseteq E^{a}(T)$ for every $T \in L(X)$. Now we introduce the generalized version of Weyl type theorems:

Definition 1.2. An operator $T \in L(X)$ is said to satisfy (gW), the *generalized Weyl's theorem*, if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T)$. $T \in L(X)$ is said to satisfy (gaW), the *generalized a-Weyl's theorem*, if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbw}}(T) = E^{a}(T)$. $T \in L(X)$ is said to satisfy (gw), the *generalized property* (w), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{usbw}}(T) = E(T)$.

Both a-Weyl's theorem and property (w) have been introduced by Rakočević [31,32]. In the following diagram we resume the relationships between all Weyl type theorems, generalized or not:

$$(gw) \quad\Rightarrow\quad (w) \quad\Rightarrow\quad (W),$$
$$(gaW) \quad\Rightarrow\quad (aW) \quad\Rightarrow\quad (W),$$

see [13, Theorem 2.3], [10,16]. Generalized property (w) and generalized a-Weyl's theorem are also independent, see [13]. Furthermore (see [13,16]),

$$(gw) \quad\Rightarrow\quad (gW) \quad\Rightarrow\quad (W),$$
$$(gaW) \quad\Rightarrow\quad (gW) \quad\Rightarrow\quad (W).$$

The following property has a relevant role in local spectral theory, see the recent monographs [28,1].

Definition 1.3. An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

The operator T is said to have SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.

It follows from the identity theorem for analytic functions that T has SVEP at every point of the boundary of the spectrum. In particular, T and T^* have SVEP at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \quad\Rightarrow\quad T \text{ has SVEP at } \lambda, \tag{2}$$

and dually

$$q(\lambda I - T) < \infty \quad\Rightarrow\quad T^* \text{ has SVEP at } \lambda. \tag{3}$$

Two important T-invariant subspaces of T are defined as follows. The *quasi-nilpotent part* of $T \in L(X)$ is defined as the set

$$H_0(T) := \left\{x \in X\colon \lim_{n\to\infty} \left\|T^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. The *analytic core* of $T \in L(X)$ is defined $K(T) := \{x \in X\colon$ there exist $c > 0$ and a sequence $(x_n)_{n\geqslant 1} \subseteq X$ such that $Tx_1 = x$, $Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$, and $\|x_n\| \leqslant c^n\|x\|$ for all $n \in \mathbb{N}\}$. Note that $T(K(T)) = K(T)$ [1, Chapter 1]. Moreover (see [6]),

$$H_0(\lambda I - T) \text{ closed} \quad\Rightarrow\quad T \text{ has SVEP at } \lambda. \tag{4}$$

Remark 1.4. If $\lambda I - T$ is semi-Fredholm then the implications (2), (3), and (4) are equivalences, see [1, Chapter 3].

Weyl's theorem may be characterized as follows:

Theorem 1.5. *(See [2].) An operator $T \in L(X)$ satisfies Weyl's theorem if only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{w}}(T)$ and $p_{00}(T) = \pi_{00}(T)$.*

Note that

$$T \text{ has SVEP at } \lambda \notin \sigma_{\mathrm{w}}(T) \quad\Leftrightarrow\quad T^* \text{ has SVEP at } \lambda \notin \sigma_{\mathrm{w}}(T), \tag{5}$$

see [5, Theorem 2.3].

In the sequel, $\Pi(T)$ denotes the set of all poles of the resolvent of T. Obviously, $\Pi(T) \subseteq E(T)$.

Theorem 1.6. *(See [7].) An operator $T \in L(X)$ satisfies generalized Weyl's theorem if only if T has SVEP at every $\lambda \notin \sigma_{\mathrm{usbw}}(T)$ and $E(T) = \Pi(T)$.*

2. Polaroid type operators

We now consider some variants of the polaroid property.

Definition 2.1. An operator $T \in L(X)$ is said to be *a-polaroid* if every $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent; T is said to be *left polaroid* if for every $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$, $p := p(\lambda I - T) < \infty$ and $T^{p+1}(X)$ is closed; and T is said to be *right polaroid* if for every $\lambda \in \operatorname{iso}\sigma_{\mathrm{s}}(T)$, $q := q(\lambda I - T) < \infty$ and $T^{q+1}(X)$ is closed.

Trivially, every a-polaroid operator is left polaroid, while a polaroid operator may be not left polaroid, see [3]. We have $\operatorname{iso}\sigma(T) \subseteq \sigma_a(T)$ for every $T \in L(X)$, since the boundary of $\sigma(T)$ is contained in $\sigma_a(T)$ [1, Theorem 2.42]; hence,

$$T \text{ } a\text{-polaroid} \quad \Rightarrow \quad T \text{ polaroid.} \tag{6}$$

Left and right polaroid operators are dual each other: T is left polaroid (respectively, right polaroid) if and only T^* is right polaroid (respectively, left polaroid), see [3]. It is well known that λ is a pole of the resolvent of T if and only if λ is a pole of the resolvent of T^*. Since $\sigma(T) = \sigma(T^*)$ then T is polaroid if and only if T^* is polaroid. It should be noted that the condition $p_{00}(T) = \pi_{00}(T)$ which appears in Theorem 1.5 is equivalent to saying that there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T) \tag{7}$$

(see [8, Theorem 2.2]). Also the condition of being polaroid may be characterized by means of the quasi-nilpotent part:

Theorem 2.2. *If $T \in L(X)$ the following statements hold*:

(i) *T is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \operatorname{iso}\sigma(T). \tag{8}$$

(ii) *If T is left polaroid then there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \operatorname{iso}\sigma_a(T). \tag{9}$$

Proof. Suppose T satisfies (8) and that λ is an isolated point of $\sigma(T)$. Since λ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p\big(K(\lambda I - T)\big) = K(\lambda I - T).$$

So $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leqslant p$, hence λ is a pole of the resolvent, so that T is polaroid. Conversely, suppose that T is polaroid and λ is an isolated point of $\sigma(T)$. Then λ is a pole, and if p is its order then $H_0(\lambda I - T) = \ker(\lambda I - T)^p$, see Theorem 3.74 of [1].

(ii) See Theorem 2.4 of [11]. □

Trivially, for every polaroid operator the equality $p_{00}(T) = \pi_{00}(T)$ is satisfied. The following result gives a very simple and useful framework for establishing Weyl's theorem for several classes of operators:

Theorem 2.3. *It $T \in L(X)$ is polaroid and either T or T^* has SVEP then both T and T^* satisfy Weyl's theorem.*

Proof. The polaroid condition for T entails that $p_{00}(T) = \pi_{00}(T)$. If T is polaroid then T^* is polaroid and hence $p_{00}(T^*) = \pi_{00}(T^*)$. Weyl's theorem for T and T^* then follows from Theorem 1.5. □

Theorem 2.4. *(See [3].) Let $T \in L(X)$ be polaroid. Then we have*:

(i) *If T^* has SVEP then (W), (aW), (w), (gW), (gaW) and (gw) hold for T, while T^* satisfies (gW).*
(ii) *If T has SVEP then (W), (aW), (w), (gW), (gaW) and (gw) hold for T^*, while T satisfies (gW).*

In the sequel we shall use the expression T *satisfies all Weyl type theorems* in the case that T satisfies (W), (aW), (w), (gW), (gaW) and (gw).

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non-constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Theorem 2.5. *Let $f \in \mathcal{H}_{nc}(\sigma(T))$. If T is polaroid then $f(T)$ is polaroid.*

Proof. Let $\lambda_0 \in \operatorname{iso}\sigma(f(T))$. The spectral mapping theorem implies $\lambda_0 \in \operatorname{iso} f(\sigma(T))$. Let us show that $\lambda_0 \in f(\operatorname{iso}\sigma(T))$.

Select $\mu_0 \in \sigma(T)$ such that $f(\mu_0) = \lambda_0$. Denote by Ω the connected component of the domain of f which contains μ_0 and suppose that μ_0 is not isolated in $\sigma(T)$. Then there exists a sequence $(\mu_n) \subset \sigma(T) \cap \Omega$ of distinct scalars such that $\mu_n \to \mu_0$. Since $K := \{\mu_0, \mu_1, \mu_2, \ldots\}$ is a compact subset of Ω, the principle of isolated zeros of analytic functions says to us that f may assume the value $\lambda_0 = f(\mu_0)$ only a finite number of points of K; so for n sufficiently large $f(\mu_n) \neq f(\mu_0) = \lambda_0$, and since $f(\mu_n) \to f(\mu_0) = \lambda_0$ it then follows that λ_0 is not an isolated point of $f(\sigma(T))$, a contradiction. Hence $\lambda_0 = f(\mu_0)$, with $\mu_0 \in \operatorname{iso}\sigma(T)$. Since T is polaroid, μ_0 is a pole of T and by [11, Theorem 2.9]; hence λ_0 is a pole for $f(T)$, which proves that $f(T)$ is polaroid. □

3. Polaroid type operators under quasi-affinities

If $T \in L(X)$ and $S \in L(Y)$ the *commutator* $C(S,T)$ is the mapping on $L(X,Y)$ defined by $C(S,T)(A) := SA - AT$. By induction it is easily to show the binomial identity

$$C(S,T)^n(A) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A T^k. \tag{10}$$

Obviously, $C(\lambda I - S, \lambda I - T)^n(A) = (-1)^n C(S,T)^n(A)$ for all $\lambda \in \mathbb{C}$. From (10) we easily obtain

$$C(S,T)^n(A)x = S^n Ax \quad \text{for all } x \in \ker T. \tag{11}$$

Let now consider the *higher order intertwining condition* defined by

$$C(S,T)^n(A) = 0 \quad \text{for some } n \in \mathbb{N}.$$

Clearly, this notion is a generalization of the condition $C(S,T)(A) = 0$ which appears in the definition of $T \prec S$.

Lemma 3.1. *Let $T \in L(X)$, $S \in L(Y)$ and suppose that for some injective map $A \in L(X,Y)$ there exists an integer $n \in \mathbb{N}$ for which $C(S,T)^n(A) = 0$. If S has SVEP at λ_0 then T has SVEP at λ_0. In particular, if $T \in L(X)$ and $S \in L(Y)$ are intertwined by an injective map $A \in L(X,Y)$ then localized SVEP carries over from S to T.*

Proof. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open neighborhood of λ_0 and $f : \mathcal{U} \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$, for all $\lambda \in \mathcal{U}$. Since $f(\lambda) \in \ker(\lambda I - T)$ taking into account (11) we then obtain

$$\begin{aligned} 0 &= (\lambda I - S)\big[C(S,T)^n(A)f(\lambda)\big] = (\lambda I - S)\big[C(\lambda I - S, \lambda I - T)^n(A)f(\lambda)\big] \\ &= (\lambda I - S)^{n+1} Af(\lambda). \end{aligned}$$

Now, $(\lambda I - S)^{n+1} Af(\lambda) = (\lambda I - S)[(\lambda I - S)^n Af(\lambda)]$ on $\mathcal{U}$ and the SVEP of S at λ_0 implies $(\lambda I - S)^n Af(\lambda) = 0$. Repeating this argument we easily deduce that $(\lambda I - S)(A(f\lambda)) = 0$. Since S has SVEP at λ_0 it then follows that $Af(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ and the injectivity of A entails $f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$. Therefore T has the SVEP at λ_0. The last assertion is clear. □

The following example shows that converse of Lemma 3.1 does not hold, i.e. if $T \prec S$ the SVEP from T may be not transmitted to S.

Example 3.2. Let C denote the Cesàro matrix. C is a lower triangular matrix such that the nonzero entries of the n-th row are n^{-1} $(n \in \mathbf{N})$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Let $1 < p < \infty$ and consider the matrix C as an operator C_p acting on ℓ_p. Let q be such that $1/p + 1/q = 1$. In [33] it has been proved that $\sigma(C_p)$ is the closed disc Γ_q, where

$$\Gamma_q := \big\{\lambda \in \mathbb{C}\colon |\lambda - q/2| \leqslant q/2\big\}.$$

Moreover, it has been proved in [23] that for each $\mu \in \operatorname{int}\Gamma_q$ the operator $\mu I - C_p$ is an injective Fredholm operator with $\beta(C_p) = 1$. Consequently, every $\mu \in \operatorname{int}\Gamma_q$ belongs to the surjectivity spectrum $\sigma_s(C_p)$.

Let $C_p^* \in L(\ell_q)$ denote the conjugate operator of C_p. Obviously, $\sigma_s(C_p)$ clusters at every $\mu \in \operatorname{int}\Gamma_q$ and since $\mu I - C_p$ is Fredholm it then follows that C_p^* does not have SVEP at these points μ, see [1, Theorem 3.27]. Every operator has SVEP at the boundary of the spectrum, and since $\sigma(C_p^*) = \sigma(C_p) = \Gamma_q$ it then follows that C_p^* has SVEP at λ precisely when $\lambda \notin \operatorname{int}\Gamma_q$. Choose $1 < p' < p < \infty$ and let q' be such that $1/p' + 1/q' = 1$. Then $1 < q < q' < \infty$. If we denote by $A : \ell_q \to \ell_{q'}$ the natural inclusion then we have $C_{p'}^* A = AC_p^*$ and clearly A is an injective operator with dense range, i.e., $C_p^* \prec C_{p'}^*$. As noted before the operator C_p^* has SVEP at every point outside of Γ_q, in particular at the points $\lambda \in \Gamma_{q'} \setminus \Gamma_q$, while $C_{p'}^*$ fails SVEP at the points $\lambda \in \Gamma_{q'} \setminus \Gamma_q$ which do not belong to the boundary of $\Gamma_{q'}$.

Recall from the classical functional calculus that to every $\lambda \in \operatorname{iso}\sigma(T)$ we can associate to T and the spectral set $\{\lambda\}$, the *spectral projection* $P_T(\lambda)$ defined by

$$P_T(\lambda) := \frac{1}{2\pi i} \int_{\Gamma} (\mu I - T)^{-1}\, d\mu,$$

where Γ denotes an oriented closed path which separates λ from the remaining part of the spectrum.

Lemma 3.3. *Suppose that $T \in L(X)$ and $S \in (Y)$ are intertwined by $A \in L(X, Y)$. If $\lambda \in \operatorname{iso}\sigma(T) \cap \operatorname{iso}\sigma(S)$ then $P_T(\lambda)$ and $P_S(\lambda)$ are also intertwined by A, i.e. $P_S(\lambda)A = AP_T(\lambda)$.*

Proof. If T and S are intertwined by $A \in L(X, Y)$ we have $(\mu I - S)A = A(\mu I - T)$ for all $\mu \in \mathbb{C}$. Suppose that μ belongs to the resolvent of T as well to the resolvent of S. Then $A = (\mu I - S)^{-1} A(\mu I - T)$ and hence $A(\mu I - T)^{-1} = (\mu I - S)^{-1}A$, from which it easily follows that

$$\begin{aligned} P_S(\lambda)A &= \left(\frac{1}{2\pi i} \int_{\Gamma} (\mu I - S)^{-1}\, d\mu\right) A = \frac{1}{2\pi i} \int_{\Gamma} (\mu I - S)^{-1} A\, d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma} A(\mu I - T)^{-1}\, d\mu = AP_T(\lambda). \qquad \square \end{aligned}$$

Recall that $T \in L(X)$ and $S \in L(Y)$ are said to be *quasi-similar* if there exist two quasi-affinities $A \in L(X, Y)$, $B \in L(Y, X)$ for which $SA = AT$ and $BS = TB$. If $T \prec S$ a classical result due to Rosenblum shows that $\sigma(S)$ and $\sigma(T)$ must overlap, see [34]. But quasi-similarity is, in general, not sufficient to preserve the spectrum. This happens only in some special cases, for instance if T and S are quasi-similar hyponormal operators [18], or whenever T and S have totally disconnected spectra, see [24, Corollary 2.5]. Therefore, it is not quite surprising that, if $T \prec S$, the preservation of "certain" spectral properties from S to T requires that some spectral inclusions are satisfied.

Remark 3.4. Classical examples show the polaroid property is not preserved if two bounded operators are intertwined by an injective map. For instance by [22] or [26], there exist bounded linear operators U, V, B on a Hilbert space such that $BU = UV$, B and its Hilbert adjoint B' are injective, V is quasi-nilpotent and the spectrum of U the unit disc $D(0, 1)$. Let $T := V'$, $S := U'$ and $A := B'$. Then $SA = AT$, so that T and S are intertwined by the injective operator A, S is polaroid, since $\sigma(S) = \overline{\sigma(U)} = D(0, 1)$ has no isolated points, while T is also quasi-nilpotent and hence not polaroid.

The next example shows that a polaroid operator may be the quasi-affine transform of an operator which is not polaroid. In the sequel, we refer to [35] for the general properties of shift operators.

Example 3.5. Let $S \in L(\ell^2(\mathbb{N}))$ be the weighted unilateral right shift defined as

$$S(x_1, x_2, \ldots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and let $T \in L(\ell^2(\mathbb{N}))$ the unilateral right shift defined by

$$T(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots), \quad (x_n) \in \ell^2(\mathbb{N}).$$

If $A \in L(\ell^2(\mathbb{N}))$ is the operator defined by

$$A(x_1, x_2, \ldots) := \left(\frac{x_1}{1!}, \frac{x_2}{2!}, \ldots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

then A is a quasi-affinity. Clearly, $SA = AT$, T is polaroid, since $\sigma(T)$ is the closed unit disc of $\mathbb{C}$, while S is quasi-nilpotent and hence not polaroid.

Theorem 3.6. *Suppose that $T \in L(X)$, $S \in L(Y)$ are intertwined by an injective map $A \in L(X, Y)$. If S is polaroid and $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$ then T is polaroid.*

Proof. If $\sigma(T)$ has no isolated point then T is polaroid and hence there is nothing to prove. Suppose that $\operatorname{iso}\sigma(T) \neq \emptyset$ and let $\lambda \in \operatorname{iso}\sigma(T)$. Then $\lambda \in \operatorname{iso}\sigma(S)$, hence λ is a pole of the resolvent of S. Let $P_T(\lambda)$ and $P_S(\lambda)$ be the spectral projections associated to T and S with respect to $\{\lambda\}$, respectively. As we have seen in Lemma 3.3, $P_T(\lambda)$ and $P_S(\lambda)$ are intertwined by A, i.e. $P_S(\lambda)A = AP_T(\lambda)$. Since λ is a pole of the resolvent of S then $p := p(\lambda I - S) = q(\lambda I - S) < \infty$ and $\ker(\lambda I - S)^p$ coincides with the range of $P_S(\lambda)$, see [1, Theorem 3.74]. Therefore, $(\lambda I - S)^p P_S(\lambda) = 0$, and consequently

$$0 = (\lambda I - S)^p P_S(\lambda)A = (\lambda I - S)^p AP_T(\lambda) = A(\lambda I - T)^p P_T(\lambda).$$

Since A is injective, $(\lambda I - T)^p P_T(\lambda) = 0$. Now, the range of $P_T(\lambda)$ coincides with the quasi-nilpotent part $H_0(\lambda I - T)$, see [1, Theorem 3.74], so

$$H_0(\lambda I - T) = P_T(\lambda)(X) \subseteq \ker(\lambda I - T)^p.$$

The opposite inclusion also holds, since $\ker(\lambda I - T)^n \subseteq H_0(\lambda I - T)$ for all natural $n \in \mathbb{N}$. Therefore, $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ for all $\lambda \in \operatorname{iso}\sigma(T)$. By Theorem 2.2 we then conclude that T is polaroid. □

The inclusion $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$ has a crucial role in Theorem 3.6. If T, S and A are as in Remark 3.4 we have $\operatorname{iso}\sigma(S) = \emptyset$, $\operatorname{iso}\sigma(T) = \{0\}$ and the polaroid condition is not preserved by the quasi-affinity A. The example of Remark 3.4 also shows that the condition $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$ cannot be replaced by the weaker condition $\operatorname{iso}\sigma(T) \subseteq \sigma(S)$.

Corollary 3.7. *Suppose that $T \in L(X)$ and $S \in L(Y)$ are intertwined by a quasi-affinity $A \in L(X, Y)$ and $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(S)$. Then T is polaroid if and only if S is polaroid.*

Proof. By Theorem 3.6 we need only to prove that if T is polaroid then S is polaroid. Now, T^* is polaroid and $SA = AT$ implies $T^*A^* = A^*S^*$, where $A^* \in L(Y^*, X^*)$ is injective, since A has a dense range. Moreover, $\operatorname{iso}\sigma(T^*) = \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(S) = \operatorname{iso}\sigma(S^*)$. Since T^* is polaroid by Theorem 3.6 it then follows that S^* is polaroid, or equivalently S is polaroid. □

Corollary 3.8. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X, Y)$. Suppose that S is polaroid, has SVEP and $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$. Then we have:*

(i) *$f(T)$ satisfies (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*
(ii) *$f(T^*) = f(T)^*$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. (i) T has SVEP by Lemma 3.1, hence $f(T)$ has SVEP for all $f \in \mathcal{H}_{nc}(\sigma(T))$, see [1, Theorem 2.40]. Furthermore, $f(T)$ is polaroid by Theorem 2.5. By part (ii) of Theorem 2.4 then $f(T)$ satisfies generalized Weyl's theorem.

(ii) Also this follows from part (ii) of Theorem 2.4. □

The operator C_p^* considered in Example 3.2 shows that in general a polaroid operator does not satisfy SVEP. In fact, $\sigma(C_p^*)$ has no isolated points. A more trivial example is given by the left shift T on $\ell^2(\mathbb{N})$. This operator is polaroid, since $\sigma(T)$ is the unit disc of $\mathbb{C}$, and it is well known that T fails SVEP at 0.

In the sequel by a part of an operator $T \in L(X)$ we mean the restriction of T to a closed T-invariant subspace.

Definition 3.9. An operator $T \in L(X)$ is said to be *hereditarily polaroid* if every part of T is polaroid.

It is easily seen that the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity. Since every hereditarily polaroid operator has SVEP, see [20, Theorem 2.8], by Corollary 3.8 we readily obtain:

Corollary 3.10. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X, Y)$. Suppose that S is hereditarily polaroid, and $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$. Then $f(T)$ satisfies (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$, while (W) and (W), (aW), (w), (gW), (gaW), (gw) hold for $f(T^*)$.*

An interesting class of hereditarily polaroid operators is given by the $H(p)$-operators [20], where $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{12}$$

From the implication (4) we see that every operator T which belongs to the class $H(p)$ has SVEP. Moreover, from (8) it follows that every $H(p)$ operator T is polaroid. The class $H(p)$ has been introduced by Oudghiri in [29] and in [12] this class of operators has been studied for $p := p(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [29]. Therefore, algebraically p-hyponormal or algebraically M-hyponormal operators are $H(p)$. Multipliers of commutative semi-simple Banach algebras T are $H(1)$, in particular every convolution T_μ operator on $L_1(G)$, G a locally compact Abelian group is $H(1)$. Note that a convolution operator T_μ on $L_1(G)$ is a-polaroid, since $\sigma_a(T_\mu) = \sigma(T_\mu)$, see [1, Corollary 5.88]. The property of being $H(p)$ is preserved by quasi-affinities [29, Lemma 3.2]. Furthermore, if T is $H(p)$ then the every part of T is $H(p)$ [29, Lemma 3.2], so every $H(p)$ is hereditarily polaroid. Other examples of hereditarily polaroid operators are given by the *completely hereditarily normaloid* operators on Banach spaces. In particular, all paranormal operators on Hilbert spaces and the class of (p,k)-*quasi-hyponormal* operators on Hilbert spaces are hereditarily polaroid, see for details [20]. Also the *algebraically quasi-class A operators* on a Hilbert space considered in [14], are hereditarily polaroid. In fact, every part of an algebraically quasi-class A operator T is algebraically quasi-class A and every algebraically quasi-class A operator is polaroid [14, Lemma 2.3].

The next result shows that hereditarily polaroid operators are transformed, always under the assumption $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$, by quasi-affinities into a-polaroid operators.

Theorem 3.11. *Suppose that $T \in L(X)$, $S \in L(Y)$ are intertwined by an injective map $A \in L(X,Y)$. If S is hereditarily polaroid and $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma(S)$ then T^* is a-polaroid.*

Proof. By Theorem 3.6 T is polaroid, and hence also T^* is polaroid. As observed above, S has SVEP, so T has SVEP by Lemma 3.3. The SVEP for T by [1, Corollary 2.45] entails that $\sigma(T^*) = \sigma(T) = \sigma_s(T) = \sigma_a(T^*)$, and this trivially implies that T^* is a-polaroid. □

Note that quasi-similar operators may have unequal approximate point spectrum, for an example see [18].

Theorem 3.12. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X,Y)$ and suppose that $\operatorname{iso}\sigma_a(T) \subseteq \operatorname{iso}\sigma_a(S)$. If S is left polaroid then T is polaroid.*

Proof. We first show that $A(H_0(\lambda I - T)) \subseteq H_0(\lambda I - S)$. Let $x \in H_0(\lambda I - T)$. Then

$$\lim_{n\to\infty} \left\|(\lambda I - S)^n Ax\right\|^{1/n} = \lim_{n\to\infty} \left\|A(\lambda I - T)^n x\right\|^{1/n} \leqslant \lim_{n\to\infty} \left\|(\lambda I - T)^n x\right\|^{1/n} = 0,$$

thus $Ax \in H_0(\lambda I - S)$ and hence $A(H_0(\lambda I - T)) \subseteq H_0(\lambda I - S)$, as claimed.

Also here we can suppose that $\operatorname{iso}\sigma(T) \neq \emptyset$. Let $\lambda \in \operatorname{iso}\sigma(T)$. Since the approximate point spectrum of every operator contains the boundary of the spectrum, in particular every isolated point of the spectrum, then $\lambda \in \operatorname{iso}\sigma_a(T) \subseteq \operatorname{iso}\sigma_a(S)$. Since S is left polaroid by part (ii) of Theorem 2.2 there exists a positive integer p such that $H_0(\lambda I - S) = \ker(\lambda I - S)^p$. Consequently,

$$A\big(H_0(\lambda I - T)\big) \subseteq H_0(\lambda I - S) = \ker(\lambda I - S)^p,$$

so, if $x \in H_0(\lambda I - T)$ then

$$A(\lambda I - T)^p x = (\lambda I - S)^p (Ax) = 0.$$

Since A is injective then $(\lambda I - T)^p x = 0$ and hence $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)^p$. The opposite inclusion is still true, so that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$ for every $\lambda \in \operatorname{iso}\sigma(T)$, and hence by Theorem 2.2 T is polaroid. □

Also in Theorem 3.12 the assumption that $\operatorname{iso}\sigma_a(T) \subseteq \operatorname{iso}\sigma_a(S)$ is essential. For the operators S and T of Remark 3.4 we have $\sigma_a(S) = \Gamma$, Γ the unit circle of $\mathbb{C}$, so $\operatorname{iso}\sigma_a(S) = \emptyset$, while $\{0\} = \sigma_a(T) = \operatorname{iso}\sigma_a(T)$. Evidently, S is both left and a-polaroid, while T is not polaroid.

Corollary 3.13. *Let $T \in L(X)$, $S \in L(Y)$ be intertwined by an injective map $A \in L(X,Y)$. Suppose that S is left polaroid operator which has SVEP and $\operatorname{iso}\sigma_a(T) \subseteq \operatorname{iso}\sigma_a(S)$. Then we have*:

(i) *$f(T)$ satisfies (W) or equivalently (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*
(ii) *$f(T^*)$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. By Theorem 3.12 T is polaroid and by Lemma 3.1 has SVEP, so we can argue as in the proof of Corollary 3.8. □

Theorem 3.14. *Suppose that $S \in L(Y)$ and $T \in L(X)$ are intertwined by a map $A \in L(Y,X)$ which has dense range. If $\operatorname{iso}\sigma_s(T) \subseteq \operatorname{iso}\sigma_s(S)$ and S is right polaroid then T is polaroid.*

Proof. From $TA = AS$ we have $A^*T^* = S^*A^*$ with $A^* \in L(X^*,Y^*)$ injective. Since S is right polaroid then S^* is left-polaroid and by duality we have $\sigma_s(T) = \sigma_a(T^*)$ and $\sigma_s(S) = \sigma_a(S^*)$. Therefore $\operatorname{iso}\sigma_a(T^*) \subseteq \operatorname{iso}\sigma_a(S^*)$. By Theorem 3.6 it then follows that T^* is polaroid, or equivalently T is polaroid. □

Corollary 3.15. *Let $S \in L(Y)$ and $T \in L(X)$ be intertwined by map $A \in L(Y, X)$ having dense range. Suppose that S is right polaroid, S^* has SVEP and* $\operatorname{iso}\sigma_s(T) \subseteq \operatorname{iso}\sigma_s(S)$. *Then we have*:

(i) *$f(T^*)$ satisfies (gW) for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*
(ii) *$f(T)$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. By Theorem 3.14 T, and hence T^*, is polaroid, so $f(T^*)$ is polaroid for all $f \in \mathcal{H}_{nc}(\sigma(T))$, by Theorem 2.5. From $TA = AS$ we obtain $S^*A^* = A^*T^*$ with A^* is injective. By assumption S^* has SVEP so, by Lemma 3.1, T^*, and hence $f(T^*)$, has SVEP. Therefore part (i) of Theorem 2.4 applies. □

An important tool in local spectral theory is the *glocal spectral subspace* $\mathcal{X}_T(F)$. It is defined, for an operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

It is known that $H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\})$ [1, Theorem 2.20]. Recall that an operator $T \in L(X)$ is said to have Dunford's *property* (C) if, for each closed set $F \subseteq \mathbb{C}$, $\mathcal{X}_T(F)$ is closed. It is well known that property (C) implies SVEP, see [19, Proposition 1.2.19].

Let U be an open subset of $\mathbb{C}$ and denote by $\mathcal{H}(U, X)$ the Fréchet space of all analytic functions $f : U \to X$ with respect the pointwise vector space operations and the topology of locally uniform convergence. $T \in L(X)$ has the Bishop's *property* (β) if, for every open $U \subseteq \mathbb{C}$ and every sequence $(f_n) \subseteq \mathcal{H}(U, X)$ for which $(\lambda I - T)f_n(\lambda)$ converges to 0 uniformly on every compact subset of U, then also $f_n \to 0$ in $\mathcal{H}(U, X)$. Subnormal operators (i.e. restrictions of normal operator to closed invariant subspaces) have property (β). Note that

$$\text{property } (\beta) \quad \Rightarrow \quad \text{property } (C) \quad \Rightarrow \quad \text{SVEP},$$

see [28, Proposition 1.2.19].

An operator $T \in L(X)$ has the *property* (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. *Decomposable operators* may be defined as those operators that satisfy property (β) and property (δ), see [28, Theorem 2.5.19]. Note that property (δ) implies SVEP for T^*. In fact T has property (δ) if and only if T^* has property (β), see [28, Theorem 2.5.19]. Every generalized scalar operator is decomposable, see [28] for definitions and details.

Under the stronger conditions of quasi-similarity and property (β), the assumption on the isolated points of the spectra of T and S in Theorem 3.6 may be omitted:

Theorem 3.16. *Let $T \in L(X)$, $S \in L(Y)$ be quasi-similar.*

(i) *If both T and S have property (β) then T is polaroid if and only if S is polaroid. In this case, T^* is a-polaroid.*
(ii) *If both T and S are Hilbert spaces operators for which property (C) holds then T is polaroid if and only if S is polaroid. In this case, T^* is a-polaroid.*

Consequently, under the assumptions (i) *or* (ii) *on S and T, $f(T)$ satisfies (gW), while $f(T^*)$ satisfies all Weyl type theorems for all $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. (i) By a result of Putinar [30] we have $\sigma(S) = \sigma(T)$, hence $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(S)$. By Corollary 3.7 we then obtain that T is polaroid exactly when S is polaroid. Evidently, in this case T^* is polaroid. Now, property (β) implies that S has SVEP and hence, by Lemma 3.3, also T has SVEP. The SVEP for T, always by [1, Corollary 2.45], entails that $\sigma(T^*) = \sigma_a(T^*)$, and hence T^* is a-polaroid.

(ii) Also in this case, by a result of Stampfli [36], we have $\sigma(S) = \sigma(T)$, and property (C) entails SVEP, so the assertion follows by using the same argument of part (i).

The last assertion is clear from Corollary 3.8. □

It is well known that hyponormal operators on Hilbert spaces have property (β). Theorem 3.16 then applies to these operators, since they are $H(1)$ and hence polaroid. Another class of polaroid operators to which Theorem 3.16 applies is the class of all $p_* - QH$ operators studied in [21]. In fact, these operators are $H(1)$ and have property (β), see [21, Theorem 2.12 and Theorem 2.2].

Let us consider a very weak notion of intertwining which dates back to Foiaş: see [19, Chapter 4] and [28, Chapter 3]. An operator $A \in L(X, Y)$ is said to *intertwine $T \in L(X)$ and $S \in L(Y)$ asymptotically* if

$$\lim_{n\to\infty} \left\| C(S, T)^n(A) \right\|^{1/n} = 0. \tag{13}$$

Clearly, this notion is a generalization of the higher order intertwining condition $C(S,T)^n(A)=0$. Note that when T and S are generalized scalar then condition (13) holds if and only if $C(S,T)^n(A)=0$ for some $n\in\mathbb{N}$, see [19, Theorem 4.4.5]. If the pairs (S,T) and (T,S) are both asymptotically intertwined by some quasi-affinity then T and S are said to be *asymptotically quasi-similar*. We recall that if a pair (S,T) is asymptotically intertwined by $A\in L(X,Y)$ then

$$A\mathcal{X}_T(F)\subseteq\mathcal{Y}_S(F)\quad\text{for all closed sets } F\subseteq\mathbb{C}, \tag{14}$$

see Corollary 3.4.5 of [28] or [27].

A very particular case of asymptotically quasi-similar operators is defined as follows: $T,S\in L(X)$ are said to be *quasi-nilpotent equivalent* if each of the pairs (S,T) and (T,S) are asymptotically intertwined by the identity operator I on X. Note that any quasi-nilpotent operator and the 0 operator are quasi-nilpotent equivalent.

Example 3.17. The polaroid condition is not transmitted whenever S and T are asymptotically intertwined by a quasi-affinity, even in the case that the inclusion $\operatorname{iso}\sigma(T)\subseteq\operatorname{iso}\sigma(S)$ is satisfied. For instance, if $T\in L(\ell^2(\mathbb{N}))$ is defined by

$$T(x_1,x_2,\dots)=\left(\frac{x_2}{2},\frac{x_3}{3},\dots\right)\quad\text{for all } (x_n)\in\ell^2(\mathbb{N}).$$

If $S:=0$ then S is polaroid, while the quasi-nilpotent operator T is not polaroid. T and S are, as observed above, quasi-nilpotent equivalent.

Let now consider the very particular case that $C(S,T)^n(I)=0$ for some $n\in\mathbb{N}$. If T and S commute then $C(S,T)^n(I)=(S-T)^n=0$. In this case T and S differ from a commuting nilpotent operator N and, without any condition, if S is polaroid then T is also polaroid, see Theorem 2.10 of [9].

Remark 3.18. If $\alpha(T)<\infty$ then $\alpha(T^n)<\infty$ for all $n\in\mathbb{N}$. This may be easily seen by an inductive argument. Suppose that $\dim\ker T^n<\infty$. Clearly $T(\ker T^{n+1})\subseteq\ker T^n$, so the restriction $T_0:=T|\ker T^{n+1}:\ker T^{n+1}\to\ker T^n$ has kernel equal to $\ker T$. Consequently, the canonical mapping

$$\hat{T}:\ker T^{n+1}/\ker T\to\ker T^n$$

is injective. Therefore we have $\dim\ker T^{n+1}/\ker T\leqslant\dim\ker T^n<\infty$, and since $\dim\ker T<\infty$ we then conclude that $\dim\ker T^{n+1}<\infty$.

Set

$$E_\infty(S):=\{\lambda\in\operatorname{iso}\sigma(S)\colon\ \alpha(\lambda I-S)<\infty\}.$$

Theorem 3.19. *Let $T\in L(X)$ and $S\in L(Y)$ be asymptotically intertwined by an injective map $A\in L(X,Y)$ and $\operatorname{iso}\sigma(T)\subseteq E_\infty(S)$.*

(i) *If S is polaroid then T is polaroid.*
(ii) *If S is polaroid and T has SVEP then $f(T)$ satisfies (gW) while $f(T^*)$ satisfies all Weyl type theorems for all $f\in\mathcal{H}_{nc}(\sigma(T))$.*

Proof. (i) If $\lambda\in\operatorname{iso}(T)$ then $\lambda\in\operatorname{iso}\sigma(T)$. Since S is polaroid it then follows that $H_0(\lambda I-S)=\ker(\lambda I-S)^p$ for some positive integer p. Since $\lambda\in E_\infty(S)$ we have $\alpha(\lambda I-S)<\infty$, so, by Remark 3.18, $\alpha((\lambda I-S)^p)<\infty$, thus $H_0(\lambda I-S)$ is finite-dimensional. By (14) we have

$$A\big(H_0(\lambda I-T)\big)=A\big(\mathcal{X}_T(\{\lambda\})\big)\subseteq\mathcal{Y}_S(\{\lambda\})=H_0(\lambda I-S),$$

and since A is injective it then follows that $H_0(\lambda I-T)$ is finite-dimensional. From the inclusion $\ker(\lambda I-T)^n\subseteq H_0(\lambda I-T)$ for all $n\in\mathbb{N}$ it then easily follows that $p(\lambda I-T)<\infty$. But λ is an isolated point of $\sigma(T)$, so the decomposition $X=H_0(\lambda I-T)\oplus K(\lambda I-T)$ holds, consequently $K(\lambda I-T)$ is finite co-dimensional, and since $K(\lambda I-T)\subseteq(\lambda I-T)(X)$ we then conclude that $\beta(\lambda I-T)<\infty$. Therefore, $\lambda I-T$ is Fredholm. But λ is an isolated point of $\sigma(T^*)=\sigma(T)$, so T^* has SVEP at λ and, since $\lambda I-T$ is Fredholm, this implies that $q(\lambda I-T)<\infty$. Therefore, λ is a pole of the resolvent of T.

(ii) By part (i) T is polaroid and has SVEP, so Theorem 2.4 applies. □

Corollary 3.20. *Suppose that S and T are quasi-nilpotent equivalent. If S is polaroid and every eigenvalue of S has finite multiplicity then T is polaroid.*

Proof. The quasi-nilpotent equivalence preserves the spectrum, see [19, Chapter 1, Theorem 2.2], hence $\operatorname{iso}\sigma(T)=\operatorname{iso}\sigma(S)$. Now, if $\lambda\in\operatorname{iso}\sigma(S)$ then either $\lambda I-S$ is injective or λ is an eigenvalue of S. In both case $\lambda\in E_\infty(S)$. □

Example 3.17 shows that the result of Corollary 3.20 fails if the eigenvalues of S do not have finite multiplicity. Define

$$E^a_\infty(S) := \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(S)\colon \alpha(\lambda I - S) < \infty\}.$$

Clearly, $E_\infty(S) \subseteq E^a_\infty(S)$.

Theorem 3.21. *Let $T \in L(X)$ and $S \in L(Y)$ be asymptotically intertwined by an injective map $A \in L(X, Y)$ and $\operatorname{iso}\sigma(T) \subseteq E^a_\infty(S)$.*

(i) *If S is left polaroid then T is polaroid.*
(ii) *If S is polaroid and T has SVEP then $f(T)$ satisfies (gW), while $f(T^*)$ satisfies all Weyl type theorems.*

Proof. (i) If $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(S)$. S is left polaroid so, by part (ii) of Theorem 2.2, there exists a positive integer p such that $H_0(\lambda I - S) = \ker(\lambda I - S)^p$. Since $\alpha(\lambda I - S) < \infty$ it then follows, again by Remark 3.18, that $H_0(\lambda I - S)$ is finite-dimensional. The remaining part of the proof is the same of part (i) of Theorem 3.19. □

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