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Some formulations for the group steiner tree problem

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Abstract

The group Steiner tree problem consists of, given a graph G, a collection $\mathscr{R}$ of subsets of $V(G)$ and a cost $c(e)$ for each edge of G, finding a minimum-cost subtree that connects at least one vertex from each $R \in \mathscr{R}$. It is a generalization of the well-known Steiner tree problem that arises naturally in the design of VLSI chips. In this paper, we study a polyhedron associated with this problem and some extended formulations. We give facet defining inequalities and explore the relationship between the group Steiner tree problem and other combinatorial optimization problems.

Keywords: Combinatorial optimization; Polyhedral combinatorics; Integer programming formulations; Branch-and-cut algorithms

1. Introduction

Perhaps one of the most studied $\mathscr{NP}$-hard problems is the Steiner tree problem (STP), which consists of given a graph $G = (V, E)$, a cost function $c: E \to \mathbb{R}_+$ and a subset Z of V, find a minimum-cost subtree of G spanning the vertices in Z.

The STP is largely used in the routing phase of VLSI design, when one needs to connect several components together by wires. However, after a component's location has been determined on the circuit, the component itself can still be rotated and flipped. These operations will change the location of the component's pin. Therefore, it seems reasonable not to determine the rotations and flippings arbitrarily, but to find the set of rotations and flippings that will give us the minimum-cost network connecting all components together. For that purpose, Reich and Widmayer [8] proposed the *group Steiner tree problem* (GST): given a graph $G = (V, E)$, a cost function $c: E \to \mathbb{R}_+$ and a collection $\mathscr{R}$ of subsets of V, find a minimum-cost subtree of G that spans at least one vertex from each $R \in \mathscr{R}$.

We call the sets in $\mathscr{R}$ *groups*. A vertex that belongs to a group is said to be a *group vertex*. Any other vertex of G is a *non-terminal* or *Steiner vertex*.

Since the STP is a special case of the GSTP, the later is also $\mathscr{NP}$-hard. There are, however, stronger complexity results than the last one concerning the GST. Ihler [7] considered several restrictions of the problem and proved that it is $\mathscr{NP}$-hard even in stars with unit costs or trees with pairwise disjoint groups. Some approximation algorithms have been proposed for the GST. The better one to date is due to Garg et al. [6].

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Rohe and Zachariasen treat in [9] the rectilinear case of the GSTP. They propose reduction methods and describe the implementation of an exact algorithm for the problem. In [4] a simple reduction from the GST to the STP is studied and the efficiency of its use is analyzed.

Salazar [10] investigated the GST from a polyhedral point of view. He presented an integer programming formulation for the problem and used lifting techniques that allows facets of the spanning tree polytope to be used in the group Steiner tree polytope. In [12] another integer programming model is proposed and lagrangean relaxation is used to provide a lower bound.

In this paper, we investigate the dominant of a polytope associated with the GST. We present results on facet inducing inequalities and provide extended formulations for the problem.

2. The polyhedron and its facets

In this section we describe the polyhedron we study, we give a first integer programming formulation for it and we investigate some of its facets. From now on, we follow the notation of [11].

2.1. The polyhedron and a first formulation

Let $G=(V,E)$ be a graph and $\mathscr{R}$ be a group collection, i.e., a collection of subsets of V. We assume henceforth that no vertex is in the intersection of all groups. An *$\mathscr{R}$-tree* of G is a subtree of G containing at least one vertex from each group. Consider the polytope

$$P(G,\mathscr{R}) := \text{conv.hull}\{\chi^{E(T)}\colon T \text{ an } \mathscr{R}\text{-tree of } G\}. \tag{1}$$

In this paper, we study the dominant of $P(G,\mathscr{R})$. More precisely, the object of our study is the polyhedron

$$P^{\uparrow}(G,\mathscr{R}) := P(G,\mathscr{R}) + \mathbb{R}^E_+. \tag{2}$$

Where there is no risk of ambiguity, we may write P and $P^{\uparrow}$ instead of $P(G,\mathscr{R})$ and $P^{\uparrow}(G,\mathscr{R})$, respectively.

We say that a partition $\mathscr{S}=(V_1,\ldots,V_k)$ of V is an *$\mathscr{R}$-partition* if for every $i=1,\ldots,k$ there is a group $R_i\in\mathscr{R}$ such that $R_i\cap V_i=\emptyset$. If $\mathscr{S}$ is an $\mathscr{R}$-partition, we denote by $\Delta(\mathscr{S})$ the set of edges of G with endpoints in different parts of $\mathscr{S}$. Obviously, if $\mathscr{S}$ is an $\mathscr{R}$-partition then the inequality

$$x(\Delta(\mathscr{S}))\geqslant 1 \tag{3}$$

is valid for $P^{\uparrow}$, where $x(\Delta(\mathscr{S}))=\sum_{e\in\Delta(\mathscr{S})}x(e)$, as usual. We call such inequalities *$\mathscr{R}$-partition inequalities*. Let $P_1(G,\mathscr{R})$ denote the set of all vectors $x\in\mathbb{R}^E_+$ satisfying (3) for each $\mathscr{R}$-partition $\mathscr{S}$.

Proposition 1. *We have that $P^{\uparrow}=(P_1)_I$, where $(P_1)_I$ is the integer hull of P_1.*

Proof. Let $x\in P_1$. We claim that there is an $\mathscr{R}$-tree T such that $E(T)\subseteq \operatorname{supp} x$. In fact, let $V_1,\ldots,V_k$ be the vertex sets of the components of $(V,\operatorname{supp} x)$ and suppose no V_i, $i=1,\ldots,k$, contains at least one vertex from each group. Then $\mathscr{S}=(V_1,\ldots,V_k)$ is an $\mathscr{R}$-partition and $x(\Delta(\mathscr{S}))=0$, a contradiction. Hence, the claim is proved. The result follows directly from this claim. □

One could also notice that the collection $\mathscr{T}$ of all minimal $\mathscr{R}$-trees forms a clutter and that the collection of all minimal $\mathscr{R}$-partitions forms the blocking clutter of $\mathscr{T}$. Proposition 1 would then follow from this observation. Moreover, this shows that if we remove from our formulation any inequality corresponding to a minimal $\mathscr{R}$-partition then we do not have anymore an integer programming formulation for $P^{\uparrow}$.

We now consider the separation problem for the $\mathscr{R}$-partition inequalities. If there is a unitary group or if there are at most two groups, then the separation problem can be solved by a sequence of minimum-cut computations. If the graph is a tree, then the separation problem can be solved by a simple dynamic programming algorithm. In general, however, the separation problem for the $\mathscr{R}$-partition inequalities is $\mathscr{N}\mathscr{P}$-hard, as we will show now.

To show this, we consider the decision version of the separation problem:

$$\text{Given a graph } G=(V,E), \text{ a group collection } \mathscr{R} \text{ and a vector } x \in \mathbb{R}_+^E, \text{ is there an } \mathscr{R}\text{-partition } \mathscr{S} \text{ such that } x(\Delta(\mathscr{S}))<1? \tag{4}$$

The problem we shall reduce to problem (4) is the decision version of the *minimum multicut problem*:

$$\text{Given a graph } G=(V,E), \text{ a cost function } c: E \to \mathbb{R}_+, \text{ a subset } Z \text{ of } V \text{ and a number } B>0, \text{ is there a set } E' \subseteq E \text{ such that no component of } G-E' \text{ contains more than one vertex of } Z \text{ and } c(E')<B? \tag{5}$$

Dahlhaus et al. [3] have shown that problem (5) is $\mathscr{NP}$-complete even when $|Z|=3$. We have the following:

Theorem 2. *Problem* (4) *is* $\mathscr{NP}$*-complete.*

Proof. Let G, c, Z and B be as in (5). Consider the group collection $\mathscr{R} := \binom{Z}{|Z|-1}$ and the vector $x \in \mathbb{R}_+^E$, $x := B^{-1}c$. We claim that there is a set E' such that no component of $G-E'$ contains more than one vertex of Z and $c(E')<B$ if and only if there is an $\mathscr{R}$-partition $\mathscr{S}$ such that $x(\Delta(\mathscr{S}))<1$.

In fact, suppose there is such a set E'. Let $V_1, \ldots, V_k$ be the vertex sets of the components of $G-E'$. Notice that for each V_i, $i=1,\ldots,k$, there is a group R_i such that $V_i \cap R_i = \emptyset$. Therefore, $\mathscr{S} := (V_1,\ldots,V_k)$ is an $\mathscr{R}$-partition. Now $x(\Delta(\mathscr{S})) \leqslant x(E') = B^{-1}c(E') < 1$.

Suppose now that there is an $\mathscr{R}$-partition $\mathscr{S}$ such that $x(\Delta(\mathscr{S}))<1$ and let $E' := \Delta(\mathscr{S})$. By the construction of our group collection, and since $\mathscr{S}$ is an $\mathscr{R}$-partition, no component of $G-E'$ may contain more than one vertex from Z. Moreover, $c(E') = Bx(E') = Bx(\Delta(\mathscr{S})) < B$, proving the claim.

Now, since all computations can be carried out in polynomial time, the theorem follows. □

Notice that, in our reduction, the size of the group collection is exactly $|Z|$ and each group has exactly $|Z|-1$ vertices. Notice moreover that the groups have intersections. A simple technique can be used to remove such intersections. It is similar to the technique described in [6] to remove group intersections in the GST; we shall not describe it here. Since problem (5) is $\mathscr{NP}$-complete even when $|Z|=3$, it follows that:

Corollary 3. *Problem* (4) *is* $\mathscr{NP}$*-complete even when there are exactly three pairwise disjoint groups with exactly two vertices each.*

It then follows that the separation problem for the $\mathscr{R}$-partition inequalities is $\mathscr{NP}$-hard even under such assumptions.

2.2. Facets of $P^\uparrow$

In this section we investigate some facet inducing inequalities for $P^\uparrow$. Again, we have a graph $G=(V,E)$ and a group collection $\mathscr{R}$. We assume that no vertex belongs to every group. Notice that, since $P^\uparrow$ is of blocking type, it is fully dimensional.

We begin with the $\mathscr{R}$-partition inequalities. We say that an $\mathscr{R}$-partition $\mathscr{S}$ is *minimal* if there is no $\mathscr{R}$-partition $\mathscr{S}'$ such that $\Delta(\mathscr{S}')$ is a proper subset of $\Delta(\mathscr{S})$. We say that a set $V' \subseteq V$ *covers* $\mathscr{R}$ if V' contains at least one vertex from each group in $\mathscr{R}$. A subgraph H of G *covers* $\mathscr{R}$ if $V(H)$ covers $\mathscr{R}$. It is easy to see that an $\mathscr{R}$-partition $\mathscr{S}=(V_1,\ldots,V_k)$ is minimal if and only if every edge $uv \in \Delta(\mathscr{S})$ with $u \in V_i$ and $v \in V_j$ connects components of $G[V_i]$ and $G[V_j]$ which together cover $\mathscr{R}$, where $G[V_i]$ denotes the subgraph of G induced by V_i.

Theorem 4. *Let* $\mathscr{S}=(V_1,\ldots,V_k)$ *be an* $\mathscr{R}$*-partition. Inequality*

$$x(\Delta(\mathscr{S})) \geqslant 1 \tag{6}$$

induces a facet of $P^\uparrow$ *if and only if* $\mathscr{S}$ *is minimal.*

Proof. We first show necessity. If $\mathscr{S}$ is not minimal, then there is an $\mathscr{R}$-partition $\mathscr{S}'$ such that $\Delta(\mathscr{S}')$ is a proper subset of $\Delta(\mathscr{S})$. But then (6) is a sum of the non-trivial valid inequalities $x(\Delta(\mathscr{S}')) \geqslant 1$ and $x(\Delta(\mathscr{S}) \backslash \Delta(\mathscr{S}')) \geqslant 0$, and hence not a facet inducing inequality.

To see sufficiency, let $a^{\top}x \geqslant \beta$ be a facet inducing inequality of $P^{\uparrow}$ such that

$$\{x \in P^{\uparrow}: x(\Delta(\mathscr{S})) = 1\} \subseteq \{x \in P^{\uparrow}: a^{\top}x = \beta\}. \tag{7}$$

We claim that $a^{\top}x \geqslant \beta$ is a non-negative multiple of (6). To see that, let $F := E(V_1) \cup \cdots \cup E(V_k)$. Let $f \in \Delta(\mathscr{S})$ and $x_0 := \chi^{F \cup \{f\}}$. Since $\mathscr{S}$ is minimal, $x_0 \in P^{\uparrow}$. Since $P^{\uparrow}$ is of blocking type, for any edge $e \notin \Delta(\mathscr{S})$ we have that $x_0 + \chi^e \in P^{\uparrow}$. But since both x_0 and $x_0 + \chi^e$ satisfy (6) with equality, we have that $a^{\top}x_0 = \beta = a^{\top}(x_0 + \chi^e)$. It follows that $a(e) = 0$ for any $e \notin \Delta(\mathscr{S})$.

Now let $e, f \in \Delta(\mathscr{S})$, $x_1 := \chi^{F \cup \{e\}}$ and $x_2 := \chi^{F \cup \{f\}}$. Both x_1 and x_2 are in $P^{\uparrow}$ and satisfy (6) with equality, hence $a^{\top}x_1 = \beta = a^{\top}x_2$. It follows that $a(e) = a(f)$. Since e and f are arbitrary, there is a number α such that $a(e) = \alpha$ for every $e \in \Delta(\mathscr{S})$, and the claim is proved. Since $P^{\uparrow}$ is fully dimensional, (6) induces a facet of $P^{\uparrow}$. □

The following result will allow us to derive facet inducing inequalities for $P^{\uparrow}$ from facet inducing inequalities for the Steiner tree polyhedron. It is analogous to a result stated in Chopra and Rao [2]. To introduce it, consider an edge $e = uv$ of G. We denote by G/e the graph obtained from G by contracting edge e. After the contraction we do not remove neither loops nor parallel edges. Suppose w is the vertex of G/e resulting from the contraction of e and let $R \in \mathscr{R}$. Then

$$R/e := \begin{cases} R & \text{if } u, v \notin R, \\ (R \backslash \{u, v\}) \cup \{w\} & \text{otherwise.} \end{cases} \tag{8}$$

Now let $\mathscr{R}/e := \{R/e: R \in \mathscr{R}\}$. We have the following result:

Theorem 5. *Let $G = (V, E)$ be a graph and $\mathscr{R}$ a collection of groups. Let e be an edge of G and suppose inequality*

$$a^{\top}x \geqslant \beta \tag{9}$$

induces a facet of $P^{\uparrow}(G/e, \mathscr{R}/e)$. Let $\bar{a} \in \mathbb{R}^E$ be such that $\bar{a}(f) := a(f)$ if $f \in E(G/e)$ and $\bar{a}(e) := 0$. Then $\bar{a}^{\top}x \geqslant \beta$ induces a facet of $P^{\uparrow}(G, \mathscr{R})$.

Proof. First let $x_0 \in P^{\uparrow}(G, \mathscr{R})$ be the incidence vector of some $\mathscr{R}$-tree and let $\bar{x}_0$ be the restriction of x_0 to the components in $E(G/e)$. It is easy to see that $\bar{x}_0 \in P^{\uparrow}(G/e, \mathscr{R}/e)$ and hence $a^{\top}\bar{x}_0 \geqslant \beta$. But then it follows that $\bar{a}^{\top}x_0 = a^{\top}\bar{x}_0 \geqslant \beta$ and $\bar{a}^{\top}x \geqslant \beta$ is valid for $P^{\uparrow}(G, \mathscr{R})$.

Now let $t := \dim P^{\uparrow}(G/e, \mathscr{R}/e)$. Then $\dim P^{\uparrow}(G, \mathscr{R}) = t + 1$. Since (9) induces a facet of $P^{\uparrow}(G/e, \mathscr{R}/e)$, there are affinely independent vectors $x_1, \ldots, x_t \in P^{\uparrow}(G/e, \mathscr{R}/e)$ satisfying (9) with equality. For $i = 1, \ldots, t$, let $\bar{x}_i \in \mathbb{R}^E_+$ be such that $\bar{x}_i(f) := x_i(f)$ if $f \in E(G/e)$ and $\bar{x}_i(e) := 1$. It is easy to see that $\bar{x}_1, \ldots, \bar{x}_t \in P^{\uparrow}(G, \mathscr{R})$ and that they are affinely independent. Let $\bar{x}_{t+1} := \bar{x}_1 + \chi^e$. Since $P^{\uparrow}(G, \mathscr{R})$ is of blocking type, $\bar{x}_{t+1} \in P^{\uparrow}(G, \mathscr{R})$. But then $\bar{x}_1, \ldots, \bar{x}_{t+1}$are $t + 1$ affinely independent vectors in $P^{\uparrow}(G, \mathscr{R})$ satisfying $\bar{a}^{\top}x \geqslant \beta$ with equality, and the theorem follows. □

Suppose now we have a graph $G = (V, E)$ and a collection $\mathscr{R}$ of groups. Suppose also that by contracting a certain set E' of edges of G, every group in $\mathscr{R}/E'$ is unitary. Notice that, in this case, $P^{\uparrow}(G/E', \mathscr{R}/E')$ is the dominant of the Steiner tree polytope for graph G and the terminal set composed of all vertices belonging to groups in $\mathscr{R}/E'$. By Theorem 5, any facet inducing inequality of $P^{\uparrow}(G/E', \mathscr{R}/E')$ is also a facet inducing inequality of $P^{\uparrow}(G, \mathscr{R})$. In this way, any facet inducing inequality for the Steiner tree polyhedron can also be used for the group Steiner tree polyhedron.

Another interesting application of Theorem 5 explores the relation between the minimum set-covering problem and the GST. Consider a finite set A and a collection $\mathscr{J}$ of subsets of A. We say that $\mathscr{J}' \subseteq \mathscr{J}$ *covers* A if every element of A belongs to some set in $\mathscr{J}'$. The *minimum set-covering problem* is as follows: given a set A, a collection $\mathscr{J}$ of subsets of A and a cost function $c: \mathscr{J} \to \mathbb{R}_+$, find a minimum-cost set $\mathscr{J}' \subseteq \mathscr{J}$ that covers A.

Given a set A and a collection $\mathscr{J}$ of subsets of A, let

$$C^{\uparrow}(A, \mathscr{J}) := \text{conv.hull}\{\chi^{\mathscr{J}'}: \mathscr{J}' \subseteq \mathscr{J} \text{ and } \mathscr{J}' \text{ covers } A\} + \mathbb{R}^{\mathscr{J}}_+, \tag{10}$$

that is, $C^{\uparrow}(A, \mathscr{J})$ is the dominant of the set-covering polytope. Now consider the graph T and the collection $\mathscr{U}$ of groups such that

(i) T has a vertex r and a vertex v_J for each $J \in \mathscr{J}$;
(ii) for each $J \in \mathscr{J}$, T has an edge rv_J;
(iii) for each $a \in A$ there is a group $U_a \in \mathscr{U}$ given by $U_a := \{v_J \in V(T) : a \in J\}$.

Notice that there is an obvious correspondence between $\mathscr{U}$-trees and covers of A. In fact, there is an obvious relation between facets of $P^{\uparrow}(T, \mathscr{U})$ and facets of $C^{\uparrow}(A, \mathscr{J})$. This relation can be used together with Theorem 5 to find facet inducing inequalities for $P^{\uparrow}(G, \mathscr{R})$ when G is a tree.

In fact, suppose G is a tree. Let r be any vertex of G that is a non-terminal. We can contract some edges of G in such a way as to obtain a star with root r. Such a star, together with the contracted group collection, corresponds to a set-covering problem. But then, any facet inducing inequality for the polyhedron of this set-covering problem is also a facet inducing inequality for $P^{\uparrow}(G, \mathscr{R})$.

There is a wealth of information on facet inducing inequalities for the set-covering polyhedron, see [1] for a quick survey and references.

3. Extended formulations

Polyhedron P_1 provides an integer programming formulation for $P^{\uparrow}$, though we know that optimizing over P_1 is $\mathscr{NP}$-hard. In this section we present some extended formulations for $P^{\uparrow}$ over which we can optimize in polynomial time and we analyze their relative quality with respect to P_1.

Again, let $G = (V, E)$ be a graph and $\mathscr{R}$ a collection of groups. Fix a group $R_0 \in \mathscr{R}$, which will be called the *root group*. Let $P_2(G, \mathscr{R})$ denote the set of all vectors $(x, y) \in \mathbb{R}_+^E \times \mathbb{R}_+^{R_0}$ satisfying

$$\begin{aligned} & x(\delta(S)) \geqslant y(S \cap R_0) \quad \text{for all } S \subseteq V \text{ not covering } \mathscr{R}, \\ & y(R_0) \geqslant 1. \end{aligned} \tag{11}$$

Let P_2^E denote the projection of P_2 into $\mathbb{R}^E$.

Proposition 6. *We have that $P^{\uparrow} = (P_2^E)_I$.*

Unlike the separation problem for P_1, the separation problem for P_2 can be solved in polynomial time. To see that, suppose we are given a vector $(x_0, y_0) \in \mathbb{R}_+^E \times \mathbb{R}_+^{R_0}$ such that $y_0(R_0) \geqslant 1$. Fix a group $R \in \mathscr{R}$, $R \neq R_0$. Construct a directed graph $D = (V', A)$ in the following way:

(i) $V' := V \cup \{s\}$;
(ii) D has, for each edge $e = uv$ of G, arcs uv and vu;
(iiii) D has an arc rs for each $r \in R$.

Consider now a capacity function $w: A \to \mathbb{R}_+$ such that $w(uv) := x_0(uv)$, $w(vu) := x_0(uv)$ for each edge uv of G and $w(rs) := \infty$ for each $r \in R$. Consider also a demand function $b: V' \to \mathbb{R}$ such that $b(r) := -y_0(r)$ for each $r \in R_0$, $b(s) := y_0(R_0)$ and $b(v) := 0$ for all other $v \in V'$.

Theorem 7. *There is a set $S \subseteq V$ such that $R \cap S = \emptyset$ and $x_0(\delta(S)) < y_0(S \cap R_0)$ if and only if there is no feasible flow on D with capacities given by w and demands given by b.*

Proof. Suppose first there is such a set S. Then $w(\delta^{\text{out}}(S)) = x_0(\delta(S)) < y_0(S \cap R_0) = -b(S)$, hence it follows from Gale's characterization of feasible flows on networks [5] that there is no feasible flow on D.

Assume now there is no feasible flow on D. But then there must be a set $S \subseteq V'$ such that $w(\delta^{\text{out}}(S)) < -b(S)$. Since $b(s) = y_0(R_0)$, we cannot have $s \in S$. But then, since all arcs leaving vertices of R have infinite capacity, we must have $S \cap R = \emptyset$. It follows that $x_0(\delta(S)) = w(\delta^{\text{out}}(S)) < -b(S) = y_0(S \cap R_0)$, and we are done. □

Since we can decide whether there is a feasible flow in a network in polynomial time and, if there is no such flow, we can find a set S with the above properties also in polynomial time, we can solve the separation problem for inequalities $x(\delta(S)) \geqslant y(S \cap R_0)$ such that $S \cap R = \emptyset$ in polynomial time. Therefore, one can solve the separation problem for P_2 by solving $|\mathscr{R}| - 1$ flow feasibility problems.

It is not in general true that $P_2^E \subseteq P_1$. However, as the following result shows, the optimal value of P_2 approximates the optimal value of P_1.

Proposition 8. *Let* $c: E \to \mathbb{R}_+$. *Then* $\min\{c^\top x : x \in P_2^E\} \geqslant (1/2)\min\{c^\top x : x \in P_1\}$.

Proof. Let $x_0 \in P_2^E$ and let y_0 be such that $(x_0, y_0) \in P_2$. Consider an $\mathscr{R}$-partition $\mathscr{S} = (V_1, \ldots, V_k)$. Since $\mathscr{S}$ is an $\mathscr{R}$-partition, for each $i = 1, \ldots, k$ there is an $R_i \in \mathscr{R}$ such that $R_i \cap V_i = \emptyset$. But then, since $(x_0, y_0) \in P_2$, $x_0(\delta(V_i)) \geqslant y_0(V_i \cap R_0)$ for each $i = 1, \ldots, k$. Hence

$$\begin{aligned} y(R_0) &= \sum_{i=1}^{k} y_0(V_i \cap R_0) \\ &\leqslant \sum_{i=1}^{k} x_0(\delta(V_i)) \\ &= 2x_0(\Delta(\mathscr{S})). \end{aligned} \tag{12}$$

Now, since $y_0(R_0) \geqslant 1$, $x_0(\Delta(\mathscr{S})) \geqslant \frac{1}{2}$. It follows that, for any $\mathscr{R}$-partition $\mathscr{S}$, $x_0(\Delta(\mathscr{S})) \geqslant \frac{1}{2}$, and the result follows immediately. $\square$

We now turn ourselves to the problem of finding a system of inequalities defining P_2^E. To this end, denote by $\mathscr{P}(R_0)$ the collection of all subsets of R_0 and let $C'(R_0)$ be the polyhedron consisting of all vectors $z \in \mathbb{R}_+^{\mathscr{P}(R_0)}$ satisfying

$$\sum_{X \ni r} z(X) \geqslant 1 \tag{13}$$

for each $r \in R_0$.

Each vector in $C'(R_0)$ represents a fractional cover of R_0. Let $\mathscr{C}(R_0)$ be the set of vertices of $C'(R_0)$.

We say that a function $\mathscr{F}: \mathscr{P}(R_0) \to \mathscr{P}(V)$ is R_0-*complete* if $X \subseteq \mathscr{F}(X)$ and $\mathscr{F}(X)$ does not cover $\mathscr{R}$ for each $X \subseteq R_0$. We have the following result:

Theorem 9. *Polyhedron* P_2^E *is the set of all vectors* $x \in \mathbb{R}_+^E$ *satisfying*

$$\sum_{X \subseteq R_0} z(X)x(\delta(\mathscr{F}(X))) \geqslant 1 \tag{14}$$

for each $z \in \mathscr{C}(R_0)$ *and each* R_0*-complete function* $\mathscr{F}$.

Proof. First, let $x_0 \in P_2^E$ and $y_0 \in \mathbb{R}_+^{R_0}$ be such that $(x_0, y_0) \in P_2$. Let $z \in \mathscr{C}(R_0)$ and let $\mathscr{F}$ be an R_0-complete function. We have that

$$\begin{aligned} \sum_{X \subseteq R_0} z(X)x_0(\delta(\mathscr{F}(X))) &\geqslant \sum_{X \subseteq R_0} z(X)y_0(X) \\ &= \sum_{r \in R_0} \sum_{X \ni r} z(X)y_0(r) \\ &\geqslant y_0(R_0) \\ &\geqslant 1, \end{aligned} \tag{15}$$

and x_0 satisfies all inequalities (14).

Now suppose $x_0 \in \mathbb{R}_+^E$ satisfies each inequality (14). For each $X \subseteq R_0$, define

$$\mu(X) := \min\{x_0(\delta(S)) \colon X \subseteq S \text{ and } S \text{ does not cover } \mathscr{R}\}. \tag{16}$$

Consider the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & y(R_0) \\ \text{subject to} & y(X) \leqslant \mu(X) \quad \text{for each } X \subseteq R_0, \\ & y \geqslant 0. \end{array} \tag{17}$$

Of course, since $\mu(X) \geqslant 0$ for all $X \subseteq R_0$, problem (17) is feasible. We claim that there is a solution y_0 of (17) such that $y_0(R_0) \geqslant 1$. In fact, suppose there is no such solution. Then, the optimal value of (17) is less than 1. Consider the dual of (17):

$$\begin{array}{ll} \text{minimize} & \sum_{X \subseteq R_0} z(X)\mu(X) \\ \text{subject to} & \sum_{X \ni r} z(X) \geqslant 1 \quad \text{for each } r \in R_0, \\ & z \geqslant 0. \end{array} \tag{18}$$

Let z^* be a basic optimal solution to (18). Notice that $z^* \in \mathscr{C}(R_0)$. For each $X \subseteq R_0$, let $S_X \subseteq V$ be a set attaining the minimum in (16) and consider the function $\mathscr{F} \colon \mathscr{P}(R_0) \to \mathscr{P}(V)$ defined as $\mathscr{F}(X) := S_X$. Notice $\mathscr{F}$ is R_0-complete. But then

$$\begin{aligned} \sum_{X \subseteq R_0} z^*(X) x_0(\delta(\mathscr{F}(X))) &= \sum_{X \subseteq R_0} z^*(X) x_0(\delta(S_X)) \\ &= \sum_{X \subseteq R_0} z^*(X)\mu(X) \\ &< 1, \end{aligned} \tag{19}$$

a contradiction since x_0 satisfies all inequalities (14). Hence, the claim is proved.

Now let y_0 be a solution to (17) such that $y_0(R_0) \geqslant 1$. It is easy to verify that $(x_0, y_0) \in P_2$, hence $x_0 \in P_2^E$. $\square$

The next formulation we describe is stronger than both P_1 and P_2. However, it may require many more variables, more exactly, it require for each edge of our graph $|R_0| + 1$ variables and one variable for each vertex of $|R_0|$. In other words, let P_3 be the set of vectors $(x, f, y) \in \mathbb{R}_+^E \times \mathbb{R}_+^{R_0 \times E} \times \mathbb{R}_+^{R_0}$ satisfying

$$\begin{aligned} & f_r(\delta(S)) \geqslant y(r) \quad \text{for each } S \subseteq V \text{ not covering } \mathscr{R},\ r \in S,\ r \in R_0, \\ & x(e) = \sum_{r \in R_0} f_r(e) \quad \text{for each } e \in E, \\ & y(R_0) \geqslant 1. \end{aligned}$$

Let P_3^E denote the projection of P_3 into $\mathbb{R}^E$.

Proposition 10. *We have that* $P^{\uparrow} = (P_3^E)_I$.

It is clear that the separation problem over P_3 can be solved in polynomial time by a sequence of at most $|R_0|(|\mathscr{R}| - 1)$ maximum flow problems. Therefore we can optimize over P_3 in polynomial time.

As we said before, P_3 is stronger than both P_1 and P_2, as the following result shows.

Proposition 11. *We have that* $P_3^E \subseteq P_1$ *and* $P_3^E \subseteq P_2^E$.

Proof. We prove that $P_3^E \subseteq P_1$; the proof of the second assertion is analogous.

To this end, let $\bar{x} \in P_3^E$. There must be vectors $\bar{f} \in \mathbb{R}_+^{R_0 \times E}$ and $\bar{y} \in \mathbb{R}_+^{R_0}$ such that $(\bar{x}, \bar{f}, \bar{y}) \in P_3$. Now let $\mathscr{S} := (V_1, \ldots, V_p)$ be an $\mathscr{R}$-partition. Recall that no V_i covers $\mathscr{R}$ and that, of course, each vertex of R_0 belongs to exactly one set V_i. Then

$$\begin{aligned}\bar{x}(\Delta(\mathscr{S})) &= \sum_{r\in R_0} \bar{f}_r(\Delta(\mathscr{S}))\\ &\geqslant \sum_{i=1}^{p}\sum_{r\in V_i\cap R_0} \bar{f}_r(\delta(V_i))\\ &\geqslant \sum_{i=1}^{p} \bar{y}(V_i\cap R_0)\\ &= \bar{y}(R_0)\\ &\geqslant 1,\end{aligned} \tag{20}$$

and we are done. □

Polyhedron P_3 has another interesting property. To introduce it, consider $r \in R_0$ and let $\mathscr{R}[r]$ denote the collection of groups obtained from $\mathscr{R}$ by replacing group R_0 by $\{r\}$. We have then the following easy result:

Proposition 12. *Let* $c: E \to \mathbb{R}_+$ *be a cost function. Then* $\min\{c^{\mathsf{T}}x: x \in P_3^E\} = \min_{r\in R_0} \min\{c^{\mathsf{T}}x: x \in P_1(G, \mathscr{R}[r])\}$.

4. Conclusion

The GSTP arises naturally in the design of VLSI chips. We study the dominant of the group Steiner tree polytope and present some of its facet inducing inequalities. We also show how to explore the relation between the GSTP for trees and the set-covering problem.

We present a first formulation for the problem over which optimization is $\mathscr{NP}$-hard. We also present two extended formulations, one of which can be weaker than the first, more natural formulation presented. However, the second extended formulation is stronger than the two others, though it can possibly contain many more variables than the first two.

Our results can be used to derive lower bounds for the GSTP. Since our formulations take into account specific structural information of the groups, these bounds are better than the bounds obtained by reducing the GSTP to the STP as done in [4,9], in which new vertices are added, one for each group, connected to the vertices on the group it represents by edges with large costs.

We still need to investigate possible directed formulations for the GSTP. As is the case with the STP [2], these directed formulations can be stronger than the undirected ones we present.

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