When a zero-divisor graph is planar or a complete \( r \)-partite graph

S. Akbari, a,c H.R. Maimani, a,b and S. Yassemi a,d,*

\( ^a \) Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran
\( ^b \) Department of Mathematics, Shahid Rajaee University, P.O. Box 16785-163, Tehran, Iran
\( ^c \) Department of Mathematics, Sharif University of Technology, Tehran, Iran
\( ^d \) Department of Mathematics, University of Tehran, Tehran, Iran

Received 15 September 2002
Communicated by Paul Roberts

Abstract

Let \( \Gamma(R) \) be the zero-divisor graph of a commutative ring \( R \). An interesting question was proposed by Anderson, Frazier, Lauve, and Livingston: For which finite commutative rings \( R \) \( \Gamma(R) \) is planar? We give an answer to this question. More precisely, we prove that if \( R \) is a local ring with at least 33 elements, and \( \Gamma(R) \neq \emptyset \), then \( \Gamma(R) \) is not planar. We use the set of the associated primes to find the minimal length of a cycle in \( \Gamma(R) \). Also, we determine the rings whose zero-divisor graphs are complete \( r \)-partite graphs and show that for any ring \( R \) and prime number \( p, p \geq 3 \), if \( \Gamma(R) \) is a finite complete \( p \)-partite graph, then \( |Z(R)| = p^2, |R| = p^3 \), and \( R \) is isomorphic to exactly one of the rings \( \mathbb{Z}_{p^3}, \mathbb{Z}_{p^2}^{(x,y)}, \mathbb{Z}_{p^2}^{(xy,y^2)} \), where \( 1 \leq s < p \).

© 2003 Published by Elsevier Inc.

Keywords: Zero-divisor graph; Girth; Planar graph; Bipartite graph

Introduction

Let \( R \) be a commutative ring (with \( 1 \neq 0 \)) and let \( Z(R) \) be its set of zero-divisors of \( R \). We denote the set of minimal prime and maximal ideals of \( R \) by \( \text{Min}(R) \) and \( \text{Max}(R) \),

* This research was supported in part by a grant from IPM.
* Corresponding author.
E-mail addresses: s_akbari@sina.sharif.ac.ir (S. Akbari), maimani@ipm.ir (H.R. Maimani), yassemi@ipm.ir (S. Yassemi).

0021-8693/$ – see front matter © 2003 Published by Elsevier Inc.
doi:10.1016/S0021-8693(03)00370-3
respectively. Also, \( \text{Ass}(R) \) denotes the set of associated prime ideals of \( R \). By the zero-divisor graph \( \Gamma(R) \) of \( R \) we mean the graph with vertices \( \mathcal{Z}(R) \setminus \{0\} \) such that there is an (undirected) edge between vertices \( a \) and \( b \) if and only if \( a \neq b \) and \( ab = 0 \). Thus \( \Gamma(R) \) is the empty graph if and only if \( R \) is an integral domain. For a graph \( G \), let \( \chi(G) \) denote the chromatic number of the graph \( G \), i.e., the minimal number of colors which can be assigned to the vertices of \( G \) in such a way that every two adjacent vertices have different colors. For a graph \( G \), the degree of a vertex \( v \) in \( G \) is the number of edges of \( G \) incident with \( v \). We denote by \( \delta(G) \) the minimum degree of vertices of \( G \). The girth of \( G \) is the length of a shortest cycle in \( G \) and is denoted by \( \text{gr}(G) \). If \( G \) has no cycles, we define the girth of \( G \) to be infinite. An \( r \)-partite graph is one whose vertex set can be partitioned into \( r \) subsets so that no edge has both ends in any one subset. A complete \( r \)-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes \( m \) and \( n \) is denoted by \( K_{m,n} \). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use \( K_n \) for the complete graph with \( n \) vertices. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski’s Theorem says that a graph is planar if and only if it contains no subdivision of \( K_5 \) or \( K_{3,3} \), cf. [5, p. 153].

For an extended list of references and the history of this topic the reader is referred to [1–4,6,8], and [9]. An interesting question was proposed by Anderson, Frazier, Lauve, and Livingston: For which finite commutative rings \( R \) is \( \Gamma(R) \) planar? Cf. [2, Question 5.3].

In Section 1, the following results are shown:

(a) Suppose that \( (R, m) \) is a finite local ring. Then \( \Gamma(R) \) is not planar if one of the following holds:

(i) \( |R/m| \geq 4 \) and \( |R| \geq 26 \).
(ii) \( |R/m|=3 \) and \( |R| \geq 28 \).
(iii) \( |R/m|=2 \) and \( |R| \geq 33 \).

(b) If \( (R, m) \) is a local Artinian ring and \( \Gamma(R) \) is planar, then \( R \) is quasi-Frobenius, or \( R \cong \mathbb{Z}_4[x]/(2, x)^2 \) or \( R \cong \mathbb{Z}_2[x]/(x, y)^2 \).

Anderson and Livingston showed in [3, Theorem 2.4] that any two vertices in \( \Gamma(R) \) are connected by a path of length (number of edges) less than or equal to three, consequently \( \Gamma(R) \) is connected; and if \( \Gamma(R) \) contains a cycle and \( R \) is Artinian, then \( \Gamma(R) \) contains a cycle of length less than or equal to four. (In [9, 1.4] Mulay has shown that this result holds for any commutative ring; also see [6, Theorem 1.6].) In Section 2, we study the set of the associated primes of the ring to find the minimal length of a cycle in \( \Gamma(R) \). Among other things, the following results are shown:

(a) If \( R \) is a finite ring with at least 10 elements and \( |\text{Ass}(R)|=1 \), then \( \text{gr}(\Gamma(R))=3 \).
(b) If \( R \) is a ring and \( |\text{Ass}(R)| \geq 3 \), then \( \text{gr}(\Gamma(R))=3 \).
(c) If \( \text{Ass}(R) = \{p_1, p_2\}, |p_i| \geq 3 \) for \( i = 1, 2 \), and \( p_1 \cap p_2 = \{0\} \), then \( \text{gr}(\Gamma(R))=4 \).
In Section 3, we prove that if \( R \) is a finite ring and \( \Gamma(R) \) is a complete \( r \)-partite graph \((r \geq 3)\) which is not complete graph, then \( r = p^m \), where \( p \) is a prime. Also, in [3, Theorem 2.10], it was shown that for any finite commutative ring \( R \), if \( \Gamma(R) \) is complete, then either \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( R \) is a local with characteristic \( p \) or \( p^2 \), and \(|\Gamma(R)| = p^2 - 1\), where \( p \) is a prime and \( s \geq 1 \). In addition, we prove that if \( R \) is a ring such that the zero-divisor graph \( \Gamma(R) \) is complete \( r \)-partite, then we have: (1) If \( R \) is Artinian, then \( R \) is finite; (2) If \( R \) is Noetherian, then \( R \) is a subring of a ring \( F \times S \), where \( F \) is a field and \( S \) is a finite ring.

1. When a zero-divisor graph is not planar?

Let \( \Gamma(R) \) be the zero-divisor graph of \( R \) with vertex set \( V(\Gamma(R)) = \{x_1, x_2, \ldots\} \) and edge set \( E(\Gamma(R)) \). A cycle of length \( n \) is an alternating sequence of vertices and edges \( x_0, z_1, x_1, \ldots, z_n, x_n = x_0 \), where \( z_i = \{x_{i-1}, x_i\} \) is an edge with end points \( x_{i-1} \) and \( x_i \) and moreover \( x_j, 0 \leq i \leq n-1, 1 \leq j \leq n, \) are distinct.

In this section, we will give an answer to the question "For which finite commutative rings \( R \) is \( \Gamma(R) \) planar?", cf. [2]. We know that \( R \cong R_1 \times \cdots \times R_n \), where \( R_i \) is local for every \( i \). Consider the following cases:

**Case 1.** \( n \geq 4 \). The vertices of the set \( R_1 \times R_2 \times \{0\} \times \cdots \times \{0\} \) and the vertices of the set \( \{0\} \times \{0\} \times \cdots \times R_{n-1} \times R_n \) are adjacent and so \( K_{3,3} \) is a subgraph of \( \Gamma(R) \). Thus \( \Gamma(R) \) is not planar.

**Case 2.** \( n = 3 \). In this case, if one of the \( R_i \)'s, say \( R_1 \), has at least four elements, then \(|R_1 \times \{0\} \times \{0\}| \geq 4\) and also \(|\{0\} \times R_2 \times R_3| \geq 4\). Therefore \( K_{3,3} \) is a subgraph of \( \Gamma(R) \), and hence \( \Gamma(R) \) is not planar. Now assume that \(|R_i| \leq 3\) for every \( 1 \leq i \leq 3 \); this means \( R_1 \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \). Using [2, Theorem 5.1], if \( R \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \), then \( \Gamma(R) \) is planar, and in the other cases, \( R \) is not planar.

**Case 3.** \( n = 2 \). In this case, if \(|R_1| \) and \(|R_2| \) are not less than four, then it is easy to see that \( K_{3,3} \) is a subgraph of \( \Gamma(R) \), and hence it is not planar. Now assume that \(|R_1| = 2 \) or \( 3 \). This means \( |R_1| \) is isomorphic to \( \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \). Let \( R_1 \cong \mathbb{Z}_2 \). If \(|Z(R_2)| \geq 5\), then \( \Gamma(R_2) \) has \( K_{1,3} \) as a subgraph (notice that \( R_2 \) is a local ring). The ring \( \mathbb{Z}_2 \times \mathbb{F}_4 \) is the only ring that its zero-divisor graph is \( K_{1,3} \), cf. [2, Example 2.1(v)]. Therefore \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4) \) is a subgraph of \( \Gamma(\mathbb{Z}_2 \times R_2) \), and hence \( \Gamma(\mathbb{Z}_2 \times R_2) \) is not planar. Now let \(|Z(R_2)| = 4\). If \( \Gamma(R_2) \) is isomorphic to \( K_{1,2} \), then \( R_2 \) is isomorphic to \( \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \) or \( \mathbb{Z}_4[x]/(2x, x^2 - 2) \), cf. [2, Example 2.1(iii)]. Since \( \mathbb{Z}_2 \times \mathbb{Z}_8 \) is planar, cf. [2, Theorem 5.1], and \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)), \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2 - 2)) \) are isomorphic to \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8) \), we have that \( \Gamma(\mathbb{Z}_2 \times R_2) \) is planar. Let \( \Gamma(R_2) \cong K_3 \). By [2, Example 2.1(iv)] there are exactly four rings whose zero divisor graphs are \( K_3 \). It is easily checked that in all of these rings for any zero-divisor \( a \), \( a^2 = 0 \). Thus \( \Gamma(\mathbb{Z}_2 \times R_2) \) has \( K_{3,3} \) as a subgraph and so by Kuratowski’s Theorem \( \Gamma(\mathbb{Z}_2 \times R_2) \) is not planar. Finally, if \(|Z(R_2)| \leq 3\), it is not hard to see that \( \Gamma(\mathbb{Z}_2 \times R_2) \) is planar.
Let $R_1 \cong \mathbb{Z}_3$. Suppose that $|Z(R_2)| \geq 4$. Then $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_q)$ is a subgraph of $\Gamma(\mathbb{Z}_3 \times R_2)$. Thus $\Gamma(\mathbb{Z}_3 \times R_2)$ is not planar, cf. [2, Theorem 5.1]. If $|Z(R_2)| = 3$, then $\Gamma(R_2) \cong K_{1,1}$ and hence $R_2$ is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_3[1]/(x^2)$ (notice that $R_2$ is a local ring), cf. [2, Example 2.1(ii)]. We know that $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9)$ is planar and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3[1]/(x^2)) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2)$. Thus $\Gamma(\mathbb{Z}_3 \times R_2)$ is planar. If $|Z(R_2)| = 2$ then $R_2$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[1]/(x^2)$, cf. [2, Example 2.1(i)]. In this case, it is easy to see that $\mathbb{Z}_3 \times R_2$ is planar. If $R_2$ is integral domain, then it is a field. Therefore there exist invertible elements $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2) \cong K_{2,n-1}$, where $n = |R_2|$. Thus $\Gamma(\mathbb{Z}_3 \times R_2)$ is planar.

**Case 4.** $n = 1$. This is the hardest part and we study it in the rest of this section.

**Remark 1.1.** It is not hard to see that, if $(R, m)$ is a finite local ring, then there exists a prime integer $p$ and positive integers $t, l, k$ such that $\text{Char}(R) = p^t$, $|m| = p^l$, $|R| = p^k$, and $\text{Ann}(R/m) = p$.

**Theorem 1.2.** Let $(R, m)$ be a finite local ring, $m \neq 0$, and $|R/m| \geq 4$. If $|m| \geq 7$ or $|R| \geq 26$, then $\Gamma(R)$ is not planar.

**Proof.** Since $R$ is Artinian, $m$ is an associated prime ideal of $R$, and hence there exists $0 \neq x \in R$ such that $\text{Ann}(x) = m$. Since $|R/m| \geq 4$, there exists distinct invertible elements $u_1, u_2, u_3$ of $R$ such that $u_1 x$, $u_2 x$, and $u_3 x$ are distinct and $\text{Ann}(u_1 x) = \text{Ann}(u_2 x) = \text{Ann}(u_3 x) = m$. If $|m| \geq 7$, there exist distinct elements $x_1, x_2, x_3$ of $m$ such that they are adjacent to $u_1 x, u_2 x, u_3 x$. Therefore $K_{3,3}$ is a subgraph of $\Gamma(R)$ and so $\Gamma(R)$ is not planar.

If $|m| \leq 6$ and $|R| \geq 26$, then by Remark 1.1, $|m| \leq 5$, and so we have $|R/m| \geq 6$. Therefore there exist invertible elements $u_1, \ldots, u_5$ of $R$ such that $u_1 x, \ldots, u_5 x$ are distinct and $\text{Ann}(u_1 x) = \cdots = \text{Ann}(u_5 x) = m$. Thus $u_1 x, \ldots, u_5 x$ are adjacent in $\Gamma(R)$, and hence $K_5$ is a subgraph of $\Gamma(R)$. This implies that $\Gamma(R)$ is not planar. \(\square\)

**Lemma 1.3.** Let $(R, m)$ be a finite local ring such that $\Gamma(R)$ is planar. Then the following hold:

(i) If $|R/m| = 2$, then $m^4 = 0$.

(ii) If $|R/m| = 3$, then $m^3 = 0$.

**Proof.** (i) If $m = 0$, then the assertion holds. In the other case, there exists an integer $r \geq 1$ such that $|m| = 2^r$. Let $k$ be the minimum integer $k \geq 2$ such that $m^k = 0$. It is clear that for any $r$, $0 \leq t < k$, $|m^{k-t}| \geq 2^t$. We have $m^2 m^{k-2} \subseteq m^k = 0$. If $k \geq 5$, we have $|m^2| \geq 8$ and $|m^{k-2}| \geq 4$, which implies that $\Gamma(R)$ has a subgraph isomorphic to $K_{3,3}$, a contradiction. Thus $k \leq 4$, and so $m^4 = 0$.

(ii) The proof of part (ii) is the same as (i). \(\square\)

**Theorem 1.4.** Let $(R, m)$ be a finite local ring such that $\Gamma(R)$ is a planar graph. Then the following hold:

1. $\Gamma(R)$ is planar.
2. $\Gamma(R)$ is not planar.
3. $\Gamma(R)$ is not isomorphic to $K_{3,3}$. 
4. $\Gamma(R)$ is not isomorphic to $K_{3,4}$.
(i) If $|R/m| = 2$, then $|R| \leq 32$.
(ii) If $|R/m| = 3$, then $|R| \leq 27$.

**Proof.** If $m = 0$, then the assertion holds. Therefore we assume that $m \neq 0$.

(i) By Lemma 1.3, we have $m^2m^2 = 0$. Since $\Gamma(R)$ has no subgraph isomorphic to $K_5$, we have $|m^2| \leq 4$. Therefore, for any $x \in m$, we have $|mx| \leq 4$. Consider the group homomorphism $\theta : m \to mx$ defined as $\theta(a) = ax$. Hence we have $|m|/|\text{Ann}(x) \cap m| = |mx| \leq 4$ and it follows that $|\text{Ann}(x)| \geq n/4$, where $|m| = n$. Since $\Gamma(R)$ is planar, we have $\delta(\Gamma(R)) \leq 5$, cf. [5, p. 144], and this implies that $n \leq 24$, and so $n \leq 16$. Therefore we have $|R| \leq 32$.

(ii) By Lemma 1.3, we have $mm = 0$. If $|m^2| \geq 4$, then $|m| \geq 9$, and so $K_{3,3}$ is a subgraph of $\Gamma(R)$, which is a contradiction. Therefore we have $|m^2| \leq 3$, and by the same argument as above for any $x \in m$, $|\text{Ann}(x)| \geq n/3$, where $|m| = n$. Since $\Gamma(R)$ is planar, we have $\delta(\Gamma(R)) \leq 5$, and this implies that $n \leq 18$, and so $n \leq 9$. Thus we have $|R| \leq 27$. □

**Remark 1.5.** By Theorems 1.2 and 1.4, we conclude that if $(R, m)$ is a finite local ring, $m \neq 0$, and $|R| > 32$, then $\Gamma(R)$ is not planar. It is easily checked that $\Gamma(\mathbb{Z}_{27})$ is planar. Now we pose a question: Is it true that, for any local ring $R$ of cardinality 32, which is not a field, $\Gamma(R)$ is not planar?

**Theorem 1.6.** Let $(R, m)$ be a local Artinian ring such that $\Gamma(R)$ is a planar graph. Then $R$ is quasi-Frobenius, or $R \cong \mathbb{Z}_4[x]/(2, x)^2$ or $R \cong \mathbb{Z}_2[x, y]/(x, y)^2$.

**Proof.** Suppose $R$ is not quasi-Frobenius. Since $R$ is Artinian, there exists an integer $t \geq 2$ such that $m^t = 0$ and $m^{t-1} \neq 0$. Since $\dim R/m \cong \dim R/m^{t-1} = 0$, we have that $\text{Ann}(m) \neq \{0\}$, and hence $\dim m/\text{Ann}(m) \geq 2$, cf. [7, Theorem 221]

If $|R/m| \geq 3$, then $|\text{Ann}(m)| \geq 9$, and hence $\Gamma(R)$ has a subgraph isomorphic to $K_5$. Therefore $\Gamma(R)$ is not planar, which is a contradiction.

If $|R/m| = 2$, then $|\text{Ann}(m)| \geq 4$. If $\text{Ann}(m) \nsubseteq m$, then $|m| \geq 8$. Suppose $a, b, c$ are three nonzero distinct elements of $\text{Ann}(m)$ and $x, y, z \in m \setminus \text{Ann}(m)$. Then $a, b, c$ are adjacent to $x, y, z$, and hence $\Gamma(R)$ has a subgraph isomorphic to $K_{3,3}$. Thus $\Gamma(R)$ is not planar, which is a contradiction. If $\text{Ann}(m) = m$ and $|m| = 4$, then $|R| = 8$, and hence $R \cong \mathbb{Z}_4[x]/(2, x)^2$ or $R \cong \mathbb{Z}_2[x, y]/(x, y)^2$, cf. [2, Example 2.1]. In each case, $\Gamma(R) \cong K_3$ is planar. □

2. Girth of $\Gamma(R)$

In this section, we study the girth of $\Gamma(R)$ in terms of the number of elements of the set of associated prime ideals of $R$.

**Lemma 2.1** (see [4, Lemma 3.6]). Let $p_1 = \text{Ann}(x_1)$ and $p_2 = \text{Ann}(x_2)$ be two distinct elements of $\text{Ass}(R)$. Then we have $x_1x_2 = 0$. 

Proof. We can assume that there exists $r \in p_1 \setminus p_2$. Thus $rx_1 = 0 \in p_2$, and so $x_1 \in p_2$. \hfill $\square$

Corollary 2.2. If $|\text{Ass}(R)| \geq 3$, then $\text{gr}((\Gamma(R))) = 3$.

Proof. Let $p_1 = \text{Ann}(x_1)$, $p_2 = \text{Ann}(x_2)$, and $p_3 = \text{Ann}(x_3)$ belong to $\text{Ass}(R)$. Then $x_1 - x_2 = x_3 - x_1$ is a cycle of length 3. \hfill $\square$

Corollary 2.3. If $|\text{Ass}(R)| \geq 5$, then $\Gamma(R)$ is not planar.

Proof. Since $|\text{Ass}(R)| \geq 5$, then $K_5$ is a subgraph of $\Gamma(R)$, and hence $\Gamma(R)$ is not planar. \hfill $\square$

Theorem 2.4 (cf. [6, Theorem 1.14]). Let $R$ be a reduced ring. Then $\Gamma(R)$ is bipartite if and only if there exist two distinct prime ideals $p_1$ and $p_2$ of $R$ such that $p_1 \cap p_2 = \{0\}$. In addition, if $\Gamma(R)$ is bipartite, then it is a complete bipartite graph.

Proof. First, observe that $p_1, p_2 \in \text{Ass}(R)$ since $p_1 \cap p_2 = \{0\}$. We claim that $Z(R) = p_1 \cup p_2$. It is clear that $p_1 \cup p_2 \subseteq Z(R)$. On the other hand, if $x \in Z(R) \setminus p_1 \cup p_2$, then there exists $0 \neq y \in R$ such that $xy = 0$, and hence $y \in p_1 \cap p_2 = \{0\}$, which is a contradiction.

Set $V_1 = p_1 \setminus \{0\}$ and $V_2 = p_2 \setminus \{0\}$. We claim that $\Gamma(R)$ is bipartite with two parts $V_1$ and $V_2$. It is enough to show that there is no edge between two vertices in $V_1$. If $a, b \in V_1$ and $ab = 0$ then $ab \in p_2$, and hence $a \in p_2$ or $b \in p_2$, which is a contradiction. Therefore $\Gamma(R)$ is bipartite.

Conversely, let $\Gamma(R)$ be bipartite with two parts $V_1$ and $V_2$. We claim that $V_1 \cup \{0\}$ is an ideal of $R$. Let $x \in V_1 \cup \{0\}$ and $r \in R$. Then there exists $t \in V_2$ such that $xt = 0$. If $rx \neq 0$, then $rx \in V_1$ since $\Gamma(R)$ is bipartite.

Now let $x, y \in V_1 \cup \{0\}$. Then there exists $t, s \in V_2$ such that $tx = 0, sy = 0$. Since $\Gamma(R)$ is bipartite and $R$ is reduced, we have $st \neq 0$, and hence $(x - y)st = 0$. Since $st \in V_2$, we have $x - y \in V_1$ and hence, $V_1 \cup \{0\}$ is an ideal. Similarly, $V_2 \cup \{0\}$ is an ideal of $R$.

Now we show that $p_1 = V_1 \cup \{0\}$ is prime. Assume that $a, b \in R$ and $ab \in V_1 \cup \{0\}$. Then there exists $t \in V_2$ such that $abt = 0$. If $bt = 0$, then $b \in V_1 \cup \{0\}$. If $bt \neq 0$, since $bt \in V_2$, we conclude that $a \in V_1 \cup \{0\}$. Similarly, $p_2 = V_2 \cup \{0\}$ is a prime ideal of $R$.

For the last part if, $a \in V_1$ and $b \in V_2$ then $ab \in p_1 \cap p_2$, and hence $ab = 0$. Thus $\Gamma(R)$ is complete. \hfill $\square$

Remark 2.5. It is easy to check that we can replace the condition “reduced” by “$b(\Gamma(R)) > 2$” in the above theorem.

Theorem 2.6. If $\text{Ass}(R) = \{p_1, p_2\}$, $|p_i| \geq 3$ for $i = 1, 2$, and $p_1 \cap p_2 = \{0\}$, then $\text{gr}((\Gamma(R))) = 4$.

Proof. Let $p_i = \text{Ann}(x_i)$, $i = 1, 2$, and $a \in p_1 \setminus \{0, x_2\}$ and $b \in p_2 \setminus \{0, x_1\}$. Since $ab \in p_1 \cap p_2 = \{0\}$, we have $a - x_1 - x_2 - b - a$, and so $\text{gr}((\Gamma(R))) \leq 4$. Now by Theorem 2.4, since $\Gamma(R)$ is bipartite, we conclude that $\text{gr}((\Gamma(R))) = 4$. \hfill $\square$
Remark 2.7. Let $p \in \text{Spec}(R)$ be an arbitrary prime ideal of $R$. Any edge in $\Gamma(R)$ has at least one vertex in $p$. Therefore if $|p| = 2$, then $R \cong \mathbb{Z}_2 \times A$, where $A$ is an integral domain or $Z(R)$ is an annihilator ideal (and hence prime), cf. [3, Theorem 2.5]. Also, if $p \in \text{Spec}(R)$ and $|p| = 2$, then we have $\text{gr}(\Gamma(R)) = \infty$.

Theorem 2.8. If $R$ is a finite ring with at least 10 elements and $|\text{Ass}(R)| = 1$, then $\text{gr}(\Gamma(R)) = 3$.

Proof. It is well-known that $|R| \leq |Z(R)|^2$. Since $|R| \geq 10$, we have $|\Gamma(R)| \geq 3$. If $|\Gamma(R)| = 3$, since $|R| \geq 10$, then by Example 2.1 of [2] we have $\text{gr}(\Gamma(R)) = 3$. Thus we may assume that $|\Gamma(R)| \geq 4$. Set $\text{Ann}(x) = p \in \text{Ass}(R)$. If $\text{gr}(\Gamma(R)) \neq 3$, then $\Gamma(R)$ is a star graph with the center $x$ and hence $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field by [3, Theorem 2.13]. Thus $|\text{Ass}(R)| \neq 1$ which is a contradiction. \[\square\]

Theorem 2.9. Let $\text{Ass}(R) = \{p_1, p_2\}$ with $p_1 \cap p_2 \neq \{0\}$. If $|p_1 \cap p_2| > 3$, then $\text{gr}(\Gamma(R)) = 3$. If $|p_1 \cap p_2| = 2$, then $\text{gr}(\Gamma(R)) = 3$ or $\infty$, unless $R \cong R_1 \times R_2$, where $R_2$ is an integral domain and $(R_1, m)$ is a local ring with $|m| = 2$, and in this case, $\Gamma(R)$ is a complete bipartite graph joined to the center of a star.

Proof. Suppose that $p_i = \text{Ann}(x_i), i = 1, 2$. If there exists an element

$$a \in p_1 \cap p_2 \setminus \{0, x_1, x_2\},$$

then we have the cycle $a - x_1 - x_2 - a$, and so $\text{gr}(\Gamma(R)) = 3$. If $p_1 \cap p_2 = \{0, x_1, x_2\}$, then $-x_1 \in p_1 \cap p_2$, and hence $-x_1 = x_1$ or $-x_1 = x_2$. Since $p_1 \neq p_2$, we have that $-x_1 = x_1$, and hence $2|p_1 \cap p_2|$, a contradiction. Thus $|p_1 \cap p_2| = 2$. Without loss of generality we can assume that $p_1 \cap p_2 = \{0, x_1\}$. Since $x_1 \in p_1$ we have $x_1^2 = 0$.

Now, since $\{0, x_1\}$ is an ideal, for any $r \in p_2$ either $rx_1 = 0$ or $rx_1 = x_1$. If there exists $r \in p_2$ such that $r \neq x_1$ and $rx_1 = 0$, then $r - x_1 - x_2 - r$ is a cycle and hence $\text{gr}(\Gamma(R)) = 3$. Otherwise, for any $r \in p_2$ and $r \neq x_1$, we have $rx_1 = x_1$.

If $|p_1| = 2$ or $|p_2| = 2$, then by Remark 2.7, we have $\text{gr}(\Gamma(R)) = \infty$. Thus there exists an element $a \in p_2 \setminus \{0, x_1\}$. We have $ax_1 = x_1$, and so $a$ is not nilpotent. This implies that $a^2x_1 = ax_1$ and so we have $a^2 - a \in p_1 \cap p_2$. Therefore $a^2 - a = 0$ or $a^2 - a = x_1$. On the other hand $a^3x_1 = ax_1$ and so we find $a^3 - a = 0$ or $a^3 - a = x_1$. If $a^2 - a = a^3 - a$, then $a^3 = a^2$, and hence $(a^2)^2 = a^2$. It follows that the ring has a nontrivial idempotent. If $a^2 = a$, clearly $a$ is a nontrivial idempotent. If $a^3 = a$, then we have $(a^2)^2 = a^2$, and in this case, $R$ has also a nontrivial idempotent. Therefore there are commutative rings $R_1$ and $R_2$ such that $R \cong R_1 \times R_2$. There are $q_1 \in \text{Spec}(R_1)$ and $q_2 \in \text{Spec}(R_2)$ such that $p_1 = q_1 \times R_2$ and $p_2 = R_1 \times q_2$. Now the equation $|p_1 \cap p_2| = 2$ implies that $|q_1| = 2$ and $|q_2| = 1$. We claim that $Z(R) = p_1 \cup p_2$. Assume that $t \in Z(R) \setminus p_1 \cup p_2$. Since $t$ is a zero-divisor, there is a nonzero element $b \in R$ such that $tb = 0$. We have $tb \in p_1$ and so $b \in p_1$. Similarly, $b \in p_2$. Hence $b = x_1$, and so $t \in p_1$, a contradiction. Thus we have $Z(R) = p_1 \cup p_2$. Now it is easily seen that $\Gamma(R)$ is a complete bipartite graph joined to the center of a star. \[\square\]
3. Complete \( r \)-partite zero-divisor graphs

In this section, we study the following question: “For which \( r \) does there exist a ring \( R \) such that \( \Gamma(R) \) is a complete \( r \)-partite graph?”

First note that for any prime number \( p \) and any positive integer \( n \) there exists a finite ring \( R \) whose the zero-divisor graph \( \Gamma(R) \) is a complete \( p^n \)-partite graph. For example, if \( \mathbb{F}_{p^n} \) is a finite field with \( p^n \) elements, then \( R = \mathbb{F}_{p^n}[x, y]/(xy, y^2 - x) \) is the desired ring.

In Theorem 2.4, we studied the case \( r = 2 \). In this section, we assume that \( r \geq 3 \) and \( V_1, \ldots, V_r \) are the \( r \) parts of the complete \( r \)-partite graph \( \Gamma(R) \). We will show that if a zero-divisor graph is a complete \( r \)-partite graph and it is not isomorphic to complete graph \( K_r \), then \( r \) is a prime power.

**Theorem 3.1.** Let \( R \) be a ring. If \( \Gamma(R) \) is a complete \( r \)-partite graph with \( r \geq 3 \), then at most one part has more than one vertex. If \( V_i \) is one of the parts such that \( V_i = \{x\} \), then \( x^2 = 0 \). Further, \( Z(R) \in \text{Max}(R) \cap \text{Ass}(R) \).

**Proof.** Assume that there exist two distinct parts \( V_i \) and \( V_j \) with more than one element. Let \( x \in V_i \) and \( y \in V_j \) be two arbitrary elements. Since \( r \geq 3 \), there exists \( V_l \) not equal to \( V_i \) and \( V_j \). If \( z \in V_l \), then \( \text{Ann}(z) \subseteq \text{Ann}(x) \cup \text{Ann}(y) \), and hence \( \text{Ann}(z) \subseteq \text{Ann}(x) \) or \( \text{Ann}(z) \subseteq \text{Ann}(y) \). Suppose that \( \text{Ann}(z) \subseteq \text{Ann}(x) \). For any \( x' \in V_l \) not equal to \( x \), we have \( x' \in \text{Ann}(z) \setminus \text{Ann}(x) \), which is a contradiction. Therefore there exists at most one part with more than one element.

Without loss of generality, we may assume that \( |V_i| = 1 \) for \( 1 \leq i \leq r - 1 \). Therefore, for any \( i, 1 \leq i \leq r - 1 \), and for any \( x \in V_i \), we have that \( Z(R) \setminus \{x\} \subseteq \text{Ann}(x) \). Now if \( x \notin \text{Ann}(x) \), then consider an element \( y \in V_j \), \( j \neq i \).

Since \( r \geq 3 \), there exists \( z \in Z(R) \) such that \( z \) is adjacent to \( x \) and \( y \), and hence \((x + y)z = 0 \). Therefore \( x \neq x + y \in Z(R) \). Thus \( (x + y)x = 0 \) and this implies that \( x^2 = 0 \), which is a contradiction. Therefore \( x \in \text{Ann}(x) \), which means \( x^2 = 0 \). But this implies that \( \text{Ann}(x) = Z(R) \). Therefore \( Z(R) \in \text{Spec}(R) \). Since \( x^2 = 0 \), we have \( Rx \) is finite, otherwise \( R \) has an infinite clique. But \( \chi(R) \) is finite, which is a contradiction, cf. [4, Theorem 3.7].

On the other hand, \( R/\text{Ann}(x) \cong Rx \) as \( R \)-module. Therefore \( R/\text{Ann}(x) \) is a finite integral domain, and hence it is a field. Thus \( Z(R) \in \text{Max}(R) \). \( \square \)

In the rest of this section, we assume that \( r \geq 3 \) and \( |V_i| = 1 \) for any \( 1 \leq i \leq r - 1 \).

**Theorem 3.2.** Let \( R \) be a finite ring and let \( \Gamma(R) \) be a complete \( r \)-partite graph with \( r \geq 3 \). Then the following statements hold:

(a) \( R \) is a local ring.
(b) If \( |V_i| \geq 2 \), then there exists a prime integer \( p \) and positive integer \( t \) and \( k \) such that \( r = p^t \) and \( |R| = p^k \).
(c) If \( |V_i| \geq 2 \), then for any \( z \in V_r \), we have \( z^2 \neq 0 \) and \( z^3 = 0 \).
(d) For any \( x \in V_r \) and for any \( y \in Z(R) \), \( xy \notin V_r \).
Proof. (a) Since \( r \geq 3 \) and \( \Gamma(R) \) is a complete \( r \)-partite, we have \( R \not\cong \mathbb{Z}_2 \times F \), where \( F \) is a field, and so by [3, Corollary 2.7] \( R \) is a local ring.

(b) By Remark 1.1, there exists a prime integer \( p \) and an integer \( k \geq 1 \) such that \( |R| = p^k \).

In \( |V_r| \geq 2 \), then we can choose two distinct elements \( x, y \in V_r \). Now \( \text{Ann}(x) \cap \text{Ann}(y) = \bigcup_{i=1}^{r-1} V_i \cup \{0\} \) is an ideal with a prime power cardinal and so \( r \) is a prime power.

(c) Choose two different elements \( x, y \in V_r \). If there exists \( z \in V_r \) such that \( z^2 = 0 \), then \( \text{Ann}(z) = \{z\} \cup (\text{Ann}(x) \cap \text{Ann}(y)) \) is an ideal, which is a contradiction. Thus for any \( z \in V_r \), we have \( z^2 \neq 0 \).

Since \( R \) is a finite local ring, we have \( Z(R) \) is a nilpotent ideal. Suppose that \( n \) is a minimal positive integer such that \( z^n = 0 \) and \( n \geq 4 \). We have \( z^{n-1} \notin V_r \). Now if \( z^{n-2} \in V_r \), then \( (z^{n-2})^2 = z^{2n-4} = 0 \) because \( 2n - 4 \geq n \). This is a contradiction. Therefore \( z^{n-2} \notin V_r \).

(d) Let \( x, y \in V_r \) and \( xy \in V_r \). Since \( y^2 = 0 \), we have \( y^2 \notin V_r \), and hence \( (xy)y = xy^2 = 0 \), which is a contradiction. If \( x \in V_r \) and \( y \in Z(R) \setminus V_r \), then \( xy = 0 \notin V_r \).

Lemma 3.3. Let \( R \) be a finite ring, \( r \geq 2 \), and \( \Gamma(R) \) be a complete \( r \)-partite graph. Then \(|Z(R)| \leq r^2\).

Proof. Let \( x \in V_r \). Consider the function \( f : Z(R) \to \text{Ann}(x) \) such that \( f(a) = ax \) for any \( a \in Z(R) \) (note that \( f \) is well-defined by Theorem 3.2(d) above). If \( |V_r| = 1 \), then \(|Z(R)| = r + 1 \leq r^2 \). If \( |V_r| \geq 2 \), then \(|\text{Ann}(x)| = r \) and \( \ker f = \text{Ann}(x) \). Therefore \(|Z(R)| \leq r^2 \).

Theorem 3.4. Let \( R \) be a ring. If there is a prime integer \( p \geq 3 \) such that \( \Gamma(R) \) is a finite complete \( p \)-partite graph, then \(|Z(R)| = p^2 \), \(|R| = p^3 \), and \( R \) is isomorphic to exactly one of the rings \( \mathbb{Z}_{p^2}, \mathbb{Z}_{p}(x, y)/(xy, y^2 - x), \mathbb{Z}_{p}[y]/(py, y^2 - ps) \), where \( 1 \leq s < p \).

Proof. Assume that \(|V_p| \geq 2 \) and \( z_0 \in V_p \). Then by Theorem 3.2, \(|\text{Ann}(z_0)| = p \), and hence \(|Z(R)| = p^2 \) where \( t \geq 2 \). Therefore by Lemma 3.3, we have \(|Z(R)| = p^2 \). Let \( 0 \neq a \in \text{Ann}(z_0) \). We have \( R/Z(R) \cong Ra \subseteq \text{Ann}(z_0) \), and so \(|R/Z(R)| = p \), which means \(|R| = p^3 \). Let \( y_0 \in V_p \). Then \( y_0^2 = x_0 \neq 0 \), \( x_0^2 = 0 \), and \( x_0y_0 = 0 \). Consider the following three cases:

Case 1. \( \text{Char}(R) = p \). Set \( A = \{ax_0 + by_0 \mid a, b \in \mathbb{Z}_p \} \).

We have \( A \subseteq Z(R) \). We claim that \( A = Z(R) \). It is enough to show that \(|A| = p^2 \). Let \( ax_0 + by_0 = a'x_0 + b'y_0 \), where \( a, b, a', b' \in \mathbb{Z}_p \).

If \( b \neq b' \), then \( y_0 = (b - b')^{-1}(a' - a)x_0 \), which is a contradiction. Thus \( b = b' \) and hence \( (a - a')x_0 = 0 \). If \( a \neq a' \), then \( x_0 = 0 \), which is a contradiction. Therefore \(|A| = p^2 \).

With the same method we can show that \( R = \{a + \beta x_0 + \gamma y_0 \mid a, \beta, \gamma \in \mathbb{Z}_p \} \).
Therefore by considering the epimorphism, \( \varphi: \mathbb{Z}_p[x, y] \to R \), where \( \varphi(f(x, y)) = f(x_0, y_0) \), we have

\[
R \cong \frac{\mathbb{Z}_p[x, y]}{(x^2, xy, y^2 - x)} = \frac{\mathbb{Z}_p[x, y]}{(xy, y^2 - x)}.
\]

Case 2. \( \text{Char}(R) = p^2 \). Since \( p^2 = 0 \), we have \( p \in Z(R) \setminus V_p \). With the same proof as in Case 1 and noting that \( py_0 = 0 \), \( Z(R) = \{ \alpha p + \beta y_0 \mid \alpha, \beta \in \mathbb{Z}_{p^2} \} \) and \( \gamma_0^2 \notin V_p \), there exists \( s \in \mathbb{Z}_{p^2} \) such that \( p \nmid s \) and \( \gamma_0^2 = ps \). It is easy to see that

\[
R = \{ \alpha + \beta y_0 \mid \alpha, \beta \in \mathbb{Z}_{p^2} \}.
\]

Therefore we have

\[
R \cong \frac{\mathbb{Z}_p[y]}{(py, y^2 - ps)}.
\]

Case 3. \( \text{Char}(R) = p^3 \). In this case, we have \( R \cong \mathbb{Z}_{p^3} \). \( \square \)

**Theorem 3.5.** Let \( R \) be an infinite ring and let \( \Gamma(R) \) be a complete \( r \)-partite graph with \( r \geq 3 \). Then \( \text{nil}(R) = \bigcup_{i=1}^{r-1} V_i \cup \{0\} \) is a prime ideal and \( |\text{nil}(R)| = r = p^3 \), where \( p \) is a prime integer. In addition, for any \( x \in V_r \), \( Rx \subseteq V_r \cup \{0\} \).

**Proof.** Since \( \Gamma(R) \) is a complete \( r \)-partite graph, we have \( |\text{Ass}(R)| \leq 2 \) by Theorem 3.1 and Corollary 2.2. On the other hand, by Theorem 3.1, \( Z(R) \in \text{Ass}(R) \). It is clear that \( Q = \bigcup_{i=1}^{r-1} V_i \cup \{0\} \) is an ideal. Now if \( Q \) is not prime, then \( \text{Ass}(R) = \{Z(R)\} \). By [4, Theorem 4.3], we know that \( \text{Min}(R) \subseteq \text{Ass}(R) \), and so \( \text{Spec}(R) = \{Z(R)\} \). Therefore \( \text{nil}(R) = Z(R) \).

Since \( \Gamma(R) \) is a complete \( r \)-partite graph, we have \( \chi(\Gamma(R)) = r < \infty \) and by [4, Theorem 3.9] \( \text{nil}(R) = Z(R) \) is finite. But we know that \( |R| \leq |Z(R)|^2 \), and this implies that \( R \) is a finite ring, which is a contradiction. Thus \( Q \) is a prime ideal, and so \( V_r \cap \text{nil}(R) = \emptyset \). Thus \( Q = \text{nil}(R) \) by Theorem 3.1. Now if there exist \( x \in V_r \) and \( r \in R \) such that \( rx \notin V_r \cup \{0\} \), then we have \( rx^2 = 0 \). Since \( x \) is not nilpotent, we have \( x^2 \in V_r \), and so \( r \notin V_r \). Thus \( rx = 0 \), which is a contradiction.

Now we show that \( r \) is a prime power. By Theorem 3.1, \( Z(R) = m \in \text{Max}(R) \). Since \( m \text{nil}(R) = 0 \), we have that \( \text{nil}(R) \) is an \( R/m \)-vector space. For \( x \in V_1 \), we have \( R/m \cong Rx \). Since \( x^2 = 0 \) and \( R \) has no infinite clique, cf. [4, Theorem 3.7], we have that \( R/m \) is a finite field. This implies that there is a prime number \( p \) such that \( |\text{nil}(R)| = r = p^3 \). \( \square \)

**Theorem 3.6.** Let \( R \) be a ring such that the zero-divisor graph \( \Gamma(R) \) is complete \( r \)-partite with \( r \geq 3 \). Then the following hold:

(i) If \( R \) is Artinian, then \( R \) is a finite local ring.
(ii) If \( R \) is Noetherian, then \( R \) is a subring of a ring \( F \times S \), where \( F \) is a field and \( S \) is a finite ring.
Proof. For the part (i), since $R$ is Artinian, we know that $R \cong R_1 \times \cdots \times R_n$, where for each $i$, $R_i$ is local ring. By Theorem 3.1, there exists $x = (x_1, \ldots, x_n) \in R$ such that $Z(R) = \text{Ann}(x) \in \text{Max}(R)$. If $n \geq 2$ since $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in Z(R)$, we conclude that $x_i = 0$, and so $x = 0$, which is a contradiction. Thus $n = 1$, which means $R$ is local. Since $R$ is Artinian, there exist an integer $k \geq 1$ such that $m^k = 0$, where $m = Z(R)$. We know that $m^{k-1}/m^k$ is a finite dimension $R/m$-vector space. Since $R/m$ is finite, we have $m^{k-1}$ is finite. With the same argument $m^{k-2}, \ldots, m$ are finite. Therefore $R$ is finite.

For part (ii), we can assume that $R$ is infinite. Let $0 = \bigcap_{i=1}^n q_i$ be a minimal primary decomposition of the zero ideal. Since $\sqrt{q_i} \in \text{Ass}(R)$ for any $i$, and $\text{Ass}(R) = \{\text{nil}(R), Z(R)\}$, we have $n \leq 2$. But the zero ideal is not primary because for $x \in V_r$ and $y \in V_{r-1}$ we have $xy = 0$ and $y \neq 0$, but $x$ is not nilpotent by Theorem 3.5. Therefore $n = 2$, which means $0 = q_1 \cap q_2$ is a minimal primary decomposition of the zero ideal. Now the assertion follows from [1, Theorem 3.6]. □

Remark 3.7. There are infinite Noetherian rings whose zero-divisor graphs are complete $r$-partite. For example, the ring $R = \mathbb{Z}_p[x_1, \ldots, x_n]/(x_1, \ldots, x_m)(x_1, \ldots, x_n)$, where $m < n$. If $f(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$, such that $f(0, \ldots, 0) = 0$, then $I + f$ is a zero-divisor, where $I = (x_1, \ldots, x_m)(x_1, \ldots, x_n)$. It is clear that if $g(x_1, \ldots, x_m) \in \mathbb{Z}_p[x_1, \ldots, x_m] \setminus I$ with $g(0, \ldots, 0) = 0$, then $\text{Ann}(I + g) = Z(R)$. The cardinality of the set $\{I + g \mid \text{Ann}(I + g) = Z(R)\}$ is $p^m - 1$. The elements of the set $V_r$ are $I + h$, where $h \in \mathbb{Z}_p[x_1, \ldots, x_n]$, $h(0, \ldots, 0) = 0$, and $h$ has at least a monomial as

$$x_1^{i_1+1}x_2^{i_2+2}\cdots x_n^{i_n}$$

such that for any $t$, $i_t \geq 0$ and $\sum i_t \neq 0$.

If $f \in \mathbb{Z}_p[x_n]$ and $f(0) = 0$, then it is clear that $I + f \in V_r$, and hence $V_r$ is infinite. Thus $R$ is an infinite ring and $r = p^m$.

Acknowledgment

The authors wish to express their deepest gratitude to the referee for careful reading of the paper, valuable comments and fruitful suggestions.

References