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Toric singularities revisited

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Abstract

In [K. Kato, Toric singularities, Amer. J. Math. 116 (5) (1994) 1073–1099], Kato defined his notion of a log regular scheme and studied the local behavior of such schemes. A toric variety equipped with its canonical logarithmic structure is log regular. And, these schemes allow one to generalize toric geometry to a theory that does not require a base field. This paper will extend this theory by removing normality requirements.

Keywords: Toric varieties; Semigroup rings

Conventions and notation

All monoids considered in this paper are commutative and cancellative. All rings considered in this paper are commutative and unital. See Kato [2] for an introduction to log schemes. There Kato defines pre-log structures and log structures on the étale site of X. However, we will use the Zariski topology throughout this paper.

- P^* — the unit group of the monoid P.
- $\overline{P}$ — the sharp image of the monoid P, $\overline{P} = P/P^*$ is the orbit space under the natural action of P^* on P.
- P^+ — $P^+ = P \setminus P^*$ is the maximal ideal of the monoid P.

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P^{gp}	the group generated by P, that is, image of P under the left adjoint of the inclusion functor from Abelian groups to monoids.
P^{sat}	the saturation of P, that is, $\{p \in P^{\mathrm{gp}} \mid np \in P$ for some $n \in \mathbb{N}^+\}$.
$R[P]$	the monoid algebra of P over a ring R. The elements of $R[P]$ are written as "polynomials". That is, they are finite sums $\sum r_p t^p$ with coefficients in R and exponents in P.
(K)	the ideal $\beta(K)A$, where $\beta : P \to A$ is a monoid homomorphism with respect to multiplication on A and K is an ideal of P. We say such an ideal is a *log ideal* of A.
$R[[P]]$	the (P^+)-adic completion of $R[P]$.
$P_{\mathfrak{p}}$	the localization of the monoid P at the prime ideal $\mathfrak{p} \subseteq P$.
$\dim P$	the (Krull) dimension of the monoid P.

Introduction

A toric variety is a normal irreducible separated scheme X, locally of finite type over a field k, which contains an algebraic torus $T \cong (k^*)^d$ as an open set and is endowed with an algebraic action $T \times X \to X$ extending the group multiplication $T \times T \to T$. According to Oda [3]:

> The theory was started at the beginning of 1970s by Demazure [4] in connection with algebraic subgroups of the Cremona groups, by Mumford et al. [5] and Satake [6] in connection with compactifications of locally symmetric varieties, and by Miyake and Oda [7]. We were inspired by Hochster [8] as well as Sumihiro [9,10].
>
> Comprehensive surveys from various different perspectives can be found in Danilov [11], Mumford et al. [5,12] as well as [13,14].

In [1], Kato extended the theory of toric geometry over a field to an absolute theory, without base. This is achieved by replacing the notion of a toroidal embedding introduced in [5] with the notion of a log structure. A toroidal embedding is a pair (X, U) consisting of a scheme X locally of finite type and an open subscheme $U \subset X$ such that (X, U) is isomorphic, locally in the étale topology, to a pair consisting of a toric variety and its algebraic torus. Toroidal embeddings are particularly nice locally Noetherian schemes with distinguished log structures.

A log structure on a scheme X, in the sense of Fontaine and Illusie, is a morphism of sheaves of monoids $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ restricting to an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$. The theory of log structures on schemes is developed by Kato in [2]. Log structures were developed to give a unified treatment of the various constructions of de Rham complexes with logarithmic poles. In [15] Illusie recalls the question that motivated their definition:

> Let me briefly recall what the main motivating question was. Suppose S is the spectrum of a complete discrete valuation ring A, with closed (respectively generic) point s (respectively η), and X/S is a scheme with semi-stable reduction, which means that, locally for the étale topology, X is isomorphic to the closed subscheme of $\mathbb{A}^n_S$ defined by

the equation $x_1 \cdots x_n = t$, where $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$ and t is a uniformizing parameter of A. Then X is regular, X_η is smooth, and $Y = X_s$ is a divisor with normal crossings on X. In this situation, one can consider, with Hyodo, the relative de Rham complex of X over S with logarithmic poles along Y,

$$\omega^{\cdot}_{X/S} = \Omega^{\cdot}_{X/S}(\log Y)$$

([16], see also [17,18]). Its restriction to the generic fiber is the usual de Rham complex $\Omega^{\cdot}_{X_\eta/\eta}$ and it induces on Y a complex

$$\omega^{\cdot}_Y = \mathcal{O}_Y \otimes \Omega^{\cdot}_{X/S}(\log Y).$$

One has $\omega^i_Y = \bigwedge^i \omega^1_Y$, and when X is defined as above by $x_1 \cdots x_r = t$, ω^1_Y is generated as an $\mathcal{O}_Y$-module by the images of the $d\log x_i$ $(1 \leqslant i \leqslant r)$ and dx_i $(r+1 \leqslant i \leqslant n)$ subject to the single relation $\sum_{1 \leqslant i \leqslant r} d\log x_i = 0$. The analogue of $\omega^{\cdot}_Y$ over $\mathbb{C}$ (S is replaced by a disc) is the complex studied by Steenbrink in [19], which "calculates" $\mathbf{R}\psi(\mathbb{C})$. If X/S is of relative dimension 1, ω^1_Y is simply the dualizing sheaf of Y (which probably explains the notation ω, chosen by Hyodo), and of course, in this case, $\omega^{\cdot}_Y$ depends only on Y. In general, however, $\omega^{\cdot}_Y$ depends not only on Y but also "a little bit" on X, so it is natural to ask: which extra structure on Y is needed to define $\omega^{\cdot}_Y$?

Assume for simplicity that Y has simple normal crossings, i.e., Y is a sum of smooth divisors Y_i meeting transversely. Let $\mathcal{L}_i$ be the invertible sheaf $\mathcal{O}_Y \otimes \mathcal{O}_X(-Y_i)$, and $s_i : \mathcal{L}_i \to \mathcal{O}_Y$ the map deduced by extension of scalars from the inclusion $\mathcal{O}_X(-Y_i) \hookrightarrow \mathcal{O}_X$. Then it is easily seen that the data consisting of the pairs $(\mathcal{L}_i, s_i)$, together with the isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \bigotimes \mathcal{L}_i$ (coming from $Y = \sum Y_i$), suffice to define ω^1_Y. Indeed, ω^1_Y can be defined as the $\mathcal{O}_Y$-module generated locally by elements $\{dx, d\log e_i\}$, for x a local section of $\mathcal{O}_Y$ and e_i a local generator of $\mathcal{L}_i$, subject to the usual relations among the dx, plus

$$s_i(e_i)\, d\log e_i = ds_i(e_i), \tag{1a}$$

$$\sum d\log e_i = 0 \quad \text{when } 1 \in \mathcal{O}_Y \text{ is written } \bigotimes e_i. \tag{1b}$$

This construction and a subsequent axiomatic development was proposed by Deligne in [20], and independently by Faltings in [21]. In order to deal with the general case (Y no longer assumed to have simple normal crossings), it is more convenient to consider, instead of the pairs $(\mathcal{L}_i, s_i)$, the sheaf of monoids $\mathcal{M}$ on Y (multiplicatively) generated by $\mathcal{O}^*_Y$ and the e_i together with the map $\mathcal{M} \to \mathcal{O}_Y$ given by $\alpha(\bigotimes e_i^{n_i}) = \prod s_i(e_i)^{n_i}$ (replace $d\log e_i$ by $d\log e$ for $e \in \mathcal{M}$, and the relations (1) by $d\log(ef) = d\log e + d\log f$, $\alpha(e) d\log e = d\alpha(e)$, $d\log u = u^{-1}du$ if u is a unit). This is the origin of the notion of logarithmic structure, proposed by Fontaine and the speaker, which gave rise to the whole theory beautifully developed by Kato (and Hyodo) in [1,2,18,22,23].

Log structures are natural generalizations to arbitrary schemes of reduced normal crossings divisors on regular schemes. Here $\mathcal{M}$ is the sheaf of regular functions on the scheme that are invertible away from the divisor, which is a sheaf of monoids under multiplication. Log structures give rise to differentials with log poles, crystals and crystalline cohomology with log poles, and similar structures.

In [1], Kato defines a notion of log regularity and proves that fine saturated log regular schemes behave very much like toric varieties. If X is such a scheme, then X is normal and Cohen–Macaulay. In addition, Kato develops a theory of fans and subdivisions, shows how this theory can be used to resolve toric singularities, and identifies the dualizing complexes of such schemes.

In Section 3 we extend this theory by relaxing the requirement that the monoids be saturated, thereby relaxing the requirement that the schemes be normal. This is accomplished by casting the theory in terms of log structures alone. We do not appeal to the traditional theory of cones to keep track of the combinatorics since such an appeal forces the restriction to saturated monoids. Most of our results are local; that is, they are concerned with the singularities that occur on these schemes. We define the notion of a toric log regular scheme. Toric varieties are examples of such schemes, and toric log regular schemes behave like toric varieties in many ways, though they need not be Cohen–Macaulay. Although most of Kato's methods go through with only minor modifications, there is one significant change. His induction proof of [1, Proposition (7.2)] involves a reduction to the case where a chart comes from a one-dimensional affine semigroup. Once he has reduced to this case, he uses the fact that $\mathbb{N}$ is the only such monoid. In Lemma 55, we introduce a faithful flat descent argument to handle the one-dimensional case.

In Section 1 we collect the algebraic preliminaries will we use in later sections and the aspects of basic log geometry that we will emphasize. In Section 2 we define and study t-flatness.

The notion of t-flatness is due to O. Gabber. As such, this presentation owes much to him.

1. Preliminaries

See Kato [2] for an introduction to log schemes. There Kato defines pre-log structures and log structures on the étale site of X. However, we will use the Zariski topology throughout this paper. See Niziol [24] for a brief comparison of log structures on the Zariski and étale sites. In Kato [22], log structures on locally ringed spaces are defined and (for the most part) the Zariski topology is used.

1.1. *Fine log scheme basics*

Definition 1. [22, Definition 1.2.3] Given a pre-log structure $\beta : \mathcal{M} \to \mathcal{O}_X$ on a scheme X, the *log structure associated to this pre-log structure* $\mathcal{M}^\alpha$ is defined to be the colimit of the diagram

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on X, equipped with the homomorphism of sheaves of monoids induced by β and the inclusion $\mathcal{O}_X^* \subseteq \mathcal{O}_X$.

Remark 2. [2, Section 1.3] If $G \xleftarrow{\beta} P \xrightarrow{\gamma} Q$ is a diagram of monoids and G is a group, then the colimit of this diagram is $(G \oplus Q)/\sim$, where $\sim$ is the congruence given by $(g,q) \sim (g',q')$ if there exist $p, p' \in P$ such that $g + \beta(p) = g' + \beta(p')$ and $q + \gamma(p) = q' + \gamma(p')$. That is, (g,q) and (g',q') differ by an element of P^{gp}. In particular, we may replace P with any submonoid of $(\gamma^{\mathrm{gp}})^{-1}(Q)$ that generates P^{gp} without changing the colimit.

Example 3. If P is a monoid, A is a ring, and $\beta : P \to A$ is a monoid homomorphism with respect to multiplication on A, then β induces a homomorphism of sheaves of monoids on $\operatorname{Spec} A$ from $P_{\operatorname{Spec} A}$ the constant sheaf on $\operatorname{Spec} A$ with stalk P to $\mathcal{O}_{\operatorname{Spec} A}$. We will denote the associated log scheme by $\operatorname{Spec}(P \xrightarrow{\beta} A)$.

A coherent log structure $\mathcal{M}$ on a scheme X is integral if and only if locally on X, $\mathcal{M}$ is isomorphic to the log structure associated to the pre-log structure $P_X \to \mathcal{O}_X$ for some finitely generated (cancellative) monoid P (see [22, Definition 1.2.6]).

Definition 4. If $\beta : P \to Q$ is a monoid homomorphism, we say β is *local* if $P^* = \beta^{-1}(Q^*)$. Furthermore, if A is a ring, $\mathfrak{p} \subset A$ is prime, and $\beta : P \to A$ is a monoid homomorphism with respect to multiplication on A, we say β is *local at* $\mathfrak{p}$ if the composite monoid homomorphism $P \xrightarrow{\beta} A \xrightarrow{\text{canonical}} A_{\mathfrak{p}}$ is local.

For the rest of this section, let P be a finitely generated monoid. Let A be a Noetherian ring and let $\beta : P \to A$ be a monoid homomorphism with respect to multiplication on A. Then $\operatorname{Spec}(P \xrightarrow{\beta} A)$ is a Noetherian fine log scheme. (In fact, every locally Noetherian fine log scheme is locally isomorphic to such a log scheme.) By Remark 2, $\operatorname{Spec}(P \xrightarrow{\beta} A)$ is isomorphic to $\operatorname{Spec}(Q \xrightarrow{\beta^{\mathrm{gp}}|_Q} A)$ for any $Q \subseteq P + \beta^{-1}(A^*)$ containing P. After we prove the following proposition, we may assume β is local at any particular prime $\mathfrak{p} \subset A$ as well.

Proposition 5. *If $\mathfrak{p}$ is a prime ideal of A, then there exists an element $f \in A \setminus \mathfrak{p}$ and a finitely generated monoid $Q \subseteq P^{\mathrm{gp}}$ containing P such that the map $Q \to A_f$ induced by β is local at $\mathfrak{p}$ and generates the same log structure on* $\operatorname{Spec} A_f$ *as β.*

Proof. Compose β with the canonical map $A \to A_{\mathfrak{p}}$ to get a map $\tilde{\beta} : P \to A_{\mathfrak{p}}$. Let X be a finite generating set for P, let $X_0 = \{x \in X \mid \tilde{\beta}(x) \in A_{\mathfrak{p}}^*\}$, let x_0 be the sum of the elements of X_0, let Q be the submonoid of P^{gp} generated by $X \cup \{-x_0\}$, and let $f = \beta(x_0)$. The rest is straightforward. □

1.2. Affine semigroups

A theorem of Grillet [25, Theorem 3.11] says, "Let P be a finitely generated monoid. The monoid P is cancellative, reduced and torsion-free if and only if it is isomorphic to a submonoid of $\mathbb{N}^k$ for some positive integer k." We will strengthen this theorem by showing that such a monoid can be embedded in $\mathbb{N}^d$ with $d = \operatorname{rank} P^{\mathrm{gp}}$ in such a way that a complete flag in P is taken to the standard flag in $\mathbb{N}^d$.

Definition 6. A monoid P is said to be *sharp* (or *reduced*) if its unit group is trivial.

A monoid P is said to be *torsion-free* if its difference group P^{gp} is torsion-free.

A monoid P is said to be an *affine semigroup* if it is isomorphic to a finitely generated submonoid of $\mathbb{N}^k$ for some $k \geqslant 0$.

A monoid S is said to be *saturated* (or *normal*) if for any positive integer n and any $p \in P^{\mathrm{gp}}$, $np \in P$ implies $p \in P$. We call the smallest saturated submonoid of P^{gp} containing P the *saturation* of P and denote it by P^{sat}.

Proposition 7. *Let P be a monoid.*

(1) *If P is sharp and saturated, then P is torsion-free.*
(2) *$\overline{P}$ is sharp.*
(3) *If P is saturated, then $\overline{P}$ is also saturated.*
(4) *If P is saturated, then every localization of P is also saturated.*
(5) *If P is finitely generated and saturated, then $P \cong P^* \oplus \overline{P}$.*

Proof. (1) We will prove this by contradiction. Suppose p were a nontrivial torsion element of P^{gp} and suppose p had order n. Then we would have $np = 0$. So, p would be in P. But, $\mathbb{N}p$ would be a group. So, p would be a unit in P. This contradicts the fact that P is sharp.

(2) Suppose $\overline{p} \in \overline{P}^*$ and let $p \in P$ be mapped to $\overline{p}$ by the canonical map. Since p maps to a unit, there is an element $q \in P$ such that $p + q \in P^*$. Therefore, $q + (-(p + q))$ is the inverse of p and $p \in P^*$. Hence, $\overline{p} = 0$.

(3) Suppose $\overline{p} \in \overline{P}^{\mathrm{gp}}$ and $n\overline{p} \in \overline{P}$ with $n \in \mathbb{N}$ positive. Let $p \in P^{\mathrm{gp}}$ be a pre-image of $\overline{p}$. Since $n\overline{p} \in \overline{P}$, np differs from an element of P by an element of P^*. That is, $np \in P$. Since P is saturated, $p \in P$. Thus, $\overline{p} \in \overline{P}$.

(4) Let $Q = P - S$ with S a submonoid of P, let n be a positive integer, and let q be an element of $P^{\mathrm{gp}} = Q^{\mathrm{gp}}$ such that $nq \in Q$. Write $nq = p - s$ with $p \in P$ and $s \in S$. We have $n(q + s) = p + (n - 1)s$. So, $n(q + s) \in P$. Since P is saturated, $q + s \in P$. So, $q \in Q$.

(5) By (2), $\overline{P}$ is sharp. By (3), $\overline{P}$ is saturated since P is saturated. By (1), $\overline{P}$ is torsion-free. Furthermore, $\overline{P}^{\mathrm{gp}}$ finitely generated and hence free. Therefore,

$$0 \to P^* \to P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}} \to 0$$

is split exact. If $p \in P^{\mathrm{gp}}$ is mapped into $\overline{P}$ by the right-hand map, then p differs from an element of P by an element of P^*. Hence, $p \in P$ and the pre-image of $\overline{P}$ by the right-hand map is P. That is, any section of $P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}}$ maps $\overline{P}$ into P and $P \cong P^* \oplus \overline{P}$. □

Remark 8. In Proposition 7, neither (3) nor (4) need be true if the word "saturated" is replaced with "torsion-free". Consider

$$P = \big\langle (1,0), (1,1), (0,2), (0,-2)\big\rangle \subseteq \mathbb{Z}^2,$$

this monoid is torsion-free but is not saturated. However, $\overline{P} \cong \langle a, b \mid 2a = 2b\rangle$ is not torsion-free (since $a - b \notin \overline{P}$ and $2(a-b) = 0$) and $P \ncong P^* \oplus \overline{P}$ (since $\overline{P}$ is not torsion-free).

Consider $V = \mathbb{Q} \otimes_{\mathbb{Z}} P^{\mathrm{gp}}$ the $\mathbb{Q}$-vector space generated by the monoid P. Let $C(X)$ be the cone over the subset $X \subseteq P$ in V, that is $C(X) = \{\sum_{i=1}^n q_i p_i \in V \mid \forall i, q_i \in \mathbb{Q}_{\geqslant 0}, p_i \in P\}$. If P is torsion-free, $P \to V$ is injective so we may freely identify P with its image in V. In this case, $P^{\mathrm{sat}} = C(P) \cap P^{\mathrm{gp}}$. In particular, if P is torsion-free, $C(P) = C(P^{\mathrm{sat}})$.

Definition 9. Let P be a monoid. A submonoid $F \subseteq P$ is said to be a *face* of P if $p + p' \in F$ implies $p \in F$.

Notice that $F \subseteq P$ is a face if and only if $P \setminus F$ is a prime ideal. (We consider P to be a face and $\emptyset$ to be a prime ideal.)

Proposition 10. *Let P be a finitely generated, torsion-free monoid.*

(1) *If F is a face of P, then $C(F)$ is a face of $C(P)$.*
(2) *If F is a face of $C(P)$, then $P \cap F$ is a face of P.*
(3) *If F is a face of P, then $P \cap C(F) = F$.*
(4) *If F is a face of $C(P)$, then $C(P \cap F) = F$.*

This establishes a bijective correspondence between the faces of P and the faces of $C(P)$.

Proof. Evidently, whenever $X \subseteq P$ and $x \in C(X)$, there exists a positive integer n such that nx is in the monoid generated by X.

(1) Suppose x and y are elements of $C(P)$ such that $x + y \in C(F)$. Let n_1 be a positive integer such that $n_1 x \in P$, let n_2 be a positive integer such that $n_2 y \in P$, let n_3 be a positive integer such that $n_3(x+y) \in F$, and let m be the least common multiple of n_1, n_2 and n_3. We have $mx \in P$, $my \in P$ and $mx + my = m(x+y) \in F$. Since F is a face of P, $mx \in F$ and $my \in F$. So, $x \in C(F)$ and $y \in C(F)$.

(2) Suppose p and p' are elements of P such that $p + p' \in P \cap F$. Since F is a face, both p and p' are in F. So, $p \in P \cap F$ and $p' \in P \cap F$.

(3) Suppose p is an element of $P \cap C(F)$. Let n be a positive integer such that $np \in F$. Since $(n-1)p + p \in F$ and F is a face, $p \in F$. Evidently, $F \subseteq P \cap C(F)$.

(4) If $x \in C(P \cap F)$, then for some positive integer n, $nx \in P \cap F$. In particular, $nx \in F$. Since F is a face, $x \in F$. If $x \in F$, then for some positive integer n, $nx \in P \cap F$. In particular, $nx \in C(P \cap F)$. Since $C(P \cap F)$ is a face, $x \in C(P \cap F)$. □

In light of this, when P is finitely generated and torsion-free, we freely speak of edges (respectively facets, etc.) meaning faces whose corresponding faces in $C(P)$ are edges (respectively facets, etc.).

Definition 11. A sequence of faces $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ in a monoid P is said to be a *flag* in P. A flag in P is said to be *complete* if $F_0 = P^*$, $F_r = P$ and whenever F is a face of P lying between F_{i-1} and F_i, $F = F_{i-1}$ or $F = F_i$.

The complements in P of the faces in a flag form a chain of prime ideals. If the flag is complete, the chain of primes is saturated; that is, there is no prime properly between consecutive primes of the chain. Furthermore, if $\varphi : P \to Q$ is a homomorphism of monoids and G is a face of Q, $\varphi^{-1}(G)$ is a face of P. We will say φ takes the flag $G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_r$ in Q to the flag $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ in P via pullback if $\varphi^{-1}(G_i) = F_i$ for all $0 \leqslant i \leqslant r$.

Let $\mathbf{e}_i \in \mathbb{N}^d$ be the element with a 1 in the ith position and zeroes elsewhere. We call $\{0\} \subset \langle \mathbf{e}_1 \rangle \subset \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \subset \cdots \subset \mathbb{N}^d$ the standard flag in $\mathbb{N}^d$.

Theorem 12. *Let P be a sharp finitely generated torsion-free monoid and let $\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d$ be a complete flag in P. Then there exists an inclusion $\varphi : P \to \mathbb{N}^d$ taking the standard flag in $\mathbb{N}^d$ to the given flag in P and inducing an isomorphism $\varphi^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathbb{Z}^d$.*

Proof. Without loss of generality we may assume P is saturated, since the faces of P correspond bijectively to the faces of $C(P) = C(P^{\mathrm{sat}})$ according to Proposition 10. We proceed by induction on d. When $d = 0$, the theorem is trivial.

Suppose we have a map $\tilde{\varphi} : F_{d-1} \to \mathbb{N}^{d-1}$ that sends the flag $\{0\} = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{d-1}$ to the standard flag in $\mathbb{N}^{d-1}$ and inducing an isomorphism $\tilde{\varphi}^{\mathrm{gp}} : F_{d-1}^{\mathrm{gp}} \to \mathbb{Z}^{d-1}$. Let σ be a splitting of the inclusion $F_{d-1}^{\mathrm{gp}} \to P^{\mathrm{gp}}$. Such a splitting σ exists since F_{d-1} is a face and P is saturated by Proposition 7. Now let $\psi_1 : P \to \mathbb{Z}^{d-1}$ be the map

$$P \to P^{\mathrm{gp}} \xrightarrow{\sigma} F_{d-1}^{\mathrm{gp}} \to \mathbb{Z}^{d-1}.$$

That is, inclusion into P^{gp} followed by σ followed by the isomorphism $\tilde{\varphi}^{\mathrm{gp}}$. Notice that if $p \in F_{d-1}$, then the coordinates of $\psi_1(p)$ are non-negative.

Let $\mathfrak{p}$ be the complement of F_{d-1}. Since F_{d-1} is a facet of P, $\mathfrak{p}$ is a height one prime of P. $\overline{P_{\mathfrak{p}}} \cong \mathbb{N}$ since P was assumed to be saturated. Let $\psi_2 : P \to \mathbb{N}$ be the map

$$P \to P_{\mathfrak{p}} \to \overline{P_{\mathfrak{p}}} \xrightarrow{\cong} \mathbb{N}.$$

Notice that $\psi_2(p) = 0$ if and only if $p \in F_{d-1}$. Now I claim $\psi = (\psi_1, \psi_2) : P \to \mathbb{Z}^{d-1} \times \mathbb{N}$ is injective. If $\psi(p) = \psi(p')$, then their last coordinates are equal and $p - p' \in F_{d-1}^{\mathrm{gp}}$.

Since the map $F_{d-1}^{\text{gp}} \to \mathbb{Z}^{d-1}$ above is an isomorphism, ψ is injective. Furthermore, since some element of P maps to an element of $\mathbb{Z}^d$ whose last coordinate is one and the map $F_{d-1}^{\text{gp}} \to \mathbb{Z}^{d-1}$ above is an isomorphism, ψ^{gp} is an isomorphism.

Recall that the unique minimal generating set of an affine semigroup is called its Hilbert basis. For every element p of the Hilbert basis of P, let

$$(p_1, p_2, \dots, p_d) = \psi(p).$$

For $1 \leqslant i \leqslant d$, choose $n_i \in \mathbb{N}$ such that $p_i + n_i p_d \geqslant 0$ for every Hilbert basis element $p \in P$, and let θ be the automorphism of $\mathbb{Z}^d$ given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & n_1 \\ 0 & 1 & 0 & \cdots & 0 & n_2 \\ 0 & 0 & 1 & \cdots & 0 & n_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & n_{d-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Take $\varphi = \theta \circ \psi$. $\square$

Example 13. Let $P = \langle (0,2), (1,0), (2,-2) \rangle \subset \mathbb{N}^2$ and consider the flag $\{0\} \subset \langle (0,2) \rangle \subset P$. Suppose we already have $\tilde{\varphi} : \langle (0,2) \rangle \to \mathbb{N}$, given by $(0,2) \mapsto 1$. Let $\psi_1 : P \to \mathbb{Z}$ be given by $(0,2) \mapsto 1$, $(1,0) \mapsto 0$ and $(2,-2) \mapsto -1$. $\psi_2 : P \to \mathbb{N}$ is given by $(0,2) \mapsto 0$, $(1,0) \mapsto 1$ and $(2,-2) \mapsto 2$. Therefore, $\psi : P \to \mathbb{Z} \times \mathbb{N}$ is given by $(0,2) \mapsto (1,0)$, $(1,0) \mapsto (0,1)$ and $(2,-2) \mapsto (-1,2)$. Let $n_1 = 1$, then

$$\theta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So, φ is given by $(0,2) \mapsto (1,0)$, $(1,0) \mapsto (1,1)$ and $(2,-2) \mapsto (1,2)$.

1.3. Toroidal log schemes

Definition 14. We say a log structure $\mathcal{M}$ on X is *torsion-free* if $\overline{\mathcal{M}}$ is a sheaf of torsion-free monoids. We say a log structure $\mathcal{M}$ on X is *toroidal* if it is fine and torsion-free. We say a log scheme $(X, \mathcal{M})$ is *toroidal* if $\mathcal{M}$ is toroidal.

Claim 15. *If P is a sharp finitely generated torsion-free monoid, A is a Noetherian ring, $\beta : P \to A$ is a monoid homomorphism with respect to multiplication on A,* $\operatorname{Spec}(P \xrightarrow{\beta} A)$ *need not be toroidal. Let k be a field and consider*

$$\beta : \big\langle (1,0), (1,1), (0,2) \big\rangle \to k[x,y,z]/\big(x^2 z - y^2\big)$$

given by $(1,0) \mapsto x$, $(1,1) \mapsto y$, and $(0,2) \mapsto z$. Here $\overline{M}_{(x,y)} \cong \langle a, b \mid 2a = 2b \rangle$. See Remark 8.

Proposition 16. *If $(X, \mathcal{M}) = \operatorname{Spec}(P \xrightarrow{\beta} A)$ is a fine log scheme and $\beta : P \to A$ is local at $x = \mathfrak{p}$, then $\overline{\mathcal{M}}_x \cong \overline{P}$. Furthermore, if*

$$0 \to P^* \to P^{\mathrm{gp}} \to \overline{P}^{\mathrm{gp}} \to 0$$

is split exact, then there is a monoid homomorphism $\sigma : \overline{P} \to P$ such that the morphism $\operatorname{Spec}(\overline{P} \xrightarrow{\sigma\circ\beta} A) \to \operatorname{Spec}(P \xrightarrow{\beta} A)$ induced by σ is an isomorphism.

Proof. Since the question of whether $\overline{\mathcal{M}}_x$ is isomorphic to $\overline{P}$ is local, we may assume x is contained in every irreducible component of X. In particular, we may assume any neighborhood U of x is connected and $\Gamma(U, P_X) = P$. Let $\overline{\beta} : P \to A_{\mathfrak{p}}$ be the composition of β and the canonical map $A \to A_{\mathfrak{p}}$. By Remark 2,

$$\mathcal{M}_x \cong A_{\mathfrak{p}}^* \oplus P / \bigl\{\overline{\beta}(u)^{-1} \oplus u \mid u \in P^*\bigr\}.$$

We may now take the quotient $\mathcal{M}_x / \mathcal{M}_x^*$ to form $\overline{\mathcal{M}}_x$. Evidently,

$$\overline{\mathcal{M}}_x \cong \bigl(A_{\mathfrak{p}}^* \oplus P\bigr) / \bigl(A_{\mathfrak{p}}^* \oplus P^*\bigr).$$

But,

$$\overline{P} \cong \bigl(A_{\mathfrak{p}}^* \oplus P\bigr) / \bigl(A_{\mathfrak{p}}^* \oplus P^*\bigr).$$

The rest is straightforward. Any splitting of the exact sequence restricts to such a $\sigma : \overline{P} \to P$. $\square$

Corollary 17. *Let $(X, \mathcal{M})$ be a fine log scheme and let x be a point on X such that $\overline{\mathcal{M}}_x$ is a torsion-free monoid. Then, there exists an open neighborhood U of x and a homomorphism $\beta : \overline{\mathcal{M}}_x \to \mathcal{O}_X(U)$ such that β induces $\alpha|_U$ and the composition of the induced map $\overline{\mathcal{M}}_x \to \mathcal{M}_X(U)$ with restriction to the stalk $\mathcal{M}_x$ and the canonical map $\mathcal{M}_x \to \overline{\mathcal{M}}_x$ is the identity.*

1.4. The prime filtration theorem

Theorem 18 *(Prime Filtration Theorem). Let P be a finitely generated torsion-free monoid, let $A = \bigoplus_{p\in P} A_p$ be a Noetherian P-graded ring, and let $E = \bigoplus_{p\in P^{\mathrm{gp}}} E_p$ be a finitely generated P^{gp}-graded A-module. Then, E has a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with each $E_i / E_{i-1} \cong A/\mathfrak{p}_i$ for some homogeneous prime ideal $\mathfrak{p}_i \subset A$.

Proof. By the usual prime filtration theorem [26, Proposition 3.7], E has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with each $E_i/E_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. By induction on n, it suffices to prove that $\mathfrak{p}_1$ is homogeneous.

Since $A/\mathfrak{p}_1 \subseteq E$, $\mathfrak{p}_1$ is an associated prime of E. Hence, there is a least positive integer s such that there exists an element e of E with annihilator $\mathfrak{p}$ and P^{gp}-homogeneous elements e_i of E such that $e = \sum_{i=1}^{s} e_i$. By the results of §3 of Gilmer [27], there is a total order $\prec$ compatible with the group structure on P^{gp} since P^{gp} is torsion-free. For each i, let q_i be the degree of e_i. We may assume $q_1 \prec q_2 \prec \cdots \prec q_s$. Let f be an element of $\mathfrak{p}_1$. Write $f = \sum_{i=1}^{m} f_i$ with each f_i P-homogeneous of degree p_i and $p_1 \prec p_2 \prec \cdots \prec p_m$. We want to show that each homogeneous component f_i of f is in $\mathfrak{p}_1$. By induction on m, it suffices to prove f_1 is in $\mathfrak{p}_1$.

Notice that

$$0 = fe = f_1 e_1 + (\text{homogeneous terms of greater degree}).$$

So, $f_1 e_1$ is zero and we are done if $s = 1$, that is, if e is e_1. Notice that $f_1 e$ is $\sum_{i=2}^{s} f_1 e_i$ and has fewer nonzero homogeneous terms than e. Since s was minimal, the annihilator of $f_1 e$ properly contains $\mathfrak{p}_1$. Let f' be an element of the annihilator of $f_1 e$ that is not contained in $\mathfrak{p}_1$. We know $f' f_1$ is in $\mathfrak{p}_1$, f' is not in $\mathfrak{p}_1$, and $\mathfrak{p}_1$ is prime. Therefore, f_1 is in $\mathfrak{p}_1$. □

Remark 19. We cannot delete the torsion-free assumption in the above theorem: Let R be any ring, let a and b be distinct elements of P such that $na = nb$ for some positive integer n, and let n_0 be the least such positive integer. Now $t^a - t^b$ is an element of $R[P]$ that is annihilated by $\sum_{m=0}^{n_0-1} t^{(n_0-m-1)a+mb}$. But, $t^a - t^b$ cannot be annihilated by any nonzero homogeneous element of $R[P]$ because P is cancellative.

1.5. Combinatorial $\mathbb{Z}[P]$-modules

Lemma 20. *Let P be a finitely generated torsion-free monoid. If $\mathfrak{p}$ is a P-homogeneous prime of $\mathbb{Z}[P]$, then $\mathfrak{p} = (q) + (K) = (q, K)$ where $(q) \subset \mathbb{Z}$ and $K \subset P$ are prime.*

Proof. Fix an arbitrary P-homogeneous prime $\mathfrak{p}$ of $\mathbb{Z}[P]$. The intersection $\mathfrak{p} \cap \mathbb{Z}$ must be prime. Let $(q) = \mathfrak{p} \cap \mathbb{Z}$. If $\mathfrak{p} = q\mathbb{Z}[P]$ we are done, take K to be the empty ideal. If $\mathfrak{p} \neq q\mathbb{Z}[P]$, pick an arbitrary P-homogeneous element nt^p of $\mathfrak{p} \setminus q\mathbb{Z}[P]$ with $n \in \mathbb{Z}$ and $p \in P$. In particular, $n \notin (q) = \mathfrak{p} \cap \mathbb{Z}$. So, $n \notin \mathfrak{p}$. Hence, $t^p \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Let $K = \{p \in P \mid t^p \in \mathfrak{p}\}$ be the set of all such p. We have established $\mathfrak{p}$ is generated by q and (K). Now, it suffices to prove K is a prime ideal of P. If k is in K and p is in P, then $t^{p+k} = t^p \cdot t^k$ is in $\mathfrak{p}$ since $\mathfrak{p}$ is an ideal. So, K is an ideal. If K were not prime, there would exist p and p' in P such that $p + p'$ would be in K while neither p nor p' was in K. That is, there would exist t^p and $t^{p'}$ in $\mathbb{Z}[P]$ such that $t^p \cdot t^{p'}$ would be in $\mathfrak{p}$ while neither t^p nor t^q was in $\mathfrak{p}$ and $\mathfrak{p}$ would not be prime. Hence, K is prime. □

Definition 21. Let P be a monoid. We say A is a *sub-P-set* of P^{gp} if A is a subset of P^{gp} and A is closed under the action of P on P^{gp} given by addition. That is, $P + A = A$. We say a sub-P-set of P^{gp} is a *fractional ideal* if there exists an element p of P such that $p + A$ is contained in P. A *combinatorial $\mathbb{Z}[P]$-module* is one isomorphic to a quotient of $\mathbb{Z}[P]$-submodules $(A)/(B)$ of $\mathbb{Z}[P^{\mathrm{gp}}]$, for some sub-$P$-sets A and B of P^{gp}. If K is an ideal of P and our module is also annihilated by (K), we will say it is a *combinatorial* $\mathbb{Z}[P]/(K)$*-module*.

Corollary 22. *If P is a finitely generated torsion-free monoid and N is a finitely generated combinatorial $\mathbb{Z}[P]$-module, then N has a filtration*

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_i/N_{i-1} \cong \mathbb{Z}[P]/(K)$ for some prime ideal $K \subset P$.

Proof. We may assume $N = (A)/(B)$ where A and B are fractional ideals of P since P is a finitely generated torsion-free monoid. By Theorem 18, we know N has a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_i/N_{i-1} \cong \mathbb{Z}[P]/\mathfrak{p}_i$ for some P-homogeneous prime ideal $\mathfrak{p}_i$. In particular, for $\mathfrak{p}_1$ we may take any associated prime of N. By induction on n, it suffices to prove that N has an associated prime of the form (K) for some prime $K \subset P$.

By Lemma 20, every associated prime of a finitely generated combinatorial $\mathbb{Z}[P]$-module is of the form (q, K) for some primes $K \subset P$ and $(q) \subset \mathbb{Z}$. Evidently, each combinatorial $\mathbb{Z}[P]$-module is a free Abelian group and on any combinatorial $\mathbb{Z}[P]$-module every integer acts injectively. So, our prime is of the form (K) for some prime $K \subset P$. □

Proposition 23. *If $P \to Q$ is an inclusion of finitely generated torsion-free monoids, K is an ideal of Q, $K' = K \cap P$, and N is a combinatorial $\mathbb{Z}[Q]/(K)$-module, then N is a direct sum of combinatorial $\mathbb{Z}[P]/(K')$-modules.*

Proof. Write $N = (A)/(B)$ with A and B sub-Q-sets of Q^{gp} such that $A + K \subseteq B$. Pick a set C of coset representatives for $Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. For each $c \in C$, let $A_c = \{q - c \mid q \in A \text{ and } q - c \in P^{\mathrm{gp}}\}$ and let $B_c = \{q - c \mid q \in B \text{ and } q - c \in P^{\mathrm{gp}}\}$. For each $c \in C$, A_c and B_c are sub-P-sets of P^{gp} such that $A_c + K' \subseteq B_c$. Therefore, $(A_c)/(B_c)$ is a combinatorial $\mathbb{Z}[P]/(K')$-module. Furthermore, $(A)/(B) \cong \bigoplus_{c \in C}(A_c)/(B_c)$ as $\mathbb{Z}[P]/(K')$-modules. □

1.6. Lorenzon's algebra

Let $(X, \mathcal{M})$ be a fine log scheme. For every local section $\overline{m}$ of $\overline{\mathcal{M}}$, the pre-image of $\overline{m}$ along the canonical map $\mathcal{M} \to \overline{\mathcal{M}}$ is an $\mathcal{O}_X^*$-torsor. So, the canonical map $\mathcal{M} \to \overline{\mathcal{M}}$ is an $\overline{\mathcal{M}}$-indexed family of $\mathcal{O}_X^*$-torsors. To each $\mathcal{O}_X^*$-torsor $\mathcal{E}$, we associate the contracted product $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$, where $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ is the quotient of the product $\mathcal{E} \times \mathcal{O}_X$ by the equivalence relation $\sim$, where $(eu, f) \sim (e, uf)$ whenever e, u and f are respectively local sections of $\mathcal{E}$, $\mathcal{O}_X^*$ and $\mathcal{O}_X$. Each $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ is an invertible sheaf. The association of $\mathcal{E} \wedge_{\mathcal{O}_X^*} \mathcal{O}_X$ to $\mathcal{E}$ and the $\overline{\mathcal{M}}$-indexed family of $\mathcal{O}_X^*$-torsors $\mathcal{M} \to \overline{\mathcal{M}}$ together yield an $\overline{\mathcal{M}}$-indexed family of invertible sheaves.

Furthermore, if $\overline{m}$ and $\overline{m}'$ are local sections $\overline{\mathcal{M}}$ with corresponding invertible sheaves $\mathcal{L}_{\overline{m}}$ and $\mathcal{L}_{\overline{m}'}$, then $\mathcal{L}_{\overline{m}+\overline{m}'}$ is isomorphic to $\mathcal{L}_{\overline{m}} \otimes \mathcal{L}_{\overline{m}'}$ (see Lorenzon [28]). This family of invertible sheaves is a ring object in the comma category of sheaves of sets on X over $\overline{\mathcal{M}}$, $\mathrm{Sh}(X)/\overline{\mathcal{M}}$, and the part indexed by the identity section of $\overline{\mathcal{M}}$ is $\mathcal{O}_X$. Lorenzon calls this $\overline{\mathcal{M}}$-indexed $\mathcal{O}_X$-algebra the canonical algebra of the log scheme. Let the algebra $\mathcal{A}_X$ be the direct sum over sections of $\overline{\mathcal{M}}$ of these invertible sheaves with multiplication given by tensor product over $\mathcal{O}_X$ extended by $\mathcal{O}_X$-linearity. Lorenzon calls this algebra the algebra induced on X by the canonical algebra of the log scheme. The canonical algebra consists of the homogeneous pieces of our $\overline{\mathcal{M}}$-graded algebra.

Claim 24. *Lorenzon uses $(\mathcal{A}_X)_*$ to denote the algebra we denote by $\mathcal{A}_X$ and he uses $\mathcal{A}_X$ to denote the canonical algebra of the log scheme.*

It happens that $\mathcal{A}_X$ is the $\mathcal{O}_X$-algebra $\mathcal{O}_X[\mathcal{M}]/(t^m - \alpha(m))_{m \in \mathcal{M}^*}$. For an ideal sheaf $\mathcal{K}$ of $\mathcal{M}$, let $(\mathcal{K})$ be the image in $\mathcal{A}_X$ of the ideal sheaf $\mathcal{K}$ along the canonical homomorphism $\mathcal{M} \to \mathcal{O}_X[\mathcal{M}] \to \mathcal{A}_X$.

Example 25 *($\mathbf{P}^1$ with a marked point).* Let k be a field. Consider the point $p = (x)$ on $\mathbf{P}^1 = \mathrm{Proj}\, k[x, y]$. Let $\mathcal{M}$ be the log structure that is trivial on the open subset $\mathbf{P}^1 \setminus \{p\}$ and induced by the $\mathbb{N} \to k[x/y]$, $1 \mapsto x/y$ on the affine open subscheme $U = \mathrm{Spec}\, k[x/y]$ containing p. Here $\overline{\mathcal{M}}$ is the skyscraper sheaf with stalk $\mathbb{N}$ at p and the $\mathcal{O}^*_{\mathbf{P}^1}$-torsor associated to the natural number n is generated by $(x/y)^n$ on U and by 1 on $\mathbf{P}^1 \setminus \{p\}$. So, the invertible sheaf associated to the natural number n has local basis $(x/y)^n$ on U and 1 on $\mathbf{P}^1 \setminus \{p\}$. That is, the invertible sheaf associated to the natural number n is $\mathcal{L}(-p)^{\bigotimes n}$ and $\mathcal{A}_{\mathbf{P}^1}$ is the $\mathcal{O}_{\mathbf{P}^1}$-algebra $\bigoplus_{n \in \mathbb{N}} \mathcal{L}(-p)^{\bigotimes n}$.

Proposition 26. *Let $(X, \mathcal{M}) = \mathrm{Spec}(P \xrightarrow{\beta} A)$ be a fine log scheme and let $\beta : P \to A$ be local at $x = \mathfrak{p}$, then*

$$\mathcal{A}_{X,x} \cong \mathcal{O}_x[P]/\big(t^p - \beta(p)\big)_{p \in P^*}.$$

Proof. To prove this, we write $\mathcal{M}_x \cong \mathcal{O}^*_{X,x} \oplus P/\{\beta(u)^{-1} \oplus u \mid u \in P^*\}$ and interchange the order in which quotients are taken. Instead of first applying the congruence that forms $\mathcal{M}_x$ from $\mathcal{O}^*_{X,x} \oplus P$, we first identify the copy of $\mathcal{O}^*_{X,x}$ in $\mathcal{O}^*_{X,x} \oplus P$ with the copy of $\mathcal{O}^*_{X,x}$ in $\mathcal{O}_{X,x}$:

$$\begin{aligned}
\mathcal{A}_{X,x} &= \mathcal{O}_{X,x}[\mathcal{M}_x]/\big(t^m - \alpha(m)\big)_{m \in \mathcal{M}_x^*} \\
&= \mathcal{O}_{X,x}\big[\mathcal{O}^*_{X,x} \oplus P\big]/\big(t^{(1,p)} - t^{(\beta(p),0)}, t^{(u,0)} - u\big)_{p \in P^*, u \in \mathcal{O}^*_{X,x}} \\
&= \mathcal{O}_{X,x}[P]/\big(t^p - \beta(p)\big)_{p \in P^*}. \qquad \square
\end{aligned}$$

In particular, $\mathcal{A}_{X,x} \cong \mathcal{O}_{X,x}[\overline{\mathcal{M}}_x]$ when $\overline{\mathcal{M}}_x$ is torsion-free by Proposition 16.

2. t-Flatness

2.1. First properties of t-flatness

Definition 27. Let $(X, \mathcal{M})$ be an fine log scheme, let x be a point of X, let $\mathcal{K}$ be an ideal of $\mathcal{M}$ and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules such that $(\mathcal{K})_x\mathcal{F}_x = 0$. We say $\mathcal{F}$ is *$\mathcal{M}$-flat relative to $\mathcal{K}$ at x* if for all ideals $\mathcal{J}$ of $\mathcal{M}$ containing $\mathcal{K}$ we have

$$\operatorname{Tor}_1^{\mathcal{A}_{X,x}/(\mathcal{K})_x}\big(\mathcal{A}_{X,x}/(\mathcal{J})_x, \mathcal{F}_x\big) = 0.$$

We say $\mathcal{F}$ is $\mathcal{M}$-flat at x if $\mathcal{F}$ is $\mathcal{M}$-flat relative to the constant ideal sheaf with empty stalks at x.

Definition 28. Let P be a finitely generated monoid, let $K \subseteq P$ be an ideal, and let E be a nonzero $\mathbb{Z}[P]/(K)$-module. We say E has *t-flat dimension d relative to K* if

$$d = \sup\big\{i \mid \exists J \subseteq P \text{ containing } K, \operatorname{Tor}_i^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(J), E\big) \neq 0\big\}.$$

If $E = 0$, we say E has t-flat dimension 0 relative to K. We say E is *t-flat relative to K* if

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(J), E\big) = 0$$

for all ideals $J \subseteq P$ containing K. If E is t-flat relative to $\emptyset$, we simply say E is t-flat.

Later, Theorem 32, we will prove E is t-flat relative to K if and only if E has t-flat dimension 0 relative to K.

Proposition 29. *Let P be a finitely generated torsion-free monoid, let A be a Noetherian ring, let $\mathfrak{p} \subset A$ be prime, let $\beta : P \to A$ be a monoid homomorphism with respect to multiplication on A, let $(X, \mathcal{M}) = \operatorname{Spec}(P \xrightarrow{\beta} A)$, and let x be the point on X corresponding to $\mathfrak{p}$. Suppose $\overline{\mathcal{M}}_x$ is torsion-free and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We consider $\mathcal{F}_x$ to be a $\mathbb{Z}[P]$-module along the map $\mathbb{Z}[P] \to \mathcal{O}_{X,x}$ induced by β. If $K \subseteq P$ is an ideal such that (K) annihilates $\mathcal{F}_x$, then*

$$\operatorname{Tor}_i^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(J), \mathcal{F}_x\big) \cong \operatorname{Tor}_i^{\mathcal{A}_{X,x}/(K)_x}\big(\mathcal{A}_{X,x}/(J)_x, \mathcal{F}_x\big), \quad \forall i \geqslant 0$$

for all ideals J of P containing K.

Proof. Since both of these modules are $\mathcal{O}_{X,x}$-modules, both sides are zero if $J \nsubseteq \beta^{-1}(\mathfrak{p})$ and we may assume $A = \mathcal{O}_{X,x}$. By Proposition 16, we may assume $P \cong \overline{\mathcal{M}}_x$. In particular, we may assume $\mathcal{A}_{X,x} = A[P]$ by Proposition 26. Fix ideals $K \subseteq J \subseteq P$. Let

$$\mathbf{F}_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}[P]/(J) \to 0$$

be a P^{gp}-graded free resolution of the $\mathbb{Z}[P]/(K)$-module $\mathbb{Z}[P]/(J)$. Since $\mathbb{Z}[P]/(J)$ is a free $\mathbb{Z}$-module,

$$\mathbf{F}'_{\bullet} : \cdots \to F_2 \otimes_{\mathbb{Z}} A \to F_1 \otimes_{\mathbb{Z}} A \to F_0 \otimes_{\mathbb{Z}} A \to A[P]/(J) \to 0$$

is a P^{gp}-graded free resolution of $A[P]/(J)$ as a $A[P]/(K)$-module. Furthermore,

$$\begin{aligned} F_i \otimes_{\mathbb{Z}[P]/(K)} \mathcal{F}_x &\cong F_i \otimes_{\mathbb{Z}[P]/(K)} A[P]/(K) \otimes_{A[P]/(K)} \mathcal{F}_x \\ &\cong F_i \otimes_{\mathbb{Z}[P]/(K)} \mathbb{Z}[P]/(K) \otimes_{\mathbb{Z}} A \otimes_{A[P]/(K)} \mathcal{F}_x \\ &\cong F_i \otimes_{\mathbb{Z}} A \otimes_{A[P]/(K)} \mathcal{F}_x. \end{aligned}$$

Therefore, $\mathrm{Tor}_i^{\mathbb{Z}[P]/(K)}(\mathbb{Z}[P]/(J), \mathcal{F}_x)$, the ith cohomology module of

$$\mathbf{F}_{\bullet} \otimes_{\mathbb{Z}[P]/(K)} \mathcal{F}_x,$$

and $\mathrm{Tor}_i^{A[P]/(K)}(A[P]/(J), \mathcal{F}_x)$, the ith cohomology module of

$$\mathbf{F}'_{\bullet} \otimes_{A[P]/(K)} \mathcal{F}_x,$$

are isomorphic for all $i \geqslant 0$. □

Corollary 30. *Let P be a finitely generated torsion-free monoid, let A be a Noetherian ring, let $\mathfrak{p} \subset A$ be prime, let $\beta : P \to A$ be a monoid homomorphism with respect to multiplication on A, let $(X, \mathcal{M})$ be the log scheme* $\mathrm{Spec}(P \xrightarrow{\beta} A)$*, and let x be the point on X corresponding to $\mathfrak{p}$. Suppose $\overline{\mathcal{M}}_x$ is torsion-free and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We consider $\mathcal{F}_x$ to be a $\mathbb{Z}[P]$-module along the map $\mathbb{Z}[P] \to \mathcal{O}_{X,x}$ induced by β. If $K \subseteq P$ is an ideal such that (K) annihilates $\mathcal{F}_x$ and $\mathcal{K} \subseteq \mathcal{M}$ is the ideal generated by the image of K, then $\mathcal{F}$ is $\mathcal{M}$-flat relative to $\mathcal{K}$ at $x \in X$ if and only if $\mathcal{F}_x$ is t-flat relative to K.*

2.2. Local criterion for t-flatness

We will model the proof of our local criterion for t-flatness on the proof of the local criterion for flatness found in Matsumura [29].

Recall that a A-module E is said to be I-adically ideal-separated if $\mathfrak{a} \otimes E$ is separated for the I-adic topology for every finitely generated ideal $\mathfrak{a}$ of A. In particular, if B is a Noetherian A-algebra, $IB \subseteq \mathrm{rad}(B)$, and E is a finitely generated B-module, then E is I-adically ideal-separated.

Theorem 31. [29, Theorem 22.3] *Let I be an ideal of a ring A and let M be a A-module. Set $A_n = A/I^{n+1}$ for each integer $n \geqslant 0$, $M_n = M/I^{n+1}M$ for each integer $n \geqslant 0$,* $\mathrm{gr}(A) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$*, and* $\mathrm{gr}(M) = \bigoplus_{n \geqslant 0} I^n M/I^{n+1}M$*. Let*

$$\mu_n : \left(I^n/I^{n+1}\right) \otimes_{A_0} M_0 \to I^n M/I^{n+1}M$$

be the standard map for each $n \geqslant 0$, and let

$$\mu : \mathrm{gr}(A) \otimes_{A_0} M_0 \to \mathrm{gr}(M)$$

be the direct sum of the μ_n.

In the above notation, suppose that one of the following two conditions is satisfied: (α) I is a nilpotent ideal or (β) A is a Noetherian ring and M is I-adically ideal-separated. Then the following conditions are equivalent,

(1) *M is flat over A;*
(2) $\mathrm{Tor}_1^A(N, M) = 0$ *for every A_0-module N;*
(3) *M_0 is flat over A_0 and $I \otimes_A M = IM$;*
(3′) *M_0 is flat over A_0 and* $\mathrm{Tor}_1^A(A_0, M) = 0$;
(4) *M_0 is flat over A_0 and μ_n is an isomorphism for every $n \geqslant 0$;*
(4′) *M_0 is flat over A_0 and μ is an isomorphism;*
(5) *M_n is flat over A_n for every $n \geqslant 0$.*

In fact, the implications (1) ⇒ (2) ⇔ (3) ⇔ (3′) ⇒ (4) ⇒ (5) *hold without the assumption on M.*

Let P be a finitely generated torsion-free monoid. We consider the case where $A = \mathbb{Z}[P]/(K)$, $I = (P^+)/(K)$ and one of the following two conditions holds: there exists a Noetherian A-algebra B with $IB \subseteq \mathrm{rad}(B)$, and E is a finitely generated B-module or I is nilpotent.

Let d be the dimension of $\overline{P^{\mathrm{sat}}}$ and let $\varphi : \overline{P^{\mathrm{sat}}} \to \mathbb{N}^d$ be an inclusion as in Proposition 12. Pick d $\mathbb{Q}$-linearly independent positive real numbers

$$\{\gamma_1, \gamma_2, \ldots, \gamma_d\},$$

let the monoid homomorphism $\psi : \mathbb{N}^d \to \mathbb{R}$ be given by

$$(n_1, n_2, \ldots, n_d) \mapsto \sum_{i=1}^{d} n_i \gamma_i,$$

let the monoid homomorphism $\nu : P \to \mathbb{R}$ be the composition

$$P \xrightarrow{\text{canonical}} \overline{P^{\mathrm{sat}}} \xrightarrow{\varphi} \mathbb{N}^d \xrightarrow{\psi} \mathbb{R},$$

and let Γ be the image of ν. We order Γ with the order induced by the standard ordering of $\mathbb{R}$. Notice that Γ is well ordered.

Set $I_\gamma = (t^p)_{\gamma \leqslant \nu(p),\, p \in P \setminus K}$ for each $\gamma \in \Gamma$, $K_\gamma^+ = \{p \in P \mid \nu(p) > \gamma\}$ for each $\gamma \in \Gamma$, $I_\gamma^+ = I_{\min\{\gamma' \mid \gamma < \gamma'\}}$ for each $\gamma \in \Gamma$, $A_\gamma = A/I_\gamma^+$ for each $\gamma \in \Gamma$, $E_\gamma = E/I_\gamma^+ E$ for each $\gamma \in \Gamma$, let $\mathrm{gr}_\gamma(A) = I_\gamma / I_\gamma^+$ for each $\gamma \in \Gamma$, let $\mathrm{gr}_\gamma(E) = I_\gamma E / I_\gamma^+ E$ for each $\gamma \in \Gamma$, let

$\mathrm{gr}(A)$ be the associated graded ring of the filtration $\{I_\gamma \mid \gamma \in \Gamma\}$, let $\mathrm{gr}(E)$ be the associated graded module of the filtration $\{I_\gamma E \mid \gamma \in \Gamma\}$, let

$$\mu_\gamma : \mathrm{gr}_\gamma(A) \otimes_{A_0} E_0 \to \mathrm{gr}_\gamma(E)$$

be the multiplication map for each $n \geqslant 0$, and let

$$\mu : \mathrm{gr}(A) \otimes_{A_0} E_0 \to \mathrm{gr}(E)$$

be the direct sum of the μ_γ.

Theorem 32 *(Local Criterion for t-Flatness). Continuing the notation above,* $\mathrm{gr}(A) \cong A$ *and if B is a Noetherian A-algebra, $IB \subseteq \mathrm{rad}(B)$, and E is a finitely generated B-module or if I is nilpotent, then the following are equivalent*:

(1) *E is t-flat relative to K.*
(2) $\mathrm{Tor}_i^A(N, E) = 0$ *for all $i > 0$ and every combinatorial A-module N.*
(3) *The canonical surjection*

$$I \otimes_A E \to IE$$

is an isomorphism.
(3′) $\mathrm{Tor}_1^A(A_0, E) = 0$.
(4) *μ_γ is an isomorphism for all $\gamma \in \Gamma$.*
(4′) *μ is an isomorphism.*
(5) *E_γ is t-flat relative to $K_\gamma^+ \cup K$ for all $\gamma \in \Gamma$.*
(6) *The multiplication map*

$$\left(I^n/I^{n+1}\right) \otimes_{A_0} E_0 \to I^n E/I^{n+1}E$$

is an isomorphism for all $n \in \mathbb{N}$.
(7) *$E/I^{n+1}E$ is t-flat relative to $(n+1)P^+ \cup K$ for all $n \in \mathbb{N}$.*

In fact, (1) ⇔ (2) ⇒ (3) ⇔ (3′) ⇔ (4) ⇔ (4′) ⇔ (5) ⇔ (6) ⇔ (7) *without any extra assumptions on I or E.*

Proof. First, consider the underlying group of $\mathrm{gr}(A)$:

$$\mathrm{gr}(A) = \bigoplus_{\gamma\in\Gamma} I_\gamma/I_\gamma^+ \cong \bigoplus_{\gamma\in\Gamma}\left(\bigoplus_{\substack{\nu(p)=\gamma\\ p\in P\setminus K}} \mathbb{Z}t^p\right) \cong \bigoplus_{p\in P\setminus K} \mathbb{Z}t^p \cong A,$$

since each ideal I_γ is P-homogeneous. Furthermore, multiplication is given by

$$t^p \cdot t^q = \begin{cases} t^{p+q}, & \text{if } p+q \notin K, \\ 0, & \text{otherwise,} \end{cases}$$

in $\mathrm{gr}(A)$ since $\nu(p+q)=\nu(p)+\nu(q)$. Since $\{t^p \mid p \in P \setminus K\}$ is a $\mathbb{Z}$-basis for the free group $\mathrm{gr}(A)$, $\mathrm{gr}(A) \cong A$ as rings as well.

(2) $\Rightarrow$ (1). Evident.

(1) $\Rightarrow$ (2). First, we will treat the finitely generated combinatorial A-modules. We proceed by induction on i. Let $i=1$. If N is a finitely generated combinatorial A-module, then N has a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = N$$

with each $N_l/N_{l-1} \cong \mathbb{Z}[P]/(J_l)$ for some prime ideal $J_l \subset P$ according to Corollary 22. Since each N_l is a submodule of N, each N_l is annihilated by (K). Since, each $\mathbb{Z}[P]/(J_l) \cong N_l/N_{l-1}$, each $\mathbb{Z}[P]/(J_l)$ is annihilated by (K) as well. So, each J_l contains K. Now we proceed by induction on the length of our filtration. If $n=1$, then $N = N_1 \cong \mathbb{Z}[P]/(J_1)$ and we are done. If $n>1$, assume $\mathrm{Tor}_1^A(N', E)=0$ for every combinatorial A-module N' whose filtration has length $n-1$. We have

$$0 \to N_{n-1} \to N \to \mathbb{Z}[P]/(J_n) \to 0.$$

Once we take the tensor product with E, we obtain the long exact sequence

$$\cdots \to \mathrm{Tor}_1^A(N_{n-1}, E) \to \mathrm{Tor}_1^A(N, E) \to \mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J_n), E\big) \to \cdots.$$

Since E is t-flat relative to K, $\mathrm{Tor}_1^A(\mathbb{Z}[P]/(J_n), E)=0$. Furthermore,

$$\mathrm{Tor}_1^A(N_{n-1}, E)=0$$

by the induction hypothesis. So, we have $\mathrm{Tor}_1^A(N, E)=0$ for every finitely generated combinatorial A-module N. Now suppose $i>1$ and

$$\mathrm{Tor}_{i-1}^A(N', E)=0$$

for every finitely generated combinatorial A-module N'. Let $J \subseteq P$ be an ideal containing K. We have

$$0 \to (J)/(K) \to A \to \mathbb{Z}[P]/(J) \to 0.$$

Once we take the tensor product with E, we obtain the long exact sequence

$$\cdots \to 0 \to \mathrm{Tor}_i^A\big(\mathbb{Z}[P]/(J), E\big) \to \mathrm{Tor}_{i-1}^A\big((J)/(K), E\big) \to 0 \to \cdots.$$

Since $\mathrm{Tor}_{i-1}^A((J)/(K), E)=0$ by the induction hypothesis,

$$\mathrm{Tor}_i^A\big(\mathbb{Z}[P]/(J), E\big)=0.$$

Any combinatorial A-module is the union of its finitely generated combinatorial submodules. Since right exact functors commute with colimits and filtered colimits are exact in module categories, we are done.

(3) $\Leftrightarrow$ (3′). Notice that $\mathrm{Tor}_1^A(A_0, E)$ is the kernel of the surjective map in (3).

(1) $\Rightarrow$ (3′). (1) implies (3′) follows from the definition of t-flatness.

(3′) $\Rightarrow$ (1). Let $S = \{J \mid \mathrm{Tor}_1^A(\mathbb{Z}[P]/(J), E) \neq 0\}$. If S is nonempty, then S has a maximal element J since the ideals of P satisfy the ascending chain condition. By Lemma 33, J is prime. Let $q \in P^+ \setminus J$. Since J is prime, we have a short exact sequence

$$0 \to \mathbb{Z}[P]/(J) \xrightarrow{\cdot t^q} \mathbb{Z}[P]/(J) \to \mathbb{Z}[P]/(q, J) \to 0.$$

This sequence yields the long exact Tor sequence

$$\cdots \to \mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J), E\big) \xrightarrow{\cdot t^q} \mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J), E\big) \to 0.$$

Now apply Nakayama's lemma. We conclude

$$\mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J), E\big)_{\mathfrak{p}} = 0$$

at every prime $\mathfrak{p}$ containing an element of $P^+ \setminus J$. If I is nilpotent, then every prime contains every element of $P^+ \setminus J$. If E is a finitely generated module over some A-algebra B such that $IB \subseteq \mathrm{rad}(B)$, every maximal ideal in the support of E contains every element of $P^+ \setminus J$. In either case,

$$\mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J), E\big)_{\mathfrak{m}} = 0$$

at every maximal ideal $\mathfrak{m} \subset A$ in the support of E. Therefore,

$$\mathrm{Tor}_1^A\big(\mathbb{Z}[P]/(J), E\big) = 0$$

and E is t-flat relative to K.

(2) $\Rightarrow$ (4). Mimic the (3) $\Rightarrow$ (4) argument in Matsumura.

(4) $\Leftrightarrow$ (4′). Evident.

(4) $\Rightarrow$ (5). Mimic the (4) $\Rightarrow$ (5) argument in Matsumura.

(5) $\Rightarrow$ (1). Mimic the (5) $\Rightarrow$ (1) argument in Matsumura.

(3′) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). Fix γ and apply the previous arguments with K replaced by $K' = K_\gamma^+ \cup K$ and E replaced by $E/(K')E$. In this case, $(P^+)/(K')$ is nilpotent. So, (4) and (5) are both equivalent to

$$\mathrm{Tor}_1^{\mathbb{Z}[P]/(K')}\big(A_0, E/(K')E\big) = 0.$$

But, $\mathrm{Tor}_1^{\mathbb{Z}[P]/(K')}(A_0, E/(K')E)$ is the kernel of the canonical map

$$\varphi : \big(I/I_\gamma^+\big) \otimes_{A_\gamma} E_\gamma \to E_\gamma$$

and

$$\left(I/I_\gamma^+\right) \otimes_{A_\gamma} E_\gamma \cong \left(I/I_\gamma^+\right) \otimes_{A_\gamma} A_\gamma \otimes_A E \cong \left(I/I_\gamma^+\right) \otimes_A E.$$

So, $\operatorname{Tor}_1^{\mathbb{Z}[P]/(K')}(A_0, E/(K')E)$ is isomorphic to $\operatorname{Tor}_1^A(A_0, E)$ the kernel of

$$\left(I/I_\gamma^+\right) \otimes_A E \to E_\gamma.$$

(2) ⇒ (6). See Matsumura's proof that (3) ⇒ (4).
(6) ⇒ (7). See Matsumura's proof that (4) ⇒ (5).
(7) ⇒ (1). See Matsumura's proof that (5) ⇒ (1).
(3′) ⇔ (6) ⇔ (7). Mimic our proof that (3′) ⇔ (4) ⇔ (5). □

Lemma 33. *Continuing the above notation, if J' is an ideal of P containing K such that*

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\left(\mathbb{Z}[P]/(J'), E\right) \neq 0,$$

then there exists a prime J containing J' such that

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\left(\mathbb{Z}[P]/(J), E\right) \neq 0.$$

Proof. Let $S = \{J'' \mid \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(\mathbb{Z}[P]/(J''), E) \neq 0\}$. Since S is nonempty, S has a maximal element J since the ideals of P satisfy the ascending chain condition. Suppose J is an ideal of P containing J' such that

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\left(\mathbb{Z}[P]/(J), E\right) \neq 0$$

and no ideal properly containing J has this property. We will prove J is prime by contradiction. Suppose J were not prime. Using Theorem 18, write

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n = \mathbb{Z}[P]/(J)$$

such that each $N_{i+1}/N_i \cong \mathbb{Z}[P]/(J_i)$ for some prime ideal $J_i \in P$. As before, each J_i contains J. Since J is assumed to not be prime, these containments are proper. We get a series of short exact sequences

$$0 \to N_i \to N_{i+1} \to \mathbb{Z}[P]/(J_i) \to 0.$$

Take the various long exact Tor sequences to get

$$0 \to \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N_i, E) \to \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N_{i+1}, E) \to 0$$

by the maximality of J. That is

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\left(\mathbb{Z}[P]/(J), E\right) = \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N_n, E) \cong \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N_0, E) = 0.$$

This contradicts the choice of J. So, J must be prime. $\square$

In particular, if $A = \mathbb{Z}[P]/(K)$, an A-module E is t-flat (relative to K) if and only if $E_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every log prime ideal $\mathfrak{p} = (J)/(K)$ of A.

Proposition 34. *Let $P \to Q$ be an inclusion between finitely generated torsion-free monoids, let K be an ideal of Q, let $K' = K \cap P$, let N be a combinatorial $\mathbb{Z}[Q]/(K)$-module, and let E be a $\mathbb{Z}[P]/(K')$-module. If E has t-flat dimension d relative to K, then*

$$\operatorname{Tor}_i^{\mathbb{Z}[P]/(K')}(N, E) = 0 \quad \textit{for all } i > d.$$

Proof. It suffices to prove that if E is t-flat relative to K, then

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K')}(N, E) = 0.$$

(For $d > 0$, apply the t-flat case to the dth syzygy module of E.) According to Proposition 23, N is a direct sum of combinatorial $\mathbb{Z}[P]/(K')$-modules. Since tor functors commute with direct sums,

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K')}(N, E)$$

is the direct sum of modules of the form $\operatorname{Tor}_1^{\mathbb{Z}[P]/(K')}(N', E)$ where each N' is a combinatorial $\mathbb{Z}[P]/(K')$-module. Each $\operatorname{Tor}_1^{\mathbb{Z}[P]/(K')}(N', E) = 0$ by the equivalence of (1) and (2) in Theorem 32. $\square$

Proposition 35. *Let $P \to Q$ be an inclusion between finitely generated torsion-free monoids, let E be a $\mathbb{Z}[P]/(K)$-module, and let $K' = K + Q$. If E has t-flat dimension d relative to K, then the $\mathbb{Z}[Q]/(K')$-module $E' = \mathbb{Z}[Q]/(K') \otimes_{\mathbb{Z}[P]/(K)} E$ has t-flat dimension less than or equal to d relative to K'.*

Proof. It suffices to prove that if E is t-flat relative to K, then E' is t-flat relative to K' (For $d > 0$, apply the t-flat case to the dth syzygy module of E.) Now suppose E is t-flat.

Let N be a combinatorial $\mathbb{Z}[Q]/(K')$-module and let $0 \to L \to F \to N \to 0$ be an exact sequence of $\mathbb{Z}[Q]/(K')$-modules with F free. Tensor this exact sequence with E' to get the long exact sequence

$$\begin{aligned}\cdots \to 0 \to \operatorname{Tor}_1^{\mathbb{Z}[Q]/(K')}(N, E') \to L \otimes_{\mathbb{Z}[Q]/(K')} E' \to F \otimes_{\mathbb{Z}[Q]/(K')} E' \\ \to N \otimes_{\mathbb{Z}[Q]/(K')} E' \to 0.\end{aligned}$$

Our long exact sequence can also be written as

$$\begin{aligned}\cdots \to 0 \to \operatorname{Tor}_1^{\mathbb{Z}[Q]/(K')}(N, E') \to L \otimes_{\mathbb{Z}[P]/(K)} E' \to F \otimes_{\mathbb{Z}[P]/(K)} E' \\ \to N \otimes_{\mathbb{Z}[P]/(K)} E' \to 0\end{aligned}$$

since, for any $\mathbb{Z}[Q]/(K')$-module E'',

$$E'' \otimes_{\mathbb{Z}[Q]/(K')} E' = E'' \otimes_{\mathbb{Z}[Q]/(K')} \mathbb{Z}[Q]/(K') \otimes_{\mathbb{Z}[P]/(K)} E \cong E'' \otimes_{\mathbb{Z}[P]/(K)} E.$$

On the other hand, we have the long exact sequence obtained by taking the tensor product of our short exact sequence with E over $\mathbb{Z}[P]/(K)$:

$$\cdots \to \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(F, E) \to \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N, E) \to L \otimes_{\mathbb{Z}[P]/(K)} E' \to F \otimes_{\mathbb{Z}[P]/(K)} E' \\ \to N \otimes_{\mathbb{Z}[P]/(K)} E' \to 0.$$

By Proposition 34 and the fact that tor functors commute with direct sums,

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(F, E) = 0.$$

Therefore,

$$\operatorname{Tor}_1^{\mathbb{Z}[Q]/(K')}(N, E') \cong \operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(N, E) = 0$$

and E' is t-flat relative to K'. □

Definition 36. We say a $\mathbb{Z}[P]/(K)$-module E is *weakly t-flat relative to K* if

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(P^+), E\big) = 0.$$

Lemma 37. *Continuing the notation from Theorem* 32*, let B be a Noetherian A-algebra, let E be a finitely generated B-module and let $\widehat{E}$ be the I-adic completion of E. Then E is weakly t-flat relative to K if and only if $\widehat{E}$ is t-flat relative to K.*

Proof. First, suppose E is weakly t-flat relative to K. Since B is Noetherian, so is its I-adic completion $\widehat{B}$. Furthermore, $I\widehat{B} \subseteq \operatorname{rad}(\widehat{B})$. Consider the long exact tor sequence

$$\cdots \to 0 \to \operatorname{Tor}_1^A(A_0, E) \to I \otimes_A E \to E_0 \to 0.$$

It is an exact sequence of B-modules and $\widehat{B}$ is a flat B-module. So, after taking the tensor product with $\widehat{B}$, the resultant sequence is exact. In particular,

$$\operatorname{Tor}_1^A\big(A_0, \widehat{E}\big) \cong \operatorname{Tor}_1^A(A_0, E) \otimes_B \widehat{B} = 0.$$

We obtain the t-flatness of $\widehat{E}$ by the equivalence of (1) and (3′) in Theorem 32.

On the other hand, suppose $\operatorname{Tor}_1^A(A_0, \widehat{E}) = 0$. Let S be the multiplicatively closed set $1 + IB$, then $\widehat{B}$ is faithfully flat over $S^{-1}B$ and

$$\operatorname{Tor}_1^A(A_0, E) \otimes_B \widehat{B} \cong \operatorname{Tor}_1^A\big(A_0, \widehat{E}\big) = 0.$$

So, $\mathrm{Tor}_1^A(A_0, E) \otimes_B S^{-1}B = 0$. But,

$$\mathrm{Supp}\big(\mathrm{Tor}_1^A(A_0, E)\big) \subseteq \big\{\mathfrak{p} \subset \mathbb{Z}[P]/(K) \mid I \subseteq \mathfrak{p}\big\}.$$

That is, $\mathrm{Tor}_1^A(A_0, E) = 0$. □

Lemma 38. *Let $K \subseteq J$ be proper ideals of P and let E be a $\mathbb{Z}[P]/(K)$-module. If E is weakly t-flat relative to K, then $E/(J)E$ is weakly t-flat relative to J.*

Proof. Continue the notation of Theorem 32. By the equivalence of (3) and (3′) in Theorem 32, it suffices to prove that the canonical surjective homomorphism

$$\big(I/(J)\big) \otimes_{A/(J)} \big(E/(J)E\big) \to IE/(J)E$$

is an isomorphism. Since

$$\big(I/(J)\big) \otimes_A E \cong \big(I/(J)\big) \otimes_{A/(J)} \big(A/(J)\big) \otimes_A E \cong \big(I/(J)\big) \otimes_{A/(J)} \big(E/(J)E\big),$$

it suffices to prove that the canonical map $\varphi : (I/(J)) \otimes_A E \to E/(J)E$ is injective. Take the tensor product of the short exact sequence

$$0 \to I/(J) \to A/(J) \to A_0 \to 0$$

with E over A to see $\ker\varphi \cong \mathrm{Tor}_1^A(A_0, E)$. Since E is weakly t-flat relative to K, $\ker\varphi = 0$. □

Lemma 39. *Let K, K_1 and K_2 be proper ideals of P, let $K_1 \cap K_2 \subseteq K$, and let E be a $\mathbb{Z}[P]/(K)$-module. If $E/(K_1)E$ is weakly t-flat relative to K_1 and $E/(K_2)E$ is weakly t-flat relative to K_2, then E is weakly t-flat relative to K.*

Proof. We continue the notation from Theorem 32. It suffices to prove the multiplication map

$$\big(I^n/I^{n+1}\big) \otimes_{A_0} E_0 \to I^nE/I^{n+1}E$$

is an isomorphism for all n by the equivalence of (3) and (7). For any particular n, we may assume $(n+1)P^+ \subseteq K$ when attempting to prove this map is an isomorphism. From now on, assume $(n+1)P^+ \subseteq K$. By the equivalence of (3) and (3′) it suffices to prove $I \otimes E \to E$ is injective. Take the tensor product of E with the short exact sequence

$$0 \to I \to I/(K_1) \oplus I/(K_2) \to I/(K_1 \cup K_2) \to 0 \tag{2}$$

and consider the commutative diagram

$$\begin{array}{ccccccc} I \otimes E & \xrightarrow{g} & (I/(K_1)) \otimes E \oplus (I/(K_2)) \otimes E & \longrightarrow & (I/(K_1 \cup K_2)) \otimes E & \longrightarrow & 0 \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & & & \\ E & \xrightarrow{j} & E/(K_1)E \oplus E/(K_2)E. & & & & \end{array} \tag{3}$$

We want to prove the f is injective. Since $E/(K_1)E$ is weakly t-flat relative to K_1 and $E/(K_2)E$ is weakly t-flat relative to K_2, h is injective. Since h is injective, it suffices to prove g is injective, then $h \circ g = j \circ f$ is injective and f is injective. We will prove g is injective by induction on n. If $n = 1$, $K = K_1 = K_2 = P^+$ and we are done. Now suppose $E/(J)E$ is weakly t-flat whenever $nP^+ \cup (K_1 \cap K_2) \subseteq J$. In particular, let $J = K \cup nP^+$. Notice that I, $I/(K_1) \oplus I/(K_2)$ and $I/(K_1 \cup K_2)$ are combinatorial $\mathbb{Z}[P]/(J)$-modules since I is annihilated by (J). Furthermore, the short exact sequence (2) above is an exact sequence of $\mathbb{Z}[P]/(J)$-modules. In fact, the top line of our commutative diagram (3) can be obtained by taking the tensor product over $\mathbb{Z}[P]/(J)$ of our short exact sequence (2) above with $E/(J)E$. So,

$$\mathrm{Tor}_1^A\big(I/(K_1 \cup K_2), E\big) = \mathrm{Tor}_1^{\mathbb{Z}[P]/(J)}\big(I/(K_1 \cup K_2), E/(J)E\big).$$

Since $E/(J)E$ is weakly t-flat relative to J and $I/(K_1 \cup K_2)$ is a combinatorial $\mathbb{Z}[P]/(J)$-module, we have $\mathrm{Tor}_1^{\mathbb{Z}[P]/(J)}(I/(K_1 \cup K_2), E/(J)E) = 0$. Therefore, g is an injection. □

Proposition 40. *Let A be a Noetherian $\mathbb{Z}[P]$-algebra and let E be a finitely generated A-module. There exists an ideal $K \subseteq P^+$ such that the module $E/(J)E$ is weakly t-flat relative to an ideal J if and only if $K \subseteq J$.*

Proof. Let $S = \{J \mid E/(J)E$ is weakly t-flat relative to $J\}$, let $K = \bigcap_{J \in S} J$, and let $I = (P^+)/(K)$. By Lemma 38, it suffices to prove $E/(K)E$ is weakly t-flat relative to K, for then $S = \{J \mid K \subseteq J\}$. By the equivalence of (3′) and (7) in Theorem 32, to prove $E/(K)E$ is weakly t-flat relative to K, it suffices to prove $E/I^{n+1}E$ is t-flat relative to $(n+1)P^+ \cup K$ for all $n \in \mathbb{N}$. If $J \in S$, then $E/(I^{n+1} + (J))E$ is t-flat relative to $(n+1)P^+ \cup J$ for all $n \in \mathbb{N}$ by the same equivalence. That is, $(n+1)P^+ \cup J \in S$ for all $n \in \mathbb{N}$. Furthermore, $(n+1)P^+ \cup K = \bigcap_{J \in S}((n+1)P^+ \cup J)$. Since there are only finitely many ideals of P containing $(n+1)P^+$, this intersection is finite and $E/I^{n+1}E$ is t-flat relative to $(n+1)P^+ \cup K$ by Lemma 39. □

2.3. Openness of t-flat loci

Definition 41. [30, Definition 2.4] Let $(X, \mathcal{M})$ be a locally Noetherian coherent log scheme. We say a sheaf of ideals $\mathcal{K} \subseteq \mathcal{M}$ is *coherent* if, locally on X, there exists a fine chart $P_X \to \mathcal{M}$ and an ideal $K \subseteq P$ such that $\mathcal{K}$ is the ideal sheaf generated by the image of K.

Proposition 42. [30, Proposition 2.6] *Let $(X, \mathcal{M})$ be a locally Noetherian fine log scheme, let $\mathcal{K} \subseteq \mathcal{M}$ be a coherent sheaf of ideals, let $\beta : P_X \to \mathcal{M}$ be a fine chart, and let $K = \beta^{-1}(\mathcal{K})$. Then $\mathcal{K}$ is the ideal sheaf generated by the image of K.*

Theorem 43. *Let $(X, \mathcal{M})$ be a locally Noetherian toroidal log scheme, let $\mathcal{K} \subseteq \mathcal{M}$ be a coherent ideal sheaf, and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules annihilated by $(\mathcal{K})$. Then*

$$\{x \in X \mid \mathcal{F} \textit{ is } \mathcal{M}\textit{-flat relative to } \mathcal{K} \textit{ at } x\}$$

is open.

Proof. The question is local. Fix $x \in X$. By Proposition 16, we may assume $X = \operatorname{Spec}(P \xrightarrow{\beta} A)$ where $P = \overline{\mathcal{M}}_x$, A is a Noetherian ring and $x = \mathfrak{p}$. In particular, β is local at x and we have an associated ring homomorphism $\tilde{\beta} : \mathbb{Z}[P] \to A$. By the previous proposition, we may also assume $\mathcal{K}$ is generated by the image of an ideal $K \subseteq P$. Furthermore, we may assume $\mathcal{F}$ is the sheaf associated to some finitely generated A-module E.

By Corollary 30, $\mathcal{F}$ is $\mathcal{M}$-flat relative to $\mathcal{K}$ at $x \in X$ if and only if $\mathcal{F}_x$ is t-flat relative to K. So, $\mathcal{F}$ is not $\mathcal{M}$-flat relative to $\mathcal{K}$ at $x \in X$ if and only if $\tilde{\beta}^{-1}(\mathfrak{p})$ is in the support of $\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}(\mathbb{Z}[P]/(J), E)$ for some ideal $J \subset P$ containing K. Since $\mathbb{Z}[P]/(J)$ is a finitely generated combinatorial $\mathbb{Z}[P]$-module, in order to check

$$\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(J), E\big) = 0 \quad \forall J \subset P \text{ containing } K,$$

by Lemma 33, it suffices to check only the prime ideals $J \subset P$ containing K. Since P is finitely generated, P has only finitely many primes. Therefore,

$$\big\{\operatorname{Tor}_1^{\mathbb{Z}[P]/(K)}\big(\mathbb{Z}[P]/(J), E\big) \mid K \subseteq J \subset P,\ J \text{ prime}\big\}$$

is a finite set of finitely generated modules. So, the union of the supports of these modules is closed. But x is in this union if and only if $\mathcal{F}$ is not $\mathcal{M}$-flat relative to $\mathcal{K}$ at x. So,

$$\{x \in X \mid \mathcal{F} \text{ is } \mathcal{M}\text{-flat relative to } \mathcal{K} \text{ at } x\}$$

is open. □

Proposition 44. *Let P be a finitely generated torsion-free monoid, let A be a Noetherian ring, let $\beta : P \to A$ be a monoid homomorphism with respect to multiplication, let E be a finitely generated A-module, and let $K_E : \operatorname{Spec} A \to \{$ideals of $P\}$ be the function that takes a prime $\mathfrak{p}$ to the ideal $K_E(\mathfrak{p})$ of P such that $E_{\mathfrak{p}}$ is weakly t-flat relative to J if and only if $K_E(\mathfrak{p}) \subseteq J$. Then, the image of K_E is finite.*

Proof. We will prove this by Noetherian induction. It suffices to prove that K_E is constant on some nonempty open subset of $\operatorname{Spec} A$. Let $\{X_i\}_{i=1}^n$ be the set of irreducible components of $\operatorname{Spec} A$, let $X = X_1 \setminus \bigcup_{i=2}^n X_i$ and let η be the generic point of X. Let

K be the ideal of P such that E_η is weakly t-flat relative to J if and only if $K \subseteq J$. Now consider $X_K = \{x \in X \mid K_E(x) = K\}$. Since η is in X_K, X_K is nonempty. For each point x on X, let $U(x) = \{x' \in X \mid E_{x'}$ is weakly t-flat relative to $K_E(x)\}$. Note that $U(x) = \{x' \in X \mid K_E(x') \subseteq K_E(x)\}$. By Theorem 43, $U(x)$ is open for all x. Since each $U(x)$ is open, η is an element of $U(x)$ for all x in X. That is, $K_E(\eta) \subseteq K_E(x)$ for all x in X. So, $X_K = U(\eta)$ and X_K is open. □

Example 45. Let $P = \mathbb{N}^2$, let $A = \mathbb{C}[x, y]$, let $\beta : P \to A$ be given by $(n, m) \mapsto x^n y^m$, and let $E = A/(x - y) \oplus A/(y^3)$. Here

$$K_E(\mathfrak{p}) = \begin{cases} P^+, & \text{if } \mathfrak{p} = (x, y), \\ (0,3) + P, & \text{if } \mathfrak{p} = (x - \alpha, y) \text{ with } \alpha \neq 0 \text{ or } \mathfrak{p} = (y), \\ \emptyset & \text{otherwise.} \end{cases}$$

3. Log regularity

This section generalizes much of Kato [1] by relaxing his condition (S) to condition $(*)$ below. By doing so, we work with toroidal log schemes rather than fine saturated log schemes. Most of Kato's methods go through with only minor modifications.

3.1. Definition of toric singularity

In this section, we will mainly consider log schemes which satisfy the following condition $(*)$.

$(*)$ $(X, \mathcal{M})$ is toroidal and its underlying scheme X is locally Noetherian.

Definition 46. Let $(X, \mathcal{M})$ be a log scheme satisfying condition $(*)$. We say $(X, \mathcal{M})$ is *logarithmically regular* at x, or $(X, \mathcal{M})$ has (at worst) a *toric singularity* at x, if the following two conditions are satisfied.

(i) $\mathcal{O}_{X,x}/(\mathcal{M}_x^+)$ is a regular local ring.
(ii) $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/(\mathcal{M}_x^+)) + \operatorname{rank}(\overline{\mathcal{M}}_x^{\mathrm{gp}})$.

We say $(X, \mathcal{M})$ is *log regular* if $(X, \mathcal{M})$ is log regular at each point $x \in X$.

Lemma 47. [1, Lemma (2.3)] *Let $(X, \mathcal{M})$ be a log scheme satisfying $(*)$ and let $x \in X$. Then*

$$\dim(\mathcal{O}_{X,x}) \leqslant \dim\bigl(\mathcal{O}_{X,x}/\bigl(\mathcal{M}_x^+\bigr)\bigr) + \operatorname{rank}\bigl(\overline{\mathcal{M}}_x^{\mathrm{gp}}\bigr).$$

Proof. See Kato's proof. □

3.2. Completed toric singularities

In this subsection, continue to let $(X, \mathcal{M})$ be a log scheme satisfying $(*)$.

Lemma 48. [1, Lemma (3.5)] *Let R be a ring, let π be a nonzero-divisor of R, let P and Q be affine semigroups, and let $P \to Q$ be an injective homomorphism. Let θ be an element of $R[\![P]\!]$ such that $\theta \equiv \pi \mod (P^+)$. Then $R[\![P]\!]/(\theta) \to R[\![Q]\!]/(\theta)$ is injective.*

Proof. See Kato's proof. □

Lemma 49. [1, Lemma (3.4)] *Let R be a ring, let π be a nonzero-divisor of R such that $R/(\pi)$ is an integral domain, let P be an affine semigroup. Let θ be an element of $R[\![P]\!]$ such that $\theta \equiv \pi \mod (P^+)$. Then $R[\![P]\!]/(\theta)$ is an integral domain.*

Proof. See Kato's proof. □

Theorem 50. [1, Theorem (3.2)] *Let $x \in X$. Assume $\mathcal{O}_{X,x}/(\mathcal{M}_x^+)$ is regular, let $P = \overline{\mathcal{M}}_x$, let φ be a section of $\mathcal{M}_x \to P$ as in Corollary 17, and let $f_1, \ldots, f_d \in \mathcal{O}_{X,x}$ such that $(f_i \mod \mathcal{M}_x^+)_{1 \leqslant i \leqslant d}$ is a regular system of parameters of $\mathcal{O}_{X,x}/(\mathcal{M}_x^+)$.*

(1) *If $\mathcal{O}_{X,x}$ contains a field, let k be a subfield of $\widehat{\mathcal{O}}_{X,x}$ such that $k \cong \widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x$. Then $(X, \mathcal{M})$ is log regular at x if and only if the surjective homomorphism*

$$\psi : k[\![P]\!][\![t_1, \ldots, t_d]\!] \to \widehat{\mathcal{O}}_{X,x}, \quad t_i \mapsto f_i$$

is an isomorphism.

(2) *If $\mathcal{O}_{X,x}$ does not contain a field, let R be a complete discrete valuation ring in which $p = \operatorname{char}(\mathcal{O}_{X,x}/\mathfrak{m}_x)$ is a prime element and fix a homomorphism $R \to \widehat{\mathcal{O}}_{X,x}$ which induces $R/pR \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}/\widehat{\mathfrak{m}}_x$. Then $(X, \mathcal{M})$ is log regular at x if and only if the kernel of the surjective homomorphism*

$$\psi : R[\![P]\!][\![t_1, \ldots, t_d]\!] \to \widehat{\mathcal{O}}_{X,x}, \quad t_i \mapsto f_i$$

is generated by an element θ such that

$$\theta \equiv p \quad \mod (P^+, t_1, \ldots, t_d).$$

Proof. See Kato's proof. □

Corollary 51. [1, Theorem (3.1)]

(1) *$(X, \mathcal{M})$ is log regular at x if and only if there exists a complete regular local ring R, an affine semigroup P, and an isomorphism*

$$R[\![P]\!]/(\theta) \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$$

with $\theta \in R[[P]]$ satisfying the following conditions (i) *and* (ii).

(i) *The constant term of θ belongs to $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$.*

(ii) *The inverse image of $\mathcal{M}$ on* $\operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$ *is induced by the map $P \to \widehat{\mathcal{O}}_{X,x}$.*

(2) *Assume $\mathcal{O}_{X,x}$ contains a field. Then, $(X, \mathcal{M})$ is log regular at x if and only if there exists a field k, an affine semigroup P, and an isomorphism*

$$k[[P]][[t_1, \dots, t_d]] \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$$

for some $d \geqslant 0$ satisfying the condition (ii) *in* (1).

Proposition 52. *$(X, \mathcal{M})$ is log regular at x if and only if $\mathcal{O}_{X,x}/(\mathcal{M}_x^+)$ is a regular local ring and $\mathcal{O}_{X,x}$ is t-flat.*

Proof. Since $\widehat{\mathcal{O}}_{X,x}$ is faithfully flat over $\mathcal{O}_{X,x}$, $\mathcal{O}_{X,x}$ is t-flat if and only if $\widehat{\mathcal{O}}_{X,x}$ is t-flat. By Theorem 50, $(X, \mathcal{M})$ is log regular at x if and only if $\operatorname{Spec}(\widehat{\mathcal{O}}_{X,x})$ is log regular at x. So, we may assume $\mathcal{O}_{X,x} = \widehat{\mathcal{O}}_{X,x}$. If $\mathcal{O}_{X,x}$ is t-flat, then the equivalence of (1) and (4′) in the Theorem 32 shows $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}/(\mathcal{M}_x^+)) + \operatorname{rank}(\overline{\mathcal{M}}_x^{\mathrm{gp}})$. On the other hand, if $(X, \mathcal{M})$ is log regular at x, then Theorem 50 and the equivalence of (1) and (4′) in the Theorem 32 show $\mathcal{O}_{X,x}$ is t-flat. □

Theorem 53. [1, Theorem (6.2)] *Let $P = \overline{\mathcal{M}}_x$ be an affine semigroup and take a section $\varphi : P \to \mathcal{M}_x$ of $\mathcal{M}_x \to P$ as in Corollary* 17. *Assume $\mathcal{O}_{X,x}$ contains a field k. Then $(X, \mathcal{M})$ is log regular at x if and only if $\mathcal{O}_{X,x}/(\mathcal{M}_x^+)$ is a regular local ring and the map $k[P] \to \mathcal{O}_{X,x}$ induced by φ is flat.*

Proof. This follows immediately from Theorems 52, 32, and the local criterion for flatness (see EGA III [31, 0_{III} (10.2.2)]). □

3.3. Some properties of toric singularities

Kato reminds us that if $(X, \mathcal{M})$ is a log regular scheme satisfying his condition (S) then its underlying scheme is Cohen–Macaulay and normal, see Hochster [8]. Let k be a field. If the canonical log structure on $\operatorname{Spec} k[P]$ satisfies $(*)$, it need not be Cohen–Macaulay nor normal. Consider the monoids

$$\big\langle (4,0), (3,1), (1,3), (0,4) \big\rangle \subset \mathbb{N}^2$$

and $\langle 2, 3 \rangle \subset \mathbb{N}$. Information on when affine semigroup rings are Cohen–Macaulay and its dependence on the characteristic, can be found in [32,33]. Information on the local cohomology modules and dualizing complexes of affine semigroup rings is found in [34] and [35].

Theorem 54. [1, Theorem (4.2)] *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be log regular schemes. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism whose underlying map of schemes is a closed immersion and assume $f^*\mathcal{N} \cong \mathcal{M}$. Then the underlying map of schemes of f is a regular immersion.*

Proof. See Kato's proof. □

3.4. Localization

For the duration of this subsection, let $(X, \mathcal{M})$ be a log scheme satisfying condition $(*)$.

Lemma 55. *Let A be a local ring, let P be a* 1*-dimensional affine semigroup, let $\beta : P \to A$ be a monoid homomorphism with respect to multiplication, and let $\mathfrak{p} \subset A$ be a prime ideal such that $\beta^{-1}(\mathfrak{p}) = \emptyset$. If* $\mathrm{Spec}(P \xrightarrow{\beta} A)$ *is log regular, then $A_{\mathfrak{p}}$ is a regular local ring.*

Proof. By Theorem 50, $\widehat{A} = R[[P]]/(\theta)$ for some complete regular local ring R and some θ whose constant term is contained in $\mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Note that $\mathbb{N}$ is the saturation of P and there exists an $n \in \mathbb{N}$ such that $m \in P$ whenever $m \geqslant n$. Let F be the image of P in A and consider the following diagram:

$$\begin{array}{ccccc}
A & \longrightarrow & R[\![P]\!]/(\theta) & \longrightarrow & R[\![\mathbb{N}]\!]/(\theta) \\
\downarrow & & \downarrow & & \downarrow \\
A_F & \longrightarrow & (R[\![P]\!]/(\theta))_F & \xrightarrow{=} & R(\!(\mathbb{N})\!)/(\theta) \\
\downarrow & & \downarrow & & \\
A_{\mathfrak{p}} & \longrightarrow & (A \setminus \mathfrak{p})^{-1} R[\![P]\!]/(\theta) & &
\end{array}$$

$A_{\mathfrak{p}}$ is faithfully flat under $(A \setminus \mathfrak{p})^{-1} R[\![P]\!]/(\theta)$ and the latter is regular since it is a localization of $R(\!(\mathbb{N})\!)/(\theta)$. Hence $A_{\mathfrak{p}}$ is regular by faithfully flat descent, see Matsumura [29, Theorem 23.7]. □

Theorem 56. [1, Proposition (7.2)] *Let $x \in X$ and suppose $(X, \mathcal{M})$ is log regular at x. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{M}_x$ and endow $X' =$* $\mathrm{Spec}(\mathcal{O}_{X,x}/(\mathfrak{p}))$ *with the log structure $\mathcal{M}'$ associated to $\mathcal{M}_x \setminus \mathfrak{p} \to \mathcal{O}_{X,x}/(\mathfrak{p})$. $(X', \mathcal{M}')$ is log regular at $x \in X'$.*

Proof. Follow Kato's induction proof, using Lemma 55 in the one-dimensional case. □

Corollary 57. [1, Corollary (7.3)] *With the notation as in the previous theorem,* $(\mathfrak{p})$ *is a prime ideal of height* $\dim(\mathcal{M}_x)_{\mathfrak{p}}$.

Theorem 58. [1, Proposition (7.1)] *Let $x \in X$, and assume that $(X, \mathcal{M})$ is log regular at x. Then for any $y \in X$ such that $x \in \overline{\{y\}}$, $(X, \mathcal{M})$ is log regular at y.*

Proof. Follow Kato's proof and use Lemma 55 at the last step. □

3.5. Log smooth morphisms

Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be log schemes satisfying $(*)$, and let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism. Then, the following two conditions (i) and (ii) are equivalent. We say f is log smooth if f satisfies these conditions. This equivalence was shown in Kato [2] for log structures on étale sites, and the proof there works for the present situation (log structures on Zariski sites). See Kato [22, Section 3.1.5]. So, we omit the proof here.

(i) Assume we are given a commutative diagram of log schemes

$$\begin{array}{ccc} (T, \mathcal{M}_T) & \xrightarrow{g} & (X, \mathcal{M}_X) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ (T', \mathcal{M}_{T'}) & \xrightarrow{g'} & (Y, \mathcal{M}_Y) \end{array}$$

such that $(T, \mathcal{M}_T)$ and $(T', \mathcal{M}_{T'})$ satisfy $(*)$, $i : T \to T'$ is a closed immersion, T is defined in T' by a nilpotent ideal of $\mathcal{O}_{T'}$ and $i^*\mathcal{M}_{T'} \to \mathcal{M}_T$ is an isomorphism. Then, locally on T' there is a morphism $h : (T', \mathcal{M}_{T'}) \to (X, \mathcal{M}_X)$ such that $h \circ i = g$ and $f \circ h = g'$. Furthermore, the underlying morphism of schemes $X \to Y$ is locally of finite type.

(ii) Étale locally on X and Y, there exist finitely generated, torsion-free monoids P and Q, an injective homomorphism $h : P \to Q$ such that the order of the torsion part of $Q^{\mathrm{gp}}/h^{\mathrm{gp}}(P^{\mathrm{gp}})$ is invertible on X, and a commutative diagram of log schemes of the form

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \longrightarrow & \mathrm{Spec}(Q \hookrightarrow \mathbb{Z}[Q]) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \text{induced by } h} \\ (Y, \mathcal{M}_Y) & \longrightarrow & \mathrm{Spec}(P \hookrightarrow \mathbb{Z}[P]) \end{array}$$

such that the inverse image of P on Y is $\mathcal{M}_Y$, the inverse image of Q on X is $\mathcal{M}_X$ and the induced morphism of the underlying schemes

$$X \to Y \times_{\mathrm{Spec}\,\mathbb{Z}[P]} \mathrm{Spec}\,\mathbb{Z}[Q]$$

is smooth (in the classical sense).

Theorem 59. [1, Theorem (8.2)] *Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a log smooth morphism between log schemes satisfying $(*)$, and assume $(Y, \mathcal{N})$ is log regular. Then $(X, \mathcal{M})$ is log regular.*

Proof. See Kato's proof. □

Theorem 60. [1, Proposition (8.3)] *Let k be a field and let $(X, \mathcal{M})$ be a log scheme satisfying $(*)$ such that the underlying scheme X is a k-scheme which is locally of finite type. Then*:

(1) *If $(X, \mathcal{M})$ is log smooth over* $\operatorname{Spec}(\{0\} \to k)$*, then $(X, \mathcal{M})$ is log regular.*
(2) *The converse of* (1) *is true if k is perfect.*

Proof. See Kato's proof. □

For completeness, we take note of the following proposition of Kato:

Proposition 61. [1, Proposition (12.2)] *Let $(A, \mathfrak{m})$ be a Noetherian local ring, let P be a sharp finitely generated torsion-free monoid, let $\beta : P \to A$ be local, and suppose $(X, \mathcal{M}) = \operatorname{Spec}(P \xrightarrow{\beta} A)$ is log regular. If $B = \mathbb{Z}[P^{\mathrm{sat}}] \otimes_{\mathbb{Z}[P]} A$ and $(Y, \mathcal{N}) =$* $\operatorname{Spec}(P^{\mathrm{sat}} \to B)$*, then*:

(1) *$(Y, \mathcal{N})$ is log regular.*
(2) *B is the normalization of A and it is a local ring.*
(3) *If A is already normal, then P is already saturated.*

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